

Joining means

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Dedicated to Professor Zsolt Páles on the occasion of his 60th birthday

Abstract. Modifying and generalizing some ideas from [1], we come to the notion of a marginal joint of two arbitrary means given on adjacent intervals. The construction of the joints makes use of the notion of a set-valued joiner. Also, the converse is proved: any mean can be obtained as a marginal joint of its two restrictions, produced with the use of a so-called reconstructing joiner having the smallest values in a sense. We conclude the paper by answering the question when the reconstructing joiner of the mean is a single-valued function.

1. Introduction

Let I be an interval of reals. A function $M : I \times I \rightarrow I$ is called a *mean on I* if

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$$

for all $x, y \in I$.

Take any interior point ξ of I , and put

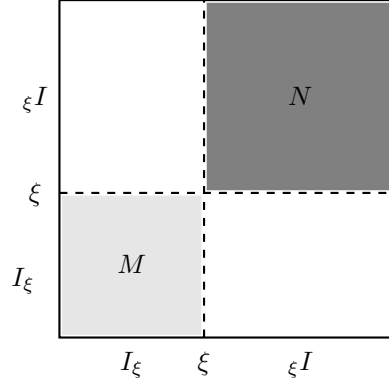
$$I_\xi := \{x \in I : x \leq \xi\}, \quad {}_\xi I := \{x \in I : \xi \leq x\} \quad (1)$$

and

$$I_\xi^\circ := \{x \in I : x < \xi\}, \quad {}_\xi I^\circ := \{x \in I : \xi < x\}. \quad (2)$$

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Our main question is as follows.

Problem 1. *Given two arbitrary means M and N on the intervals I_ξ and $_\xi I$, respectively, find a mean, say $M \oplus N$, on the interval I such that*

$$M \oplus N|_{I_\xi \times I_\xi} = M \quad \text{and} \quad M \oplus N|_{_\xi I \times _\xi I} = N.$$

Any such mean $M \oplus N$ will be called a *joint of M and N* .

Observe that, given any mean K on I , the formula

$$(M \oplus N)(x, y) = \begin{cases} M(x, y), & \text{if } (x, y) \in I_\xi \times I_\xi \\ K(x, y), & \text{if } (x, y) \in I_\xi^\circ \times _\xi I^\circ \cup _\xi I^\circ \times I_\xi^\circ \\ N(x, y), & \text{if } (x, y) \in _\xi I \times _\xi I \end{cases}$$

defines a joint of M and N . However, its values taken in the set $I_\xi^\circ \times _\xi I^\circ \cup _\xi I^\circ \times I_\xi^\circ$ need not be connected with M and N at all. Such trivial joints will not be of interest for us. In the sequel, we will focus on joints carrying information on the means M and N .

In the paper [1], Z. DARÓCZY and the authors solved Problem 1, assuming that the *marginal functions* $h_1, h_2 : I \rightarrow I$, given by

$$h_1(x) = \begin{cases} M(x, \xi), & \text{if } x \in I_\xi, \\ N(x, \xi), & \text{if } x \in _\xi I, \end{cases} \quad (3)$$

$$h_2(y) = \begin{cases} M(\xi, y), & \text{if } y \in I_\xi, \\ N(\xi, y), & \text{if } y \in _\xi I, \end{cases} \quad (4)$$

are continuous and strictly increasing. Next, we solve Problem 1 for arbitrary means.

2. Joiners

The idea is to modify and generalize some ideas from [1]. The main tool used there to produce joints $M \oplus N$ is the notion of the so-called *joining function*. Now, we replace it by its set-valued analogue, called by us a *joiner*. Using it we construct a set-valued joint with the selections being joints of the means M and N . In what follows, we assume that:

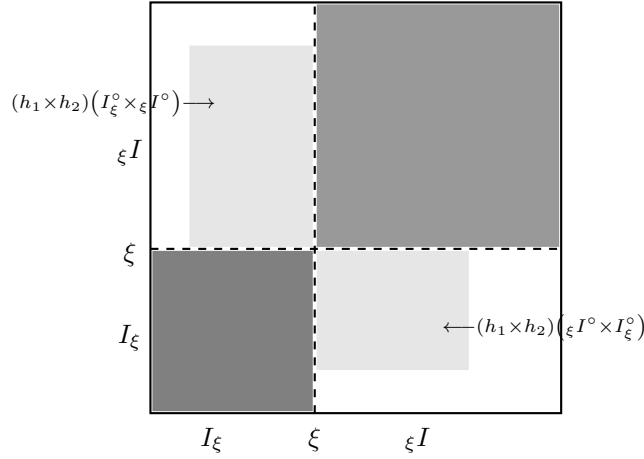
- ξ is an interior point of an interval I ;
- the intervals $I_\xi, {}_\xi I$ and $I_\xi^\circ, {}_\xi I^\circ$ are defined by (1) and (2), respectively;
- M and N are means on I_ξ and ${}_\xi I$, respectively;
- $h_1, h_2 : I \rightarrow I$ are the marginal functions given by (3) and (4), respectively.

Given any functions $f : I_\xi \rightarrow I_\xi$ and $g : {}_\xi I \rightarrow {}_\xi I$ satisfying $f(\xi) = g(\xi)$, we use the notation $f \cup g$ for the function mapping I into itself, defined by

$$(f \cup g)(x) := \begin{cases} f(x), & \text{if } x \in I_\xi, \\ g(x), & \text{if } x \in {}_\xi I. \end{cases}$$

Moreover, we define the product $h_1 \times h_2 : I \times I \rightarrow I \times I$ by the equality

$$(h_1 \times h_2)(x, y) = (h_1(x), h_2(y)).$$



Definition 1. A multifunction

$$\mathbb{K} : (h_1 \times h_2) (I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ) \rightarrow 2^I \setminus \{\emptyset\}$$

is called a *joiner of the pair* (M, N) if

$$(h_1|_{I_\xi} \cup h_2|_{{}_\xi I})^{-1} (\mathbb{K} (h_1(x), h_2(y))) \cap [x, y] \neq \emptyset \quad \text{for } (x, y) \in I_\xi^\circ \times {}_\xi I^\circ, \quad (5)$$

and

$$(h_2|_{I_\xi} \cup h_1|_{{}_\xi I})^{-1} (\mathbb{K} (h_1(x), h_2(y))) \cap [y, x] \neq \emptyset \quad \text{for } (x, y) \in {}_\xi I^\circ \times I_\xi^\circ, \quad (6)$$

or, equivalently, if for every $(x, y) \in I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ$ there is a $\kappa(x, y) \in I$ such that

$$\min\{x, y\} \leq \kappa(x, y) \leq \max\{x, y\},$$

and the function $(h_1|_{I_\xi} \cup h_2|_{{}_\xi I}) \circ \kappa$ is a selection of $K \circ (h_1 \times h_2)$.

Observe that if $\mathbb{K}_1$ is a joiner of the pair (M, N) , then so is any $\mathbb{K}_2$ such that

$$\mathbb{K}_1(x, y) \subset \mathbb{K}_2(x, y)$$

for each $(x, y) \in (h_1 \times h_2) (I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ)$.

The most trivial example of a joiner is the multifunction $\mathbb{K}$ given by $\mathbb{K}(u, v) = I$: for any point $(x, y) \in I_\xi^\circ \times {}_\xi I^\circ$ we have

$$\begin{aligned} (h_1|_{I_\xi} \cup h_2|_{{}_\xi I})^{-1} (\mathbb{K} (h_1(x), h_2(y))) &= (h_1|_{I_\xi} \cup h_2|_{{}_\xi I})^{-1} (I) \\ &= h_1^{-1} (I_\xi) \cup h_2^{-1} ({}_{{}_\xi} I) = I_\xi \cup {}_\xi I = I, \end{aligned}$$

so condition (5) holds. Similarly, (6) follows directly.

Clearly, such a joiner is not of interest. It is evident that the main point is to consider joiners and set-valued joints with relatively small values.

The following examples of joiners originated in fact in the paper [1].

Example 1. If the marginal functions $h_1, h_2 : I \rightarrow I$ are continuous and strictly increasing, and

$$K(x, y) = h_1(x) + h_2(y) - \xi, \quad (x, y) \in (h_1 \times h_2) (I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ),$$

then the formula

$$\mathbb{K}(x, y) := \{K(x, y)\} \quad (7)$$

defines a single-valued joiner of the pair (M, N) (cf. [1, Ex. 2.1]).

Example 2. Let $\varphi : I_\xi \rightarrow \mathbb{R}$ and $\psi : {}_\xi I \rightarrow \mathbb{R}$ be continuous strictly monotonic functions vanishing at ξ , and let $p, q \in (0, 1)$. Consider the quasi-arithmetic means $M = A_p^\varphi$ and $N = A_q^\psi$ on the intervals I_ξ and ${}_\xi I$, respectively:

$$M(x, y) = A_p^\varphi(x, y) = \varphi^{-1}(p\varphi(x) + (1-p)\varphi(y)), \quad x, y \in I_\xi,$$

$$N(x, y) = A_q^\psi(x, y) = \psi^{-1}(q\psi(x) + (1-q)\psi(y)), \quad x, y \in {}_\xi I.$$

For any $(x, y) \in (h_1 \times h_2)(I_\xi^\circ \times {}_\xi I^\circ)$, put

$$K(x, y) = \begin{cases} \varphi^{-1}(p(\varphi(x) + \psi(y))), & \text{if } \varphi(x) + \psi(y) < 0, \\ \psi^{-1}((1-q)(\varphi(x) + \psi(y))), & \text{if } \varphi(x) + \psi(y) \geq 0. \end{cases}$$

Similarly, having $(x, y) \in (h_1 \times h_2)({}_\xi I^\circ \times I_\xi^\circ)$, we put

$$K(x, y) = \begin{cases} \varphi^{-1}((1-p)(\psi(x) + \varphi(y))), & \text{if } \psi(x) + \varphi(y) < 0, \\ \psi^{-1}(q(\psi(x) + \varphi(y))), & \text{if } \psi(x) + \varphi(y) \geq 0. \end{cases}$$

Then $\mathbb{K}$, defined by (7), is a joiner of the pair (A_p^φ, A_q^ψ) (cf. [1, Ex. 2.2]).

More generally, one can easily check (see [1, p. 224]) that if

$$K : (h_1 \times h_2)(I_\xi \times {}_\xi I \cup {}_\xi I \times I_\xi) \rightarrow I$$

is any joining function for the pair (M, N) in the sense of the paper [1], and the marginal functions $h_1, h_2 : I \rightarrow I$ are continuous and strictly increasing, then the single-valued multifunction $\mathbb{K}$, given on $(h_1 \times h_2)(I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ)$ by (7), is a joiner of the pair (M, N) .

3. Marginal joints of means

The following result yields a pretty general procedure of joining two means.

Theorem 1. *Let $\mathbb{K}$ be any joiner of the pair (M, N) . Then the values of the multifunction $M \oplus_{\mathbb{K}} N$ defined as*

$$\begin{cases} \{M(x, y)\}, & \text{if } (x, y) \in I_\xi \times I_\xi, \\ (h_1|_{I_\xi} \cup h_2|_{{}_\xi I})^{-1}(\mathbb{K}(h_1(x), h_2(y))) \cap [x, y], & \text{if } (x, y) \in I_\xi^\circ \times {}_\xi I^\circ, \\ (h_2|_{{}_\xi I} \cup h_1|_{I_\xi})^{-1}(\mathbb{K}(h_1(x), h_2(y))) \cap [y, x], & \text{if } (x, y) \in {}_\xi I^\circ \times I_\xi^\circ, \\ \{N(x, y)\}, & \text{if } (x, y) \in {}_\xi I \times {}_\xi I, \end{cases}$$

are non-empty, and every its selection is a mean on the interval I , extending both the means M and N .

PROOF. It is enough to follow Definition 1. $\square$

Definition 2. Any mean $M \oplus_{\mathbb{K}} N$ constructed above is called a *marginal $\mathbb{K}$ -joint of the means M and N* .

4. The converse problem

Now, we would like to answer the following question.

Can any mean L on the interval I be reconstructed as a $\mathbb{K}$ -joint $M \oplus_{\mathbb{K}} N$ of the restricted means $M := L|_{I_{\xi} \times I_{\xi}}$ and $N := L|_{\xi I \times_{\xi} I}$, with a suitable joiner $\mathbb{K}$?

However, that question is not well-posed since it can be answered in the following quite trivial way. Namely, if $\mathbb{K}$ is defined by $\mathbb{K}(x, y) = I$, then the joint $M \oplus_{\mathbb{K}} N$ is given by

$$(M \oplus_{\mathbb{K}} N)(x, y) = \begin{cases} \{L(x, y)\}, & \text{if } (x, y) \in I_{\xi} \times I_{\xi} \cup \xi I \times_{\xi} I, \\ [x, y], & \text{if } (x, y) \in I_{\xi}^{\circ} \times_{\xi} I^{\circ}, \\ [y, x], & \text{if } (x, y) \in \xi I^{\circ} \times I_{\xi}^{\circ}, \end{cases}$$

and every its selection is a mean on I . Note that L is one of those selections. The reason of that phenomenon is completely clear: the values of the used joiner are too big. So, we will try to answer the following modified question.

Problem 2. *Can any mean L on the interval I be reconstructed as a $\mathbb{K}$ -joint $M \oplus_{\mathbb{K}} N$ of the restricted means $M := L|_{I_{\xi} \times I_{\xi}}$ and $N := L|_{\xi I \times_{\xi} I}$, with a suitable joiner $\mathbb{K}$ with relatively small values?*

Fix a mean L on the interval I , and put

$$M := L|_{I_{\xi} \times I_{\xi}} \quad \text{and} \quad N := L|_{\xi I \times_{\xi} I}.$$

Define marginal functions $h_1, h_2 : I \rightarrow I$ by the formulas

$$h_1(x) = L(x, \xi) \quad \text{and} \quad h_2(y) = L(\xi, y),$$

respectively. Then

$$h_1(x) = \begin{cases} M(x, \xi), & \text{if } x \in I_{\xi}, \\ N(x, \xi), & \text{if } x \in \xi I, \end{cases}$$

and

$$h_2(x) = \begin{cases} M(\xi, y), & \text{if } x \in I_{\xi}, \\ N(\xi, y), & \text{if } x \in \xi I. \end{cases}$$

The formula

$$(u, v) \sim (x, y) : \Longleftrightarrow h_1(u) = h_1(x) \quad \text{and} \quad h_2(v) = h_2(y)$$

defines an equivalence relation in the set $I_\xi^\circ \times_\xi I^\circ \cup_\xi I^\circ \times I_\xi^\circ$. Denoting by $(\mathfrak{x}_0, \mathfrak{y}_0)$ the equivalence class of the point $(x_0, y_0) \in I_\xi^\circ \times_\xi I^\circ \cup_\xi I^\circ \times I_\xi^\circ$, we have

$$\begin{aligned} (\mathfrak{x}_0, \mathfrak{y}_0) &= \{(x, y) \in I_\xi^\circ \times_\xi I^\circ \cup_\xi I^\circ \times I_\xi^\circ : h_1(x) = h_1(x_0) \quad \text{and} \quad h_2(y) = h_2(y_0)\} \\ &= \{(x, y) \in I_\xi^\circ \times_\xi I^\circ \cup_\xi I^\circ \times I_\xi^\circ : (h_1 \times h_2)(x, y) = (h_1(x_0), h_2(y_0))\} \\ &= (h_1 \times h_2)^{-1}(\{h_1(x_0), h_2(y_0)\}) \cap (I_\xi^\circ \times_\xi I^\circ \cup_\xi I^\circ \times I_\xi^\circ). \end{aligned}$$

This means that the equivalence class $(\mathfrak{x}_0, \mathfrak{y}_0)$ is the level set of the point $(h_1(x_0), h_2(y_0))$ under the product $h_1 \times h_2$ restricted to $I_\xi^\circ \times_\xi I^\circ \cup_\xi I^\circ \times I_\xi^\circ$.

The next result gives a positive answer to the question posed in this section.

Theorem 2. Let $\mathbb{K}_0 : (h_1 \times h_2)(I_\xi^\circ \times_\xi I^\circ \cup_\xi I^\circ \times I_\xi^\circ) \rightarrow 2^I \setminus \{\emptyset\}$ be given by

$$\mathbb{K}_0(h_1(x), h_2(y)) = (h_1|_{I_\xi} \cup h_2|_{I_\xi I})(L((\mathfrak{x}_0, \mathfrak{y}_0))), \quad (x, y) \in I_\xi^\circ \times_\xi I^\circ, \quad (8)$$

and

$$\mathbb{K}_0(h_1(x), h_2(y)) = (h_2|_{I_\xi} \cup h_1|_{I_\xi I})(L((\mathfrak{x}_0, \mathfrak{y}_0))), \quad (x, y) \in I_\xi^\circ \times I_\xi^\circ. \quad (9)$$

Then

- (i) $\mathbb{K}_0$ is a joiner of the pair $(L|_{I_\xi \times I_\xi}, L|_{I_\xi I \times_\xi I})$ and satisfies the condition

$$L((\mathfrak{x}_0, \mathfrak{y}_0)) \subset (L|_{I_\xi \times I_\xi} \oplus_{\mathbb{K}_0} L|_{I_\xi I \times_\xi I})(x, y), \quad (x, y) \in I_\xi^\circ \times_\xi I^\circ \cup_\xi I^\circ \times I_\xi^\circ,$$

and

- (ii) if $\mathbb{K}$ is a joiner of the pair $(L|_{I_\xi \times I_\xi}, L|_{I_\xi I \times_\xi I})$ and satisfies

$$L((\mathfrak{x}_0, \mathfrak{y}_0)) \subset (L|_{I_\xi \times I_\xi} \oplus_{\mathbb{K}_0} L|_{I_\xi I \times_\xi I})(x, y), \quad (x, y) \in I_\xi^\circ \times_\xi I^\circ \cup_\xi I^\circ \times I_\xi^\circ, \quad (10)$$

then $\mathbb{K}_0 \subset \mathbb{K}$, that is,

- (i) the mean L can be reconstructed as a selection for the marginal $\mathbb{K}_0$ -joint of its restrictions $L|_{I_\xi \times I_\xi}$ and $L|_{I_\xi I \times_\xi I}$,

and

- (ii) $\mathbb{K}_0$ is the smallest (in the sense of inclusion) joiner of the pair $(L|_{I_\xi \times I_\xi}, L|_{I_\xi I \times_\xi I})$ satisfying condition (10).

PROOF. (i) For every $(x, y) \in I_\xi^\circ \times {}_\xi I^\circ$, we have

$$\begin{aligned} & (h_1|_{I_\xi} \cup h_2|_{{}_\xi I})^{-1} [(h_1|_{I_\xi} \cup h_2|_{{}_\xi I}) (L((\mathbb{x}_0, \mathbb{y}_0)))] \cap [x, y] \\ & \supset L((\mathbb{x}_0, \mathbb{y}_0)) \cap [x, y] \ni L(x, y). \end{aligned}$$

Similarly, if $(x, y) \in {}_\xi I^\circ \times I_\xi^\circ$, then

$$\begin{aligned} & (h_2|_{I_\xi} \cup h_1|_{{}_\xi I})^{-1} [(h_2|_{I_\xi} \cup h_1|_{{}_\xi I}) (L((\mathbb{x}_0, \mathbb{y}_0)))] \cap [y, x] \\ & \supset L((\mathbb{x}_0, \mathbb{y}_0)) \cap [y, x] \ni L(x, y). \end{aligned}$$

Thus $\mathbb{K}_0$ is a joiner of the pair $(L_{I_\xi \times I_\xi}, L_{{}_\xi I \times {}_\xi I})$ and

$$(L_{I_\xi \times I_\xi} \oplus_{\mathbb{K}_0} L_{{}_\xi I \times {}_\xi I}) \supset L((\mathbb{x}_0, \mathbb{y}_0)),$$

for each $(x, y) \in I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ$.

(ii) Take any joiner $\mathbb{K}$ of the pair $(L_{I_\xi \times I_\xi}, L_{{}_\xi I \times {}_\xi I})$ satisfying condition (10). Then, for any $(x, y) \in I_\xi^\circ \times {}_\xi I^\circ$, we have

$$\begin{aligned} \mathbb{K}_0(h_1(x), h_2(y)) &= (h_1|_{I_\xi} \cup h_2|_{{}_\xi I}) (L((\mathbb{x}_0, \mathbb{y}_0))) \\ &\subset (h_1|_{I_\xi} \cup h_2|_{{}_\xi I}) [(L_{I_\xi \times I_\xi} \oplus_{\mathbb{K}} L_{{}_\xi I \times {}_\xi I})(x, y)] \\ &= (h_1|_{I_\xi} \cup h_2|_{{}_\xi I}) [(h_1|_{I_\xi} \cup h_2|_{{}_\xi I})^{-1} (\mathbb{K}(h_1(x), h_2(y))) \cap [x, y]] \\ &\subset \mathbb{K}(h_1(x), h_2(y)). \end{aligned}$$

Repeating the calculation for an arbitrary point $(x, y) \in {}_\xi I^\circ \times I_\xi^\circ$, we come to the assertion. $\square$

Definition 3. The multifunction $\mathbb{K}_0$, introduced in Theorem 2, is called ξ -reconstructing joiner for the mean L .

5. Reconstructing joiner with singleton values

The following question arises naturally.

Problem 3. *Is it possible that the reconstructing joiner is in fact a single-valued function?*

Below we give a full answer to this question, providing a simple characterization of means with single-valued reconstructing joiners.

We say that a mean L *preserves its ξ -margins* if the equalities $L(u_1, \xi) = L(u_2, \xi)$ and $L(\xi, v_1) = L(\xi, v_2)$ imply that

$$\begin{aligned} & L(\min\{L(u_1, v_1), \xi\}, \max\{L(u_1, v_1), \xi\}) \\ &= L(\min\{L(u_2, v_2), \xi\}, \max\{L(u_2, v_2), \xi\}) \end{aligned}$$

for all $(u_1, v_1), (u_2, v_2) \in I_\xi^\circ \times {}_\xi I^\circ$, and

$$\begin{aligned} & L(\max\{L(u_1, v_1), \xi\}, \min\{L(u_1, v_1), \xi\}) \\ &= L(\max\{L(u_2, v_2), \xi\}, \min\{L(u_2, v_2), \xi\}) \end{aligned}$$

for all $(u_1, v_1), (u_2, v_2) \in {}_\xi I^\circ \times I_\xi^\circ$.

Remark 1. Observe that for symmetric means the above defining condition becomes much simpler: the equalities $L(u_1, \xi) = L(u_2, \xi)$ and $L(v_1, \xi) = L(v_2, \xi)$ imply that

$$L(L(u_1, v_1), \xi) = L(L(u_2, v_2), \xi)$$

for all $(u_1, v_1), (u_2, v_2) \in I_\xi^\circ \times {}_\xi I^\circ$ and $(u_1, v_1), (u_2, v_2) \in {}_\xi I^\circ \times I_\xi^\circ$.

Remark 2. If the marginal functions $L(\cdot, \xi)$ and $L(\xi, \cdot)$ are one-to-one, then the mean L preserves its ξ -margins.

Example 3. Take $I = \mathbb{R}$ and $\xi = 0$, and put

$$L(x, y) = \max\{x, y\}, \quad x, y \in I.$$

Of course, the marginal function $L(\cdot, 0)$ is not one-to-one. Nevertheless, L preserves its 0-margins. To see this, take any $(u_1, v_1), (u_2, v_2) \in I_0^\circ \times {}_0 I^\circ$ satisfying $L(u_1, 0) = L(u_2, 0)$ and $L(v_1, 0) = L(v_2, 0)$. Then $u_1 < 0 < v_1$ and $u_2 < 0 < v_2$, whence also $L(u_1, v_1) = v_1$ and $L(u_2, v_2) = v_2$. Therefore,

$$L(L(u_1, v_1), 0) = L(v_1, 0) = L(v_2, 0) = L(L(u_2, v_2), 0).$$

A similar condition holds whenever $(u_1, v_1), (u_2, v_2) \in {}_0 I^\circ \times I_0^\circ$. Thus, by Remark 1, the mean L preserves its 0-margins.

Example 4. Of course, not every mean preserves its margins. To see this, take $\xi = 1$ and define L as the contraharmonic mean on the interval $(0, +\infty)$ by the equality

$$L(x, y) = \frac{x^2 + y^2}{x + y}.$$

Suppose that L preserves its 1-margins. As it is symmetric, we may use Remark 1. Take an arbitrary $v \in (1, +\infty)$ and put $u_1 = \frac{1}{5}, u_2 = \frac{2}{3}, v_1 = v_2 = v$. Then $(u_1, v_1), (u_2, v_2) \in I_1^\circ \times {}_1I^\circ$,

$$L(u_1, 1) = L\left(\frac{1}{5}, 1\right) = \frac{13}{15} = L\left(\frac{2}{3}, 1\right) = L(u_2, 1),$$

and, of course, $L(v_1, 1) = L(v_2, 1) = L(v, 1)$. Thus, by Remark 1,

$$L(L(u_1, v), 1) = L(L(u_2, v), 1). \quad (11)$$

Since

$$\lim_{v \rightarrow \infty} L(u, v) = \lim_{v \rightarrow \infty} \frac{u^2 + v^2}{u + v} = +\infty, \quad u \in (0, +\infty),$$

we may choose v in such a way that $L(u_1, v) > 1$ and $L(u_2, v) > 1$. Then, as the restriction of $L(\cdot, 1)$ to $(1, +\infty)$ is one-to-one, we have $L(u_1, v) = L(u_2, v)$. Taking into account that $u_1 = \frac{1}{5}$ and $u_2 = \frac{2}{3}$, we see that

$$\frac{25v^2 + 1}{25v + 5} = \frac{9v^2 + 4}{9v + 6},$$

whence

$$15v^2 - 13v - 2 = 0,$$

that is, $v \in \{-\frac{2}{15}, 1\}$, a contradiction.

The main result of this section reads as follows.

Theorem 3. *The ξ -reconstructing joiner of the mean L has only singletons among the values if and only if L preserves its ξ -margins.*

PROOF. Assume that the reconstructing joiner $\mathbb{K}_0$ given by (8) and (9) is single-valued. Take any points $(u_1, v_1), (u_2, v_2) \in I_\xi^\circ \times {}_\xi I^\circ$ satisfying $L(u_1, \xi) = L(u_2, \xi)$ and $L(\xi, v_1) = L(\xi, v_2)$. Consider the following four possible cases:

- (a) $L(u_1, u_2) \leq \xi$ and $L(v_1, v_2) \leq \xi$;
- (b) $L(u_1, u_2) \leq \xi < L(v_1, v_2)$;
- (c) $L(v_1, v_2) \leq \xi < L(u_1, u_2)$;
- (d) $\xi < L(u_1, u_2)$ and $\xi < L(v_1, v_2)$.

According to (8), we have

$$\mathbb{K}_0(h_1(u_1), h_2(v_1)) = h_1(L((\mathfrak{u}_1, \mathfrak{v}_1)) \cap I_\xi) \cup h_2(L((\mathfrak{u}_1, \mathfrak{v}_1)) \cap {}_\xi I^\circ).$$

Therefore, as the above set is a singleton, we conclude that:

$$h_1(L(u_1, v_1)) = h_1(L(u_2, v_2)),$$

i.e. $L(L(u_1, v_1), \xi) = L(L(u_2, v_2), \xi)$ in case (a),

$$h_1(L(u_1, v_1)) = h_2(L(u_2, v_2)),$$

i.e. $L(L(u_1, v_1), \xi) = L(\xi, L(u_2, v_2))$ in case (b),

$$h_2(L(u_1, v_1)) = h_1(L(u_2, v_2)),$$

i.e. $L(\xi, L(u_1, v_1)) = L(L(u_2, v_2), \xi)$ in case (c),

$$h_2(L(u_1, v_1)) = h_2(L(u_2, v_2)),$$

i.e. $L(\xi, L(u_1, v_1)) = L(\xi, L(u_2, v_2))$ in case (d). In all cases (a)–(d), the obtained equalities mean that

$$\begin{aligned} & L(\min\{L(u_1, v_1), \xi\}, \max\{L(u_1, v_1), \xi\}) \\ &= L(\min\{L(u_2, v_2), \xi\}, \max\{L(u_2, v_2), \xi\}). \end{aligned}$$

A similar reasoning gives the second desired equality in the case when $(u_1, v_1), (u_2, v_2) \in {}_\xi I^\circ \times I_\xi^\circ$. So L preserves its ξ -margins.

A careful analysis of the above proof shows that also the converse implication holds true. $\square$

Finally, we notice the following immediate consequence of Theorem 3 and Remark 2.

Corollary 1. *If the marginal functions $L(\cdot, \xi)$ and $L(\xi, \cdot)$ are one-to-one, then the ξ -reconstructing joiner of L is single-valued.*

References

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