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Generalized symmetric $*$-rings and Jacobson's Lemma for Moore–Penrose inverse

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Abstract. It is well known as Jacobson's Lemma that $1-ba$ is invertible in a ring if so is $1-ab$. Moreover, if $c=(1-ab)^{-1}$, then $(1-ba)^{-1}=1+bca$. However, the analogous statement for Moore–Penrose inverse in a $*$-ring is not true in general. Note that Jacobson's Lemma for Moore–Penrose inverse holds true in a symmetric $*$-ring. In this paper, we study symmetric $*$-rings and introduce the notion of a generalized symmetric $*$-ring. A $*$-ring R is called generalized symmetric if $1-(u^*-u)^2$ is invertible for all units u in R. When $1-ab$ is Moore–Penrose invertible in such a ring, we provide sufficient and necessary conditions under which $1-ba$ has a Moore–Penrose inverse $(1-ba)^\dagger$ and give a formula for $(1-ba)^\dagger$.

1. Introduction

Let R be an associative ring with identity. It is well known as Jacobson's Lemma that, for any $a,b\in R$, $1-ab$ is invertible if and only if so is $1-ba$ (see, e.g., [3]). Moreover, if $c=(1-ab)^{-1}$, then $(1-ba)^{-1}=1+bca$. Similarly, $1-ab$ is regular if and only if $1-ba$ is regular. One can verify that $1+bca$ is an inner inverse of $1-ba$ if c is an inner inverse of $1-ab$ (see, e.g., [2, p. 160]). Jacobson's Lemma for Drazin inverse was established in [2, Theorem 3.6] and

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[3, Theorem 2.2]. The reflexive inverse and group inverse version of Jacobson's Lemma were also proved in [2, Theorem 3.4] and [2, Theorem 3.5], respectively. In 2012, Zhuang *et al.* [13, Theorem 2.3] extended Jacobson's Lemma to the case of generalized Drazin inverse.

However, Jacobson's Lemma for Moore–Penrose inverse can fail in a $*$-ring (i.e., a ring with involution). In fact, there exists a $*$-ring R with $a, b \in R$ such that $1-ab$ is Moore–Penrose invertible while $1-ba$ is not (see [2, Example 3.10]). This motivates us to investigate some $*$-rings in which Jacobson's Lemma for Moore–Penrose inverse holds true. Given a, b in a $*$-ring R such that $1-ab$ has Moore–Penrose inverse, it is also of interest to consider necessary and sufficient conditions under which $1-ba$ has Moore–Penrose inverse. Note that even if every element is Moore–Penrose invertible in a $*$-ring, the formula $(1-ba)^\dagger = 1+b(1-ab)^\dagger a$ does not hold in general (see Example 5). Therefore, it is natural to ask what is the formula of the Moore–Penrose inverse of $1-ba$ provided both $1-ab$ and $1-ba$ are Moore–Penrose invertible.

Recall that a $*$-ring R is said to be *symmetric* if $1+x^*x$ is invertible for all $x \in R$ (see, e.g., [1, 5]). Note that symmetric $*$-rings possess some attractive properties. For example, every regular element in a symmetric $*$-ring is Moore–Penrose invertible (see [7, Theorem 2], [9, Theorem 3.2] or [12, Theorem 1.4]). Using this result, it is easy to deduce that Jacobson's Lemma for Moore–Penrose inverse holds true in a symmetric $*$-ring (see Proposition 3). However, only C^*-algebras are frequently mentioned as examples of symmetric $*$-rings in the literature. In this note, we provide two more classes of such examples. In addition, the notion of a generalized symmetric $*$-ring is introduced (see Definition 1). When $1-ab$ is Moore–Penrose invertible in a generalized symmetric $*$-ring, we provide sufficient and necessary conditions under which $1-ba$ has a Moore–Penrose inverse $(1-ba)^\dagger$ and give a formula for $(1-ba)^\dagger$ (see Theorem 6).

2. Preliminaries

In this section, we briefly recall some definitions and set some notations for the readers' convenience.

By a *ring* we will mean an associative ring with identity. The set of all units in a ring R is denoted by $\mathrm{U}(R)$. Given a, b in a ring R, recall that b is called an *inner inverse* of a if $aba = a$. In this case, a is said to be (*von Neumann*) *regular*. A ring R is called (*von Neumann*) *regular* if all elements in R are regular. In a ring R, an element b is called a *reflexive inverse* of a if $aba = a$ and $bab = b$.

A $*$-*ring* is a ring R with an involution, i.e., a map $* : R \to R$ written as $a \mapsto a^*$ such that $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all $a, b \in R$. Note that a $*$-ring is also called an *involutive ring* or an *involutory ring* in the literature. We will regard a commutative ring R as a $*$-ring with respect to the identity map $* : a \mapsto a$ unless otherwise specified.

An element a in a $*$-ring R is said to be *Moore–Penrose invertible* if there exists $b \in R$ such that $aba = a$, $bab = b$, $(ab)^* = ab$ and $(ba)^* = ba$. Such an element b is unique (if it exists) and is called the *Moore–Penrose inverse* of a, which is indicated by $a^\dagger$.

In [5], a $*$-ring R is said to satisfy the *k-term star-cancellation law* (SC_k) if the involution $*$ of R is *k-proper* in the sense of Berberian [1], i.e., $a_1^*a_1 + \cdots + a_k^*a_k = 0$ implies $a_1 = \cdots = a_k = 0$, for any $a_1, \ldots, a_k \in R$. Recall that a $*$-ring R is called a $*$-*regular* ring if it is regular and satisfies SC_1 or, equivalently, if every element in R has Moore–Penrose inverse (see [5] and [8, Theorem 5.4]).

Given a ring R and a positive integer n, the $n \times n$ matrix ring over R is denoted by $\mathrm{M}_n(R)$. The identity matrix in $\mathrm{M}_n(R)$ is denoted by I_n, or simply by I. In case R is a $*$-ring, $\mathrm{M}_n(R)$ is also a $*$-ring with involution defined by $(a_{ij})^* = (a_{ji}^*)$ for all $(a_{ij}) \in \mathrm{M}_n(R)$. The ring of integers, the field of real numbers and the field of complex numbers are denoted by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. As usual, $\mathbb{Z}_n$ stands for the factor ring of $\mathbb{Z}$ modulo n, i.e., $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, where n is a positive integer.

3. Main results

Let us start with two classes of examples of symmetric $*$-rings, namely $\mathbb{Z}_n$ for some positive integers n, and $\mathbb{R}\mathbb{Z}_n$ for any positive integers n.

Example 1. (1) Suppose p is a prime number and s is a positive integer. Then $\mathbb{Z}_{p^s} = \mathbb{Z}/p^s\mathbb{Z}$ is a symmetric $*$-ring (i.e., $1 + x^2 \equiv 0 (\mathrm{mod}\ p)$ has no solution in $\mathbb{Z}$) if and only if $p \equiv -1 (\mathrm{mod}\ 4)$ by the well-known Euler's Criterion (see, e.g., [4, Theorem 3.13]).

(2) Let $n = p_1^{s_1} p_2^{s_2} \cdots p_m^{s_m}$, where $p_1, p_2, \ldots, p_m$ are pairwise different prime numbers and $s_1, s_2, \ldots, s_m$ are positive integers. Then $\mathbb{Z}_n \cong \prod_{i=1}^m \mathbb{Z}_{p_i^{s_i}}$ is a symmetric $*$-ring if and only if each $p_i \equiv -1 (\mathrm{mod}\ 4)$.

Let R be a ring and G a group. Then the group ring RG consists of all the sums of the form $\sum_{g \in G} a_g g$ with each $a_g \in R$ such that only finitely many $a_g \neq 0$. When R is an involutive ring with involution $a \mapsto \overline{a}$, the group ring RG is a $*$-ring with $(\sum_{g \in G} a_g g)^* = \sum_{g \in G} \overline{a_g} g^{-1}$. We refer the reader to [10] for more details on group rings.

Example 2. Let n be a positive integer and consider the cyclic group $\mathbb{Z}_n$. Then the group ring $\mathbb{R}\mathbb{Z}_n$ is a symmetric $*$-ring with respect to the involution induced from the identity involution on $\mathbb{R}$.

Indeed, let $P = \left(\begin{smallmatrix} 0 & I_{n-1} \\ 1 & 0 \end{smallmatrix}\right)$, where I_{n-1} is the $(n-1)\times(n-1)$ identity matrix over $\mathbb{R}$. It is easy to see that $\mathbb{R}\mathbb{Z}_n$ is isomorphic to $S = \{\sum_{i=0}^{n-1} a_i P^i \mid \text{each } a_i \in \mathbb{R}\}$, which is a subring of the $n \times n$ matrix ring $\mathrm{M}_n(\mathbb{R})$. The involution $*$ on S corresponding to the natural involution on $\mathbb{R}\mathbb{Z}_n$ is given by $(\sum_{i=0}^{n-1} a_i P^i)^* = \sum_{i=0}^{n-1} a_i P^{-i}$.

For any $A = \sum_{i=0}^{n-1} a_i P^i \in S$, it follows that

$$A^* = \sum_{i=0}^{n-1} a_i P^{-i} = \sum_{i=0}^{n-1} a_i (P^i)^{\mathrm{T}} = A^{\mathrm{T}},$$

where A^{T} stands for the transpose of A. It is a well-known fact in linear algebra that $I + A^{\mathrm{T}}A$ is a positive definite symmetric real matrix. Thus $I + A^*A = I + A^{\mathrm{T}}A$ is invertible in $\mathrm{M}_n(\mathbb{R})$. We need to show that $I + A^*A$ is also invertible in S. For any invertible matrix $B \in \mathrm{M}_n(\mathbb{R})$, it follows as a corollary of the Hamilton–Cayley Theorem that $B^{-1} = b_0 I + b_1 B + \cdots + b_{n-1} B^{n-1}$ for some $b_0, b_1, \ldots, b_{n-1} \in \mathbb{R}$. From this fact one can see that $(I + A^*A)^{-1} \in S$, since $I + A^*A \in S$. This shows S is a symmetric $*$-ring. Therefore, the group ring $\mathbb{R}\mathbb{Z}_n$ is a symmetric $*$-ring.

Recall that a $*$-ring R is said to be *$*$-regular* if every element in R is Moore–Penrose invertible.

Proposition 1. *Let R be a $*$-ring and define the involution on $\mathrm{M}_2(R)$ by $(a_{ij})^* = (a_{ji}^*)$. Then*

(1) *R is a symmetric $*$-ring if Jacobson's Lemma holds in $\mathrm{M}_2(R)$, i.e., if the existence of $(I - AB)^\dagger$ implies that of $(I - BA)^\dagger$, where A, $B \in \mathrm{M}_2(R)$.*

(2) *R is a regular symmetric $*$-ring if and only if $\mathrm{M}_2(R)$ is a $*$-regular ring.*

Proof. (1) For any $a \in R$, let $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & a \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{M}_2(R)$. Then $AB = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $(I - AB)^\dagger = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. By hypotheses, $(I - BA)^\dagger$ also exists. Let $C = I - BA = \left(\begin{smallmatrix} 0 & -a \\ 0 & 1 \end{smallmatrix}\right)$. According to [8, Theorem 5.4], we have $C = XC^*C$ for some $X \in \mathrm{M}_2(R)$. Let $X = \left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)$. Then

$$\left(\begin{smallmatrix} 0 & -a \\ 0 & 1 \end{smallmatrix}\right) = C = XC^*C = \left(\begin{smallmatrix} 0 & x_2(1+a^*a) \\ 0 & x_4(1+a^*a) \end{smallmatrix}\right).$$

Hence $x_4(1 + a^*a) = 1$, and $(1 + a^*a)x_4^* = [x_4(1 + a^*a)]^* = 1$. This shows $1 + a^*a \in U(R)$. Therefore, R is a symmetric $*$-ring.

(2) It is well known that $\mathrm{M}_2(R)$ is a $*$-regular ring if and only if R is a regular $*$-ring satisfying SC_2 (see, e.g., [6]). In view of [5, Theorem 3], a $*$-ring R is regular and satisfies SC_2 if and only if R is a regular symmetric $*$-ring. Therefore, the result follows. □

Next, we extend the class of symmetric $*$-rings to a larger one and consider the relationship between them.

Definition 1. Let R be a $*$-ring. If $1-(u^*-u)^2 \in \mathrm{U}(R)$ for all $u \in \mathrm{U}(R)$, then R is called a *generalized symmetric* $*$-ring.

Let us illustrate the notion of generalized symmetric $*$-rings by some concrete examples.

Example 3. (1) Every commutative ring R is a generalized symmetric $*$-ring with respect to the involution $*$ given by $x^* = x$.

(2) Every symmetric $*$-ring R is generalized symmetric. Indeed, for any $u \in \mathrm{U}(R)$, it follows that $1-(u^*-u)^2 = 1+(u^*-u)^*(u^*-u) \in \mathrm{U}(R)$.

(3) $\mathbb{Z}_5 = \mathbb{Z}/5\mathbb{Z}$ is a generalized symmetric $*$-ring. But $\mathbb{Z}_5$ is not a symmetric $*$-ring according to Example 1.

(4) $\mathbb{C}$ is a generalized symmetric $*$-ring but not a symmetric $*$-ring with respect to the involution $*$ given by $x^* = x$. Note that $\mathbb{C}$ is a symmetric $*$-ring with respect to the involution given by the complex conjugate.

(5) Let $R = \mathbb{Z}_2\langle x, y\rangle/(x^2-x, y^2-y, xyx)$ be the ring generated over $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ by $\{x, y\}$ with the relations $\{x^2-x, y^2-y, xyx\}$. Let

$$X = x + (x^2-x, y^2-y, xyx) \quad \text{and} \quad Y = y + (x^2-x, y^2-y, xyx).$$

Define the involution on R such that $1^* = 1$, $X^* = X$, $Y^* = Y$, $(XY)^* = YX$, $(YX)^* = XY$ and $(YXY)^* = YXY$. For any

$$a = a_1 + a_2X + a_3Y + a_4XY + a_5YX + a_6YXY \in R,$$

it follows that $a^* - a = a^* + a = (a_4 + a_5)(XY + YX)$. So we have

$$1-(a^*-a)^2 = 1+(a_4+a_5)^2(XY+YX)^2 = 1+(a_4+a_5)^2YXY \in \mathrm{U}(R),$$

since $[1+(a_4+a_5)^2YXY]^2 = 1$. This shows R is a generalized symmetric $*$-ring. But R is not a symmetric $*$-ring, since $1+1^*1 = 0$.

Proposition 2.

(1) *Let R be a $*$-ring and define the involution on $\mathrm{M}_2(R)$ by $(a_{ij})^* = (a_{ji}^*)$. If $\mathrm{M}_2(R)$ is a generalized symmetric $*$-ring, then R is a symmetric $*$-ring.*

(2) *Let R be a commutative ring and define the involution on $\mathrm{M}_2(R)$ by $(a_{ij})^* = (a_{ji})$. Then $\mathrm{M}_2(R)$ is a generalized symmetric $*$-ring if and only if R is a symmetric $*$-ring.*

PROOF. (1) For any $a \in R$, let $A = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$. Then

$$\left(\begin{smallmatrix} 1+aa^* & 0 \\ 0 & 1+a^*a \end{smallmatrix}\right) = I - (A^* - A)^2$$

is invertible, since $\mathrm{M}_2(R)$ is a generalized symmetric $*$-ring. Hence $1+a^*a \in \mathrm{U}(R)$.

(2) If $\mathrm{M}_2(R)$ is a generalized symmetric $*$-ring, then R is a symmetric $*$-ring by (1). Conversely, assume that R is a symmetric $*$-ring. Then for any $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(R)$, we have

$$I - (A^* - A)^2 = \left(\begin{smallmatrix} 1+(b-c)^2 & 0 \\ 0 & 1+(b-c)^2 \end{smallmatrix}\right),$$

where $1 + (b-c)^2 \in \mathrm{U}(R)$. Thus $I - (A^* - A)^2$ is invertible. This shows that $\mathrm{M}_2(R)$ is a generalized symmetric $*$-ring. □

Example 4. Let $1 < n \in \mathbb{Z}$ and define the involution on $\mathrm{M}_2(\mathbb{Z}_n)$ by $(a_{ij})^* = (a_{ji})$. Then $\mathrm{M}_2(\mathbb{Z}_n)$ is not a symmetric $*$-ring, since otherwise $\mathbb{Z}_n$ is a symmetric $*$-ring by Proposition 2. In view of Example 1(2), n has at least one odd prime factor p. By virtue of [4, Lemma 2.9], we have integers a and b with $a^2 + b^2 + 1 \equiv 0 (\mathrm{mod}\ p)$. Now let $A = \left(\begin{smallmatrix} \overline{a} & \overline{0} \\ \overline{b} & \overline{0} \end{smallmatrix}\right)$, where $\overline{i} = i + n\mathbb{Z} \in \mathbb{Z}_n$ for $i = a, b, 0$. Then $I + A^*A = \left(\begin{smallmatrix} \overline{a^2+b^2+1} & \overline{0} \\ \overline{0} & \overline{1} \end{smallmatrix}\right)$ is invertible. So we have $\overline{a^2 + b^2 + 1} \in \mathrm{U}(\mathbb{Z}_n)$, which contradicts $a^2 + b^2 + 1 \equiv 0 (\mathrm{mod}\ p)$.

Now, we consider Jacobson's Lemma for Moore–Penrose inverse. The following result is essentially due to the authors of [7], [9] and [12].

Proposition 3. *Let R be a symmetric $*$-ring and $a, b \in R$. If $1 - ab$ is Moore–Penrose invertible, then so is $1 - ba$.*

PROOF. According to [7, Theorem 2], [9, Theorem 3.2] or [12, Theorem 1.4], every regular element in R is Moore–Penrose invertible, since R is a symmetric $*$-ring. Note that $1 - ab$ is regular in R if and only if so is $1 - ba$. Thus, the result follows. □

Note that when $1-ab$ is invertible in a ring, it follows that

$$(1-ba)^{-1} = 1 + b(1-ab)^{-1}a.$$

The next example shows that, even if every element is Moore–Penrose invertible in a $*$-ring, the formula $(1-ba)^\dagger = 1 + b(1-ab)^\dagger a$ does not hold in general.

Example 5. Consider the involution on $\mathrm{M}_2(\mathbb{C})$ induced from the conjugate complex involution on $\mathbb{C}$. It is well known that every matrix in $\mathrm{M}_2(\mathbb{C})$ is Moore–Penrose invertible. Let $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Then $I - AB = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $I - BA = \left(\begin{smallmatrix} 0 & 0 \\ -1 & 1 \end{smallmatrix}\right)$. By computation, we have $(I-AB)^\dagger = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $(I-BA)^\dagger = \left(\begin{smallmatrix} 0 & -1/2 \\ 0 & 1/2 \end{smallmatrix}\right)$. These show $(I-BA)^\dagger \neq I + B(I-AB)^\dagger A$. In fact, one can check that $(I-BA)^\dagger \neq I + BCA$ for any $C \in \mathrm{M}_2(\mathbb{C})$.

To provide some conditions under which the Moore–Penrose invertibility of $1-ba$ and the formula for $(1-ba)^\dagger$ can be interpreted in terms of $(1-ab)^\dagger$, we need the following lemmas, which are of interest in their own right.

Lemma 4. *Let e be an idempotent in a $*$-ring R. If $u = 1+(e-e^*)(e-e^*)^* \in \mathrm{U}(R)$, then e has Moore–Penrose inverse and $e^\dagger = u^{-1}e^* = e^*u^{-1}$.*

Proof. Since $e^2 = e$ and $u = 1 + (e-e^*)(e-e^*)^*$, it follows that $u = u^*$, $eu = ee^*e = ue$ and $e^*u = ue^*$. Moreover, since $u \in \mathrm{U}(R)$, one can verify that $(u^{-1})^* = u^{-1}$, $eu^{-1} = u^{-1}e$, $u^{-1}e^* = e^*u^{-1}$ and $e = ee^*(eu^{-1}) = (u^{-1}e)e^*e$. Now it is straightforward to check that $e^\dagger = u^{-1}e^* = e^*u^{-1}$. □

We remark that Lemma 4 appeared in the proof of [12, Theorem 1.4] (see also [6, p. 183, (vi)]).

Lemma 5. *Let R be a $*$-ring and b be a reflexive inverse of $a \in R$. If both ab and ba are Moore–Penrose invertible, then a is also Moore–Penrose invertible and $a^\dagger = (ba)^\dagger b(ab)^\dagger$.*

Proof. Let $x = (ba)^\dagger b(ab)^\dagger$. Then we have

$$ax = a[(ba)^\dagger b(ab)^\dagger] = (aba)[(ba)^\dagger(bab)(ab)^\dagger] = (ab)(ab)^\dagger.$$

Hence $(ax)^* = ax$ and $axa = [(ab)(ab)^\dagger](aba) = a$. Similarly, we have $xa = (ba)^\dagger(ba)$, $(xa)^* = xa$ and $xax = [(ba)^\dagger(ba)][(ba)^\dagger b(ab)^\dagger] = (ba)^\dagger b(ab)^\dagger = x$. Thus, $a^\dagger = x = (ba)^\dagger b(ab)^\dagger$. □

Theorem 6. *Let R be a generalized symmetric $*$-ring and $a, b \in R$. Suppose that $\alpha = 1-ab$ has Moore–Penrose inverse. Then the following statements are equivalent:*

(1) $\beta = 1 - ba$ *has Moore–Penrose inverse;*

(2) *both* $u = 1 - [(bpa)^* - bpa]^2$ *and* $v = 1 - [(bqa)^* - bqa]^2$ *are invertible, where* $p = 1 - \alpha^\dagger\alpha$ *and* $q = 1 - \alpha\alpha^\dagger$.

In this case, $\beta^\dagger = u^{-1}e^*[1 + b(\alpha^\dagger - pq)a]f^*v^{-1}$, *where* $e = 1 - bpa$ *and* $f = 1 - bqa$.

Proof. Let $\overline{\beta} = 1 + b(\alpha^\dagger - pq)a$. Then $\overline{\beta}$ is a reflexive inverse of β by [2, Theorem 3.4]. Moreover, $\overline{\beta}\beta = \beta + b(\alpha^\dagger - pq)a\beta = \beta + b(\alpha^\dagger - pq)\alpha a = \beta + b\alpha^\dagger\alpha a = 1 - bpa$. Similarly, $\beta\overline{\beta} = 1 - bqa$.

(1)⇒(2) Let $w_1 = \beta^*\beta + bpa$ and $w_2 = \beta\beta^* + bqa$. Then $w_1 = \beta^*\beta + 1 - \overline{\beta}\beta$ and $w_2 = \beta\beta^* + 1 - \beta\overline{\beta}$. Since β has Moore–Penrose inverse, it follows that w_1 and w_2 are units by [11, Theorem 1.1]. Consequently, $u = 1 - [(bpa)^* - bpa]^2 = 1 - (w_1^* - w_1)^2$ and $v = 1 - [(bqa)^* - bqa]^2 = 1 - (w_2^* - w_2)^2$ are units as R is a generalized symmetric $*$-ring.

(2)⇒(1) Since $e = 1 - bpa = \overline{\beta}\beta$ is an idempotent and $1 + (e - e^*)(e - e^*)^* = 1 - [(bpa)^* - bpa]^2 = u$ is a unit, it follows that e has Moore–Penrose inverse and $e^\dagger = u^{-1}e^*$ by Lemma 4. Similarly, $f = \beta\overline{\beta}$ has Moore–Penrose inverse and $f^\dagger = f^*v^{-1}$. In view of Lemma 5, we have β is Moore–Penrose invertible and $\beta^\dagger = e^\dagger\overline{\beta}f^\dagger = u^{-1}e^*[1 + b(\alpha^\dagger - pq)a]f^*v^{-1}$. □

As an application of Theorem 6, we obtain the following examples of generalized symmetric $*$-rings in which Jacobson's Lemma for Moore–Penrose inverse holds.

Example 6. (1) Let R be a commutative symmetric $*$-ring. Define the involution on $\mathrm{M}_2(R)$ by $(a_{ij})^* = (a_{ji})$. By Proposition 2, $\mathrm{M}_2(R)$ is a generalized symmetric $*$-ring. For any $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(R)$, it follows that $I - (A^* - A)^2 = \left(\begin{smallmatrix} 1+(b-c)^2 & 0 \\ 0 & 1+(b-c)^2 \end{smallmatrix}\right)$ is invertible, since $1 + (b-c)^2 \in \mathrm{U}(R)$. Therefore, Jacobson's Lemma for Moore–Penrose inverse holds in $\mathrm{M}_2(R)$ by Theorem 6.

(2) Let R be the ring in Example 3(5). We have seen that $1 - (a^* - a)^2 \in \mathrm{U}(R)$ for all $a \in R$. Thus Jacobson's Lemma for Moore–Penrose inverse holds in R by Theorem 6.

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