

A new characterization of Clifford torus

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Abstract. We extend a previous sharp upper bound of the first strong stability eigenvalue due to ALÍAS *et al.* [1], for the context of a closed submanifold immersed with nonzero parallel mean curvature vector field in the Euclidean sphere, and through this result, we obtain a new characterization for the Clifford torus.

1. Introduction

Given a closed submanifold M^n immersed in the unit Euclidean sphere $\mathbb{S}^{n+p}$ with parallel mean curvature vector field h (which means that h is parallel as a section of the normal bundle of M^n), its *strong stability operator* is defined by

$$J = -\Delta - |\Phi|^2 - n(1 + H^2), \quad (1)$$

where Δ stands for the Laplacian operator on M^n , $|\Phi|$ denotes the length of the traceless second fundamental Φ , and $H = |h|$ is the mean curvature of M^n . We observe that, when $p = 1$, J arises to the classical Jacobi operator established in [2].

We note that J belongs to a class of operators which are usually referred to as Schrödinger operators, that is, operators of the form $\Delta + q$, where q is any continuous function on M^n . The *first strong stability eigenvalue* λ_1^J of M^n is defined as being the smallest real number λ which satisfies the equation $Jf - \lambda f =$

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0 in M^n , for some nonzero $f \in C^\infty(M)$. As it is well known, λ_1^J has the following min-max characterization:

$$\lambda_1^J = \inf \left\{ \frac{\int_M f J f \, dM}{\int_M f^2 \, dM}; f \in C^\infty(M), f \neq 0 \right\}, \quad (2)$$

where dM stands for the volume element with respect to the metric induced of M^n .

In his seminal work [9], SIMONS studied the first strong stability eigenvalue of a minimal closed hypersurface M^n immersed in $\mathbb{S}^{n+1}$. In this setting, he proved that either $\lambda_1^J = -n$, and M^n is a totally geodesic sphere, or $\lambda_1^J \leq -2n$, otherwise. Later on, WU in [10] characterized the equality $\lambda_1^J = -2n$ by showing that it holds only for the minimal Clifford torus. Shortly thereafter, PERDOMO [7] provides a new proof of this spectral characterization by the value of λ_1^J . Afterwards, ALÍAS, BARROS and BRASIL JR. [1] extended these results to the case of constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$, characterizing Clifford torus via the value of λ_1^J .

Proceeding with this picture, we obtain the following extension of the main result of [1] for the context of higher codimension.

Theorem 1.1. *Let M^n be a closed submanifold immersed in $\mathbb{S}^{n+p}$, $n \geq 4$, with nonzero parallel mean curvature vector field. If the normalized scalar curvature of M^n satisfies $R \geq 1$, then*

- (i) *either $\lambda_1^J = -n(1 + H^2)$ (and M^n is totally umbilical),*
- (ii) *or*

$$\lambda_1^J \leq -2n(1 + H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}} H \max_M |\Phi|.$$

Moreover, the equality occurs if and only if M^n is a Clifford torus $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$, with $r^2 \leq \frac{n-2}{n}$.

The proof of Theorem 1.1 is given in Section 3.

2. Some preliminaries and key lemmas

Let M^n be an n -dimensional connected submanifold immersed in a unit Euclidean sphere $\mathbb{S}^{n+p}$. Let $\{\omega_B\}$ be the corresponding dual coframe, and $\{\omega_{BC}\}$ the connection 1-forms on $\mathbb{S}^{n+p}$. We choose a local field of orthonormal frame $\{e_1, \dots, e_{n+p}\}$ in $\mathbb{S}^{n+p}$, with dual coframe $\{\omega_1, \dots, \omega_{n+p}\}$, such that, at each point

of M^n , $e_1, \dots, e_n$ are tangent to M^n , and $e_{n+1}, \dots, e_{n+p}$ are normal to M^n . We will use the following convention for indices

$$1 \leq A, B, C, \dots \leq n+p, \quad 1 \leq i, j, k, \dots \leq n \quad \text{and} \quad n+1 \leq \alpha, \beta, \gamma, \dots \leq n+p.$$

With restricting on M^n , the second fundamental form A , the curvature tensor R and the normal curvature tensor $R^\perp$ of M^n are given by

$$\begin{aligned} \omega_{i\alpha} &= \sum_j h_{ij}^\alpha \omega_j, \quad A = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha, \\ d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \\ d\omega_{\alpha\beta} &= \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \frac{1}{2} \sum_{k,l} R_{\alpha\beta kl}^\perp \omega_k \wedge \omega_l. \end{aligned}$$

The Gauss equation is

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha).$$

In particular, the components of the Ricci tensor R_{ik} and the normalized scalar curvature R are given, respectively, by

$$R_{ik} = (n-1)\delta_{ik} + n \sum_\alpha H^\alpha h_{ik}^\alpha - \sum_{\alpha,j} h_{ij}^\alpha h_{jk}^\alpha \quad (3)$$

and

$$R = \frac{1}{(n-1)} \sum_i R_{ii}. \quad (4)$$

From (3) and (4), we get the following relation

$$n(n-1)R = n(n-1) + n^2 H^2 - S, \quad (5)$$

where $S = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2$ is the squared norm of the second fundamental form, and, being $h = \sum_\alpha H^\alpha e_\alpha = \frac{1}{n} \sum_\alpha (\sum_k h_{kk}^\alpha) e_\alpha$ the mean curvature vector field, $H = |h|$ is the mean curvature function of M^n .

The Ricci equation is given by

$$R_{\alpha\beta ij}^\perp = \sum_k (h_{ik}^\alpha h_{kj}^\beta - h_{jk}^\alpha h_{ki}^\beta). \quad (6)$$

From now on, we will deal with submanifolds M^n of $\mathbb{S}^{n+p}$ having *nonzero parallel mean curvature vector field*, which means that the mean curvature function H is, in fact, a positive constant, and that the corresponding mean curvature vector field h is parallel as a section of the normal bundle. In this context, we can choose a local orthonormal frame $\{e_1, \dots, e_{n+p}\}$ such that $e_{n+1} = \frac{h}{H}$. Thus,

$$H^{n+1} = \frac{1}{n} \operatorname{tr}(h^{n+1}) = H \quad \text{and} \quad H^\alpha = \frac{1}{n} \operatorname{tr}(h^\alpha) = 0, \alpha \geq n+2. \quad (7)$$

We will also consider the following symmetric tensor

$$\Phi = \sum_{\alpha, i, j} \Phi_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha, \quad (8)$$

where $\Phi_{ij}^\alpha = h_{ij}^\alpha - H\delta_{ij}$. Consequently, we have that

$$\Phi_{ij}^{n+1} = h_{ij}^{n+1} - H\delta_{ij} \quad \text{and} \quad \Phi_{ij}^\alpha = h_{ij}^\alpha, \quad n+2 \leq \alpha \leq n+p. \quad (9)$$

Let $|\Phi|^2 = \sum_{\alpha, i, j} (\Phi_{ij}^\alpha)^2$ be the square of the length of Φ . From (5), it is not difficult to verify that Φ is traceless with

$$|\Phi|^2 = S - nH^2. \quad (10)$$

From [5, Lemma 4.1] we obtain the following Simons type formula:

Lemma 1. *Let M^n be an n -dimensional ($n \geq 2$) submanifold immersed with nonzero parallel mean curvature vector field in the Euclidean sphere $\mathbb{S}^{n+p}$. Then, we have*

$$\frac{1}{2} \Delta |\Phi|^2 = |\nabla \Phi|^2 + n |\Phi|^2 + n \sum_{\beta, i, j, k} H h_{ij}^{n+1} h_{jk}^\beta h_{ki}^\beta - \sum_{i, j, k, l} \left(\sum_{\alpha} h_{ij}^\alpha h_{kl}^\alpha \right)^2 - \sum_{i, j, \alpha, \beta} (R_{\alpha\beta ij}^\perp)^2.$$

The next key lemma is due to BARROS *et al.* (see [3, Lemma 1]).

Lemma 2. *Let M^n be a Riemannian manifold isometrically immersed into a Riemannian manifold N^{n+p} . Consider $\Psi = \sum_{\alpha, i, j} \Psi_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha$ a traceless symmetric tensor satisfying Codazzi equation. Then the following inequality holds:*

$$|\nabla |\Psi|^2|^2 \leq \frac{4n}{n+2} |\Psi|^2 |\nabla \Psi|^2,$$

where $|\Psi|^2 = \sum_{\alpha, i, j} (\Psi_{ij}^\alpha)^2$ and $|\nabla \Psi|^2 = \sum_{\alpha, i, j, k} (\Psi_{ijk}^\alpha)^2$. In particular, the conclusion holds for the tensor Φ defined in (8).

In order to prove Theorem 1.1, we will also need two algebraic lemmas. The proof of them can be found in [8] and [6], respectively.

Lemma 3. *Let $B, C : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be symmetric linear maps such that $BC - CB = 0$ and $\text{tr} B = \text{tr} C = 0$, then*

$$-\frac{n-2}{\sqrt{n(n-1)}}|B|^2|C| \leq \text{tr}(B^2C) \leq \frac{n-2}{\sqrt{n(n-1)}}|B|^2|C|.$$

Lemma 4. *Let $B^1, B^2, \dots, B^n$ be symmetric $(n \times n)$ -matrices. Set $S_{\alpha\beta} = \text{tr}(B^\alpha B^\beta)$, $S_\alpha = S_{\alpha\alpha}$, $S = \sum_\alpha S_\alpha$, then*

$$\sum_{\alpha,\beta} |B^\alpha B^\beta - B^\beta B^\alpha|^2 + \sum_{\alpha,\beta} S_{\alpha\beta}^2 \leq \frac{3}{2} \left(\sum_\alpha S_\alpha \right)^2.$$

3. Proof of Theorem 1.1

From Lemma 1 we have that

$$\begin{aligned} \frac{1}{2}\Delta|\Phi|^2 &= |\nabla\Phi|^2 + n|\Phi|^2 + n \sum_{\beta,i,j,k} H h_{ij}^{n+1} h_{jk}^\beta h_{ki}^\beta \\ &\quad - \sum_{i,j,k,l} \left(\sum_\alpha h_{ij}^\alpha h_{kl}^\alpha \right)^2 - \sum_{i,j,\alpha,\beta} (R_{\alpha\beta ij}^\perp)^2. \end{aligned} \quad (11)$$

From (7) and (9) we get

$$\begin{aligned} \sum_{i,j,k,\beta} H h_{ij}^{n+1} h_{jk}^\beta h_{ki}^\beta &= \sum_{i,j,k} H h_{ij}^{n+1} h_{jk}^{n+1} h_{ki}^{n+1} + \sum_{\beta=n+2}^{n+p} \sum_{i,j,k} H h_{ij}^{n+1} \Phi_{jk}^\beta \Phi_{ki}^\beta \\ &= H \text{tr}(\Phi^{n+1} + HI)^3 + \sum_{\beta=n+2}^{n+p} \sum_{i,j,k} H \Phi_{ij}^{n+1} \Phi_{jk}^\beta \Phi_{ki}^\beta + \sum_{\beta=n+2}^{n+p} H^2 |\Phi^\beta|^2 \\ &= H \text{tr}(\Phi^{n+1})^3 + 3H^2 |\Phi^{n+1}|^2 + nH^4 + \sum_{\beta=n+2}^{n+p} H^2 |\Phi^\beta|^2 \\ &\quad + \sum_{\beta=n+2}^{n+p} \sum_{i,j,k} H \Phi_{ij}^{n+1} \Phi_{jk}^\beta \Phi_{ki}^\beta. \end{aligned} \quad (12)$$

Since $\text{tr } \Phi^\alpha = 0$ and $\Phi^{n+1}\Phi^\beta - \Phi^\beta\Phi^{n+1} = 0, n+2 \leq \beta \leq n+p$, from Lemma 3 we obtain

$$\begin{aligned}
& H\text{tr}(\Phi^{n+1})^3 + 3H^2|\Phi^{n+1}|^2 + nH^4 + \sum_{\beta=n+2}^{n+p} H^2|\Phi^\beta|^2 + \sum_{\beta=n+2}^{n+p} \sum_{i,j,k} H\Phi_{ij}^{n+1}\Phi_{jk}^\beta\Phi_{ki}^\beta \\
& \geq -\frac{n-2}{\sqrt{n(n-1)}}H|\Phi^{n+1}|^3 + 2H^2|\Phi^{n+1}|^2 + H^2|\Phi|^2 + nH^4 \\
& \quad - \frac{n-2}{\sqrt{n(n-1)}} \sum_{\beta=n+2}^{n+p} H|\Phi^{n+1}||\Phi^\beta|^2 \\
& = 2H^2|\Phi^{n+1}|^2 + H^2|\Phi|^2 + nH^4 - \frac{n-2}{\sqrt{n(n-1)}}H|\Phi^{n+1}||\Phi|^2. \tag{13}
\end{aligned}$$

Hence, from (12) and (13) we have

$$\sum_{\beta,i,j,k} Hh_{ij}^{n+1}h_{jk}^\beta h_{ki}^\beta \geq 2H^2|\Phi^{n+1}|^2 + H^2|\Phi|^2 + nH^4 - \frac{n-2}{\sqrt{n(n-1)}}H|\Phi^{n+1}||\Phi|^2. \tag{14}$$

From Ricci equation (6) we get

$$\begin{aligned}
& \sum_{i,j,k,l} \left(\sum_{\alpha} h_{ij}^\alpha h_{kl}^\alpha \right)^2 + \sum_{\alpha,\beta,i,j} (R_{\alpha\beta ij}^\perp)^2 \\
& = \sum_{\alpha,\beta} (\text{tr}(A^\alpha A^\beta))^2 + \sum_{\alpha \neq n+1, \beta \neq n+1, i,j} (R_{\alpha\beta ij}^\perp)^2 \\
& = [\text{tr}(A^{n+1} A^{n+1})]^2 + 2 \sum_{\beta \neq n+1} [\text{tr}(A^{n+1} A^\beta)]^2 \\
& \quad + \sum_{\alpha \neq n+1, \beta \neq n+1} |A^\alpha A^\beta - A^\beta A^\alpha|^2 + \sum_{\alpha \neq n+1, \beta \neq n+1} (\text{tr}(A^\alpha A^\beta))^2. \tag{15}
\end{aligned}$$

But, using (9) and Lemma 4, we obtain

$$\begin{aligned}
& \frac{3}{2} \left(\sum_{\beta \neq n+1} |\Phi^\beta| \right)^2 \geq \frac{3}{2} \left(\sum_{\beta \neq n+1} \text{tr}(A^\beta A^\alpha) \right)^2 \\
& \geq \sum_{\alpha \neq n+1, \beta \neq n+1} [\text{tr}(A^\alpha A^\beta)]^2 + \sum_{\alpha \neq n+1, \beta \neq n+1} |A^\alpha A^\beta - A^\beta A^\alpha|^2. \tag{16}
\end{aligned}$$

Hence, from (15) and (16) we have

$$\sum_{i,j,k,l} \left(\sum_{\alpha} h_{ij}^\alpha h_{kl}^\alpha \right)^2 + \sum_{\alpha,\beta,i,j} (R_{\alpha\beta ij}^\perp)^2$$

$$\begin{aligned}
&\leq [\operatorname{tr}(A^{n+1}A^{n+1})]^2 + 2 \sum_{\beta \neq n+1} [\operatorname{tr}(A^{n+1}A^\beta)]^2 + \frac{3}{2} \left(\sum_{\beta \neq n+1} |\Phi^\beta|^2 \right)^2 \\
&= |\Phi^{n+1}|^4 + 2nH^2|\Phi^{n+1}|^2 + n^2H^4 + 2 \sum_{\beta \neq n+1} [\operatorname{tr}(\Phi^{n+1}\Phi^\beta)]^2 + \frac{3}{2}(|\Phi|^2 - |\Phi^{n+1}|^2)^2 \\
&\leq \frac{5}{2}|\Phi^{n+1}|^4 + 2nH^2|\Phi^{n+1}|^2 + n^2H^4 + \frac{3}{2}|\Phi|^4 + 2|\Phi^{n+1}|^2(|\Phi|^2 - |\Phi^{n+1}|^2) - 3|\Phi|^2|\Phi^{n+1}|^2 \\
&= \frac{1}{2}|\Phi^{n+1}|^4 + 2nH^2|\Phi^{n+1}|^2 + n^2H^4 - |\Phi|^2|\Phi^{n+1}|^2 + \frac{3}{2}|\Phi|^4. \tag{17}
\end{aligned}$$

Therefore, from (11), (14) and (17) we get

$$\begin{aligned}
&\frac{1}{2}\Delta|\Phi|^2 \\
&\geq n|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi^{n+1}||\Phi|^2 + nH^2|\Phi|^2 - \frac{1}{2}|\Phi^{n+1}|^4 + |\Phi|^2|\Phi^{n+1}|^2 - \frac{3}{2}|\Phi|^4 \\
&= (|\Phi| - |\Phi^{n+1}|) \left(\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^2 - \frac{1}{2}(|\Phi| - |\Phi^{n+1}|)(|\Phi| + |\Phi^{n+1}|)^2 \right) \\
&\quad + |\Phi|^2 \left(-|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + n(1 + H^2) \right). \tag{18}
\end{aligned}$$

On the other hand, we note that the following algebraic inequality (3.5) of [4] also holds:

$$(|\Phi| - |\Phi^{n+1}|)(|\Phi| + |\Phi^{n+1}|)^2 \leq \frac{32}{27}|\Phi|^3. \tag{19}$$

Moreover, since $R \geq 1$, we use (5) and (10), in order to obtain

$$n^2H^2 = S + n(n-1)(R-1) \geq S = |\Phi|^2 + nH^2,$$

which gives us

$$H \geq \frac{1}{\sqrt{n(n-1)}}|\Phi|. \tag{20}$$

Thus, from (19) and (20) we conclude that

$$\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^2 - \frac{1}{2}(|\Phi| - |\Phi^{n+1}|)(|\Phi| + |\Phi^{n+1}|)^2 \geq \left(\frac{n-2}{n-1} - \frac{16}{27} \right) |\Phi|^3. \tag{21}$$

But, taking into account our assumption that $n \geq 4$, we have that

$$\frac{n-2}{n-1} - \frac{16}{27} > 0. \tag{22}$$

Consequently, inserting (13), (21) and (22) in (18), we get that

$$\begin{aligned} \frac{1}{2}\Delta|\Phi|^2 &\geq |\nabla\Phi|^2 - |\Phi|^2 P_H(|\Phi|) + (|\Phi| - |\Phi^{n+1}|) \left(\frac{n-2}{n-1} - \frac{16}{27} \right) |\Phi|^3 \\ &\geq |\nabla\Phi|^2 - |\Phi|^2 P_H(|\Phi|), \end{aligned} \quad (23)$$

where

$$P_H(x) = |\Phi|^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi| - n(1+H^2).$$

If M^n is totally umbilical, then $|\Phi|^2 = 0$ and $J = -\Delta - n(1+H^2)$, where H is constant, so $\lambda_1^J = \lambda_1^{-\Delta} - n(1+H^2) = -n(1+H^2)$, whose corresponding eigenfunctions are the constant functions. On the other hand, following the ideas of [1], when M^n is not umbilical, for an arbitrary $\varepsilon > 0$, we consider the positive smooth function $f_\varepsilon \in C^\infty(M)$ given by

$$f_\varepsilon = \sqrt{\varepsilon + |\Phi|^2}. \quad (24)$$

With a straightforward computation, from (24) we have that

$$f_\varepsilon \Delta f_\varepsilon = \frac{1}{2} \Delta|\Phi|^2 - \frac{1}{4(\varepsilon + |\Phi|^2)} |\nabla|\Phi|^2|^2. \quad (25)$$

Thus, from (25) and (23) we get

$$f_\varepsilon \Delta f_\varepsilon \geq |\nabla\Phi|^2 - |\Phi|^2 P_H(|\Phi|) - \frac{1}{4(\varepsilon + |\Phi|^2)} |\nabla|\Phi|^2|^2. \quad (26)$$

Hence, applying Lemma 2 to Φ , from (26) we obtain

$$\begin{aligned} f_\varepsilon \Delta f_\varepsilon &\geq -|\Phi|^2 P_H(|\Phi|) + |\nabla\Phi|^2 - \frac{n}{(n+2)} \frac{|\Phi|^2}{(\varepsilon + |\Phi|^2)} |\nabla\Phi|^2 \\ &= -|\Phi|^2 P_H(|\Phi|) + \left(1 - \frac{n}{(n+2)} \frac{|\Phi|^2}{(\varepsilon + |\Phi|^2)} \right) |\nabla\Phi|^2 \\ &\geq -|\Phi|^2 P_H(|\Phi|) + \left(1 - \frac{n}{n+2} \right) |\nabla\Phi|^2 \geq -|\Phi|^2 P_H(|\Phi|) + \frac{2}{n+2} |\nabla\Phi|^2. \end{aligned} \quad (27)$$

Then, from (27) we have that

$$\begin{aligned} f_\varepsilon J(f_\varepsilon) &= -f_\varepsilon \Delta f_\varepsilon - (|\Phi|^2 + n(1+H^2))(\varepsilon + |\Phi|^2) \\ &\leq |\Phi|^2 P_H(|\Phi|) - \frac{2}{n+2} |\nabla\Phi|^2 - (\varepsilon + |\Phi|^2)(|\Phi|^2 + n(1+H^2))(\varepsilon + |\Phi|^2). \end{aligned} \quad (28)$$

From (2) and (28) we obtain

$$\begin{aligned} \lambda_1^J \int_M f_\varepsilon^2 dM &= \lambda_1^J \int_M (\varepsilon + |\Phi|^2) dM \leq \int_M f_\varepsilon J(f_\varepsilon) dM \\ &\leq \int_M |\Phi|^2 P_H(|\Phi|) dM - \frac{2}{n+2} \int_M |\nabla \Phi|^2 dM \\ &\quad - \int_M (\varepsilon + |\Phi|^2) (|\Phi|^2 + n(1 + H^2)) dM. \end{aligned}$$

Finally, letting $\varepsilon \rightarrow 0$ in this last inequality, we have

$$\begin{aligned} \lambda_1^J \int_M |\Phi|^2 dM &\leq \int_M (|\Phi|^2 P_H(|\Phi|) - |\Phi|^4 - n(H^2 + 1)|\Phi|^2) dM - \frac{2}{n+2} \int_M |\nabla \Phi|^2 dM \\ &\leq -2n(H^2 + 1) \int_M |\Phi|^2 dM + \frac{n(n-2)}{\sqrt{n(n-1)}} H \int_M |\Phi|^3 dM. \end{aligned} \quad (29)$$

Hence, from (29) we get

$$\lambda_1^J \leq -2n(H^2 + 1) + \frac{n(n-2)}{\sqrt{n(n-1)}} H \max_M |\Phi|.$$

Now, suppose $\lambda_1^J = -2n(1 + H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}} H \max_M |\Phi|$. Thus, from (29) we get that $|\nabla \Phi| \equiv 0$, and using once more Lemma 2, we conclude that $|\Phi|$ must be a positive constant.

On the other hand, from equation (1) it follows that

$$\lambda_1^{-\Delta} - (|\Phi|^2 + n(1 + H^2)) = \lambda_1^J = -2n(1 + H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}} H |\Phi|.$$

Thus, since M^n is closed, we obtain

$$0 = \lambda_1^{-\Delta} = |\Phi|^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} H |\Phi| - n(1 + H^2) = P_H(|\Phi|).$$

Hence, all the inequalities along this proof must be equalities. In particular, taking into account (22), from (23) we conclude that $|\Phi| = |\Phi^{n+1}|$, and consequently, $\Phi^\alpha = 0$, for all $n+2 \leq \alpha \leq n+p$. Thus, since e_{n+1} is parallel in the normal bundle of M^n , we are in position to apply [11, Theorem 1] to conclude that M^n is, in fact, isometrically immersed in a $(n+1)$ -dimensional totally geodesic submanifold $\mathbb{S}^{n+1}$ of $\mathbb{S}^{n+p}$. Therefore, we can use [1, Theorem 2.2] to finish our proof.

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