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# On the Diophantine equation $(x+1)^k + (x+2)^k + \cdots + (lx)^k = y^n$

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*To my wife and my daughter*

**Abstract.** Let $k, l \geq 2$ be fixed integers. In this paper, firstly, we prove that all solutions of the equation $(x+1)^k + (x+2)^k + \cdots + (lx)^k = y^n$ in integers $x, y, n$ with $x, y \geq 1, n \geq 2$ satisfy $n < C_1$, where $C_1 = C_1(l, k)$ is an effectively computable constant. Secondly, we prove that all solutions of this equation in integers $x, y, n$ with $x, y \geq 1, n \geq 2, k \neq 3$ and $l \equiv 0 \pmod 2$ satisfy $\max\{x, y, n\} < C_2$, where $C_2$ is an effectively computable constant depending only on $k$ and $l$.

## 1. Introduction

In 1956, J. J. Schäffer [14] considered the equation

$$1^k + 2^k + \cdots + x^k = y^n. \tag{1.1}$$

He proved that for fixed $k \geq 1$ and $n \geq 2$, (1.1) has at most finitely many solutions in positive integers $x$ and $y$, unless

$$(k, n) \in \{(1, 2), (3, 2), (3, 4), (5, 2)\},$$

where, in each case, there are infinitely many such solutions.

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Schäffer's proof used an ineffective method due to Thue and Siegel, so his result is also ineffective. This means that the proof does not provide any algorithm to find all solutions. Applying Baker's method, K. Győry, R. Tijdeman and M. Voorhoeve [6] proved a more general and effective result in which the exponent $n$ is also unknown.

Let $k \geq 2$ and $r$ be fixed integers with $k \notin \{3, 5\}$ if $r = 0$, and let $s$ be a square-free odd integer. In [6], they proved that the equation

$$s(1^k + 2^k + \cdots + x^k) + r = y^n$$

in positive integers $x, y \geq 2$, $n \geq 2$ has only finitely many solutions, and all these can be effectively determined. Of particular importance is the special case when $s = 1$ and $r = 0$. They also showed that for given $k \geq 2$ with $k \notin \{3, 5\}$, equation (1.1) has only finitely many solutions in integers $x, y \geq 1$, $n \geq 2$, and all these can be effectively determined. The following striking result is due to Voorhoeve, Győry and Tijdeman [17].

Let $R(x)$ be a fixed polynomial with integer coefficients, and let $k \geq 2$ be a fixed integer such that $k \notin \{3, 5\}$. In [17], the same authors proved that the equation

$$1^k + 2^k + \cdots + x^k + R(x) = by^n$$

in integers $x, y \geq 2$, $n \geq 2$ has only finitely many solutions, and an effective upper bound can be given for $n$. Later, various generalizations and analogues of the results of Győry, Tijdeman and Voorhoeve have been established by several authors [1], [2], [3], [4], [5], [8], [11], [16]. For a survey of these results, we refer to [7] and the references given there.

Here we present the result of B. Brindza [2]. For brevity, let us set $S_k(x) = 1^k + 2^k + \cdots + x^k$, $A = \mathbb{Z}[x]$, $\kappa = (k+1) \prod_{(p-1)|(k+1)!} p$ ($p$ prime). Let

$$F(y) = Q_n y^n + \cdots + Q_1 y + Q_0 \in A[y].$$

Consider the equation

$$F(S_k(x)) = y^n \tag{1.2}$$

in integers $x, y \geq 2$, $n \geq 2$. Let $Q_i(x) = \kappa^i K_i(x)$, where $K_i(x) \in \mathbb{Z}[x]$ for $i = 2, 3, \ldots, m$. In [2], Brindza proved that if $Q_i(x) \equiv 0 \pmod{\kappa^i}$, for $i = 2, 3, \ldots, m$; $Q_1(x) \equiv \pm 1 \pmod 4$ and $k \notin \{1, 2, 3, 5\}$, then all solutions of (1.2) satisfy $\max\{x, y, n\} < C_1$, where $C_1$ is an effectively computable constant depending only on $F$ and $k$.

Recently, Cs. Rakaczki [12] gave a generalization of the results of Győry, Tijdeman and Voorhoeve, as well as an extension of the result of Brindza to the case when the polynomials $Q_i(x)$ are arbitrary constant polynomials.

Let $F(x)$ be a polynomial with rational coefficients and $d \neq 0$ be an integer. Suppose that $F(x)$ is not an $n$-th power. In [12], Rakaczki showed that the equation

$$F(S_k(x)) = dy^n$$

has only finitely many integer solutions $x, y \geq 2$, $n \geq 2$, which can be effectively determined provided that $k \geq 6$.

Let $k > 1$, $r, s \neq 0$ be fixed integers. Then, apart from the cases when (i) $k = 3$ and either $r = 0$ or $s + 64r = 0$, and (ii) $k = 5$ and either $r = 0$ or $s - 324r = 0$, Rakaczki proved that the equation

$$s(1^k + 2^k + \cdots + x^k) + r = y^n$$

in integers $x > 0$, $y$ with $|y| \geq 2$, and $n \geq 2$ has only finitely many solutions, which can be effectively determined.

Recently, Z. Zhang [18] studied the Diophantine equation

$$(x-1)^k + x^k + (x+1)^k = y^n, n > 1,$$

and completely solved it for $k = 2, 3, 4$. Now, we consider a more general equation. Let

$$G(x) = (x+1)^k + (x+2)^k + \cdots + (lx)^k.$$

In this paper, we are interested in the solutions of the equation

$$G(x) = y^n \tag{1.3}$$

in integers $x, y \geq 1$ and $n \geq 2$.

**Theorem 1.** *Let $k, l \geq 2$ be fixed integers. Then all solutions of equation* (1.3) *in integers $x, y \geq 1$ and $n \geq 2$ satisfy $n < C_1$, where $C_1$ is an effectively computable constant depending only on $l$ and $k$.*

**Theorem 2.** *Let $k, l \geq 2$ be fixed integers such that $k \neq 3$. Then all solutions of equation* (1.3) *in integers $x, y, n$ with $x, y \geq 1$, $n \geq 2$, and $l \equiv 0 \pmod 2$ satisfy $\max\{x, y, n\} < C_2$, where $C_2$ is an effectively computable constant depending only on $l$ and $k$.*

We organize this paper as follows. In Section 2, firstly, we recall the general results that we will need. Secondly, we give two new lemmas and prove that these lemmas imply our theorems. In Section 3, we discuss the number of solutions in integers $x, y \geq 1$ of (1.3), where $n > 1$ is fixed, $k \in \{1, 3\}$ and $l \equiv 0 \pmod 2$, and reformulate this case. In the last section, we give the proofs of Theorems 1 and 2.

## 2. Auxiliary results

**Lemma 1.** $(x+1)^k + (x+2)^k + \cdots + (lx)^k = \dfrac{B_{k+1}(lx+1) - B_{k+1}(x+1)}{k+1}$, *where*

$$B_q(x) = x^q - \frac{1}{2}qx^{q-1} + \frac{1}{6}\binom{q}{2}x^{q-2} + \cdots = \sum_{i=0}^{q}\binom{q}{i}B_i x^{q-i}$$

*is the q-th Bernoulli polynomial with* $q = k+1$.

Proof. It is an application of the equality

$$\sum_{n=M}^{N-1} n^k = \frac{1}{k+1}(B_{k+1}(N) - B_{k+1}(M)),$$

which is given by Rademacher in [13, pp. 3–4]. □

Now, we give an important result of Brindza which is an effective version of LeVeque's theorem [9]

**Lemma 2** (Brindza)**.** *Let* $H(x) \in \mathbb{Q}[x]$,

$$H(x) = a_0x^N + \cdots + a_N = a_0\prod_{i=1}^{n}(x-\alpha_i)^{r_i},$$

*with* $a_0 \neq 0$ *and* $\alpha_i \neq \alpha_j$ *for* $i \neq j$. *Let* $0 \neq b \in \mathbb{Z}$, $2 \leq m \in \mathbb{Z}$ *and define* $t_i = \frac{m}{(m,r_i)}$. *Suppose that* $\{t_1, \ldots, t_n\}$ *is not a permutation of the n-tuples*

$$\text{(a)}\{\text{t}, 1, \ldots, 1\}, \text{t} \geq 1; \qquad \text{(b)}\{2, 2, 1, \ldots, 1\}.$$

*Then all solutions* $(x, y) \in \mathbb{Z}^2$ *of the equation*

$$H(x) = by^m$$

*satisfy* $\max\{|x|, |y|\} < C$, *where* $C$ *is an effectively computable constant depending only on* $H$, $b$ *and* $m$.

Proof. See B. Brindza [2]. □

**Lemma 3** (Schinzel and Tijdeman)**.** *Let* $0 \neq b \in \mathbb{Z}$, *and let* $P(x) \in \mathbb{Q}[x]$ *be a polynomial with at least two distinct zeroes. Then the equation*

$$P(x) = by^n$$

*in integers* $x, y > 1$, $n$ *implies that* $n < C$, *where* $C = (P, b)$ *is an effectively computable constant.*

PROOF. See A. SCHINZEL and R. TIJDEMAN [15]. □

**Lemma 4.** *For $k \in \mathbb{Z}^+$, let $B_k(x)$ be the $k$-th Bernoulli polynomial. Then the polynomial*

$$G(x) = \frac{B_{k+1}(lx+1) - B_{k+1}(x+1)}{k+1}$$

*has at least two distinct zeroes.*

PROOF. By Lemma 1, we have $G(x) = \left(\frac{l^{k+1}-1}{k+1}\right) x^{k+1} + \left(\frac{l^k-1}{2}\right) x^k + \cdots + cx$, where $c$ is a rational number. Now, one can observe that the coefficient of $x^k$ is nonzero and that $x = 0$ is a zero of $G(x)$. Let us also assume that there is no other zero of $G(x)$. Thus we have

$$G(x) = \left(\frac{l^{k+1}-1}{k+1}\right) x^{k+1},$$

which is a contradiction. □

**Lemma 5** (Voorhoeve, Győry and Tijdeman)**.** *Let $q \geq 2$, $R^*(x) \in \mathbb{Z}[x]$ and set*

$$Q(x) = B_q(x) - B_q + qR^*(x).$$

*Then*

(i) *$Q(x)$ has at least three zeros of odd multiplicity, unless $q \in \{2, 4, 6\}$.*

(ii) *for any odd prime $p$, at least two zeros of $Q(x)$ have multiplicities relatively prime to $p$.*

PROOF. See M. VOORHOEVE, K. GYŐRY and R. TIJDEMAN [17]. □

**Lemma 6.** *For $q \geq 2$, let $B_q(x)$ be the $q$-th Bernoulli polynomial. Let*

$$P(x) = B_q(lx+1) - B_q(x+1), \tag{2.1}$$

*where $l$ is even. Then*

(i) *$P(x)$ has at least three zeros of odd multiplicity unless $q \in \{2, 4\}$.*

(ii) *for any odd prime $p$, at least two zeros of $P(x)$ have multiplicities relatively prime to $p$.*

PROOF. We shall follow the proof of [17, Lemma 5]. By the Staudt–Clausen theorem (see RADEMACHER [13, p. 10]), the denominators of the Bernoulli numbers $B_i$, $B_{2k}$ $(k = 1, 2, \dots)$ are even but not divisible by 4. Choose the minimal

$d \in \mathbb{N}$ such that both the polynomials $d(B_q(lx+1) - B_q(x+1))$ and $dB_q(x)$ are in $\mathbb{Z}[x]$. Using the equality $B_q(x+1) = B_q(x) + qx^{q-1}$ (see [13, pp. 4–5]), we have

$$dP(x) = d\left(\sum_{i=0}^{q} \binom{q}{i} \left[(lx+1)^{q-i} - x^{q-i}\right] B_i - qx^{q-1}\right). \tag{2.2}$$

Hence, by the choice of $d$ and by the Staudt–Clausen theorem, we have $d\binom{q}{i}B_i \in \mathbb{Z}$ and $\binom{q}{2k}dB_{2k} \in \mathbb{Z}$ for $k = 1, 2, \ldots, \frac{q-1}{2}$. If $d$ is odd, then necessarily $\binom{q}{i}$ and $\binom{q}{2k}$ must be even for $k = 1, 2, \ldots, \frac{q-1}{2}$. Write $q = 2^\mu r$, where $\mu \geq 1$ and $r$ is odd. Then $\binom{q}{2^\mu}$ is odd, giving a contradiction unless $r = 1$. So

$$d \text{ is odd} \Longleftrightarrow q = 2^\mu \text{ for some } \mu \geq 1.$$

If $q \neq 2^\mu$ for any $\mu \geq 1$, then

$$d \equiv 2 \pmod 4. \tag{2.3}$$

We distinguish three cases.

*Case 1.* Suppose $q = 2^\mu$, for some $\mu \geq 1$, so that $d$ is odd. We first prove (i), so we may assume that $\mu \geq 3$. Considering (2.2) modulo 4, we have

$$dP(x) \equiv d\sum_{i=0}^{q-2} \binom{q}{i} (lx+1)^{q-i} B_i - d\sum_{i=0}^{\frac{q-2}{2}} \binom{q}{2i} B_{2i} x^{q-2i} \pmod 4. \tag{2.4}$$

Firstly, let $l \equiv 0 \pmod 4$. Then we obtain

$$d\sum_{i=0}^{q-2} \binom{q}{i} (lx+1)^{q-i} B_i \equiv d\sum_{i=0}^{q-2} \binom{q}{i} B_i \equiv d\sum_{i=0}^{\frac{q-2}{2}} \binom{q}{2i} B_{2i} \pmod 4. \tag{2.5}$$

It is easy to see that $\sum_{i=1}^{q} \binom{q}{q-i} B_{q-i} = 0$. Hence we get

$$\sum_{i=1}^{\frac{q-2}{2}} \binom{q}{2i} B_{2i} = -B_0 - qB_1. \tag{2.6}$$

By using (2.5) and (2.6), one gets

$$d\sum_{i=0}^{q-2} \binom{q}{i} (lx+1)^{q-i} B_i \equiv d\left(\binom{q}{0} B_0 + \sum_{i=1}^{\frac{q-2}{2}} \binom{q}{2i} B_{2i}\right) \equiv 0 \pmod 4.$$

Then we deduce by (2.4) the following:

$$dP(x) \equiv -d\sum_{i=0}^{\frac{q-2}{2}} \binom{q}{2i} B_{2i} x^{q-2i} \pmod 4. \tag{2.7}$$

Secondly, let $l \equiv 2 \pmod 4$. Then we obtain

$$d\sum_{i=0}^{q-2} \binom{q}{i} (lx+1)^{q-i} B_i \equiv d\sum_{i=0}^{q-2} \binom{q}{i} (2x+1)^{q-i} B_i \pmod 4. \tag{2.8}$$

Then the RHS of (2.8) becomes

$$\begin{aligned} d\sum_{i=0}^{q-2} \binom{q}{i} (2x+1)^{q-i} B_i = d(B_0.(2x+1)^q + qB_1.(2x+1)^{q-1} \\ + \sum_{i=1}^{\frac{q-2}{2}} \binom{q}{2i} (2x+1)^{q-2i} B_{2i}. \end{aligned} \tag{2.9}$$

Since $2x+1$ is odd and $q = 2^\mu$, $\mu \geq 3$, is even, considering (2.9) modulo 4 and using (2.6), (2.8) becomes

$$d\sum_{i=0}^{q-2} \binom{q}{i} (lx+1)^{q-i} B_i \equiv 0 \pmod 4.$$

So in all cases (2.4) reduces to (2.7).

Note that $\binom{q}{2i}$ is divisible by 8 unless $2i$ is divisible by $2^{\mu-2}$. We have therefore for some odd $d'$, writing $t = \frac{1}{4}q$,

$$dP(x) \equiv d'x^{4t} + 2x^{3t} + dx^{2t} + 2x^t \pmod 4. \tag{2.10}$$

Write $dP(x) = R^2(x)S(x)$, where $R(x), S(x) \in \mathbb{Z}[x]$ and $S$ contains each factor of odd multiplicity of $P$ in $\mathbb{Z}[x]$ exactly once. Assume that $\deg S(x) \leq 2$. Since

$$R^2(x)S(x) \equiv x^{4t} + x^{2t} \equiv x^{2t}(x^{2t}+1) \pmod 2,$$

$R^2(x)$ must be divisible by $x^{2t-2} \pmod 2$. So

$$R(x) = x^{t-1}R_1(x) + 2R_2(x), \qquad R^2(x) = x^{2t-2}R_1^2(x) + 4R_3(x),$$

for certain $R_1, R_2, R_3 \in \mathbb{Z}[x]$. If $q > 8$, then $t > 2$, so the last identity is incompatible with (2.10) because of the term $2x^t$. Hence $\deg S(x) \geq 3$, which proves (i). If $q = 8$, then by (2.7)

$$dP(x) \equiv 3x^8 + 2x^6 + x^4 + 2x^2 \pmod 4.$$

From here, we follow the proof in the corrigendum paper [17]. This fact can also be reduced from (2.7). So, the proof of (i) is completed where $q = 2^\mu$, $\mu \geq 3$.

To prove (ii), let $p$ be an odd prime and write $P(x) = (R(x))^p S(x)$, where $R, S \in \mathbb{Z}[x]$ and all the roots of multiplicity divisibly by $p$ are incorporated in $(R(x))^p$. We have, writing $\delta = \frac{1}{2}q$, by (2.10)

$$dP(x) \equiv (R(x))^p S(x) \equiv x^\delta(x^\delta + 1) \equiv x^\delta(x+1)^\delta \pmod 2.$$

Since $\delta$ is prime to $p$, $S$ has at least two different zeros, proving (ii) in Case 1.

*Case 2.* Suppose $q$ is even and $q \neq 2^\mu$ for any $\mu$. Then $d \equiv 2 \pmod 4$, and hence considering (2.2) in modulo 2, we get

$$dP(x) \equiv d\sum_{i=0}^{q} \binom{q}{i}(1 - x^{q-i})B_i \pmod 2.$$

Since $B_i d\binom{q}{i} \equiv \binom{q}{i} \pmod 2$ for $i = 1, 2, 3, \ldots, q$, we have

$$dP(x) \equiv \sum_{k=1}^{\frac{q-2}{2}} \binom{q}{2k} x^{2k} = \sum_{t=1}^{q-1} \binom{q}{t} x^t \equiv (x+1)^q - x^q - 1 \pmod 2.$$

Write $q = 2^\mu r$, where $r > 1$ is odd. Then

$$dP(x) \equiv (x+1)^q - x^q - 1 \equiv ((x+1)^r - x^r - 1)^{2^\mu} \pmod 2.$$

Since $r > 1$ is odd, $(x+1)^r - x^r - 1$ has $x$ and $x+1$ as simple factors $\pmod 2$. Thus

$$dP(x) \equiv x^{2^\mu}(x+1)^{2^\mu} K(x) \pmod 2,$$

where $K(x)$ is neither divisible by $x$ nor by $(x+1) \pmod 2$. As in the preceding case, $P(x)$ must have two roots of multiplicity prime to $p$. This proves part (ii) of the lemma.

In order to prove part (i), first we consider the case $q = 6$. In this case,

$$dP(x) \equiv (2l^6 + 2)x^6 + (2l^5 + 2)x^5 + (l^4 + 3)x^4 + (3l^2 + 1)x^2 \pmod 4.$$

Since $l$ is even, we can write

$$dP(x) \equiv 2x^6 + 2x^5 + 3x^4 + x^2 \pmod 4.$$

So, $P(x)$ has at least three simple roots. To prove our claim, suppose $dP$ can be written as

$$dP(x) \equiv S(x)R^2(x) \pmod 4, \tag{2.11}$$

with $\deg S \leq 2$.

If $\deg S = 0$, then clearly $S$ is an odd constant, so $R^2(x) \equiv x^4 + x^2 \pmod 2$. Hence $R(x) \equiv x^2 + x \pmod 2$ and $R^2(x) \equiv x^4 + 2x^3 + x^2 \pmod 4$, which is a contradiction. If $\deg S = 1$, then either $S(x) \equiv x$ or $S(x) \equiv x + 1 \pmod 2$. In both cases, the quotient of $P$ and $S$ cannot be written as a square $\pmod 2$. If $\deg S = 2$, then either $S(x) \equiv x^2$ or $S(x) \equiv x^2 + x$ or $S(x) \equiv x^2 + 1 \pmod 2$, since $x^2 + x + 1$ does not divide $P \pmod 2$. In the first case, $R(x) \equiv x + 1 \pmod 2$, hence $R^2(x) \equiv x^2 + 2x + 1 \pmod 4$, which is a contradiction. In the second case, the quotient of $P$ and $S$ is not even a square $\pmod 2$. In the third case, $R(x) \equiv x \pmod 2$, hence $R^2(x) \equiv x^2 \pmod 4$, which is a contradiction. We conclude that $dP$ cannot be written in form (2.11) with $\deg S < 3$, proving our claim.

Secondly, as $q = 2$ and 4 are the exceptional cases, $q = 6$ case is just treated above. Finally, the case $q = 8$ was treated in Case 1, and we may assume that $q \geq 10$. Considering (2.2) modulo 4, where $d \equiv 2 \pmod 4$, we have

$$\begin{aligned} dP(x) \equiv d\sum_{i=0}^{q-2} \binom{q}{i} (lx+1)^{q-i} B_i - dqB_1 x^{q-1} \\ - d\sum_{i=0}^{\frac{q-2}{2}} \binom{q}{2i} B_{2i} x^{q-2i} \pmod 4. \end{aligned} \tag{2.12}$$

Firstly, let $l \equiv 0 \pmod 4$. Then we obtain

$$d\sum_{i=0}^{q-2} \binom{q}{i} (lx+1)^{q-i} B_i \equiv dqB_1 + d\sum_{i=0}^{\frac{q-2}{2}} \binom{q}{2i} B_{2i} \pmod 4. \tag{2.13}$$

We know that $\sum_{i=1}^{q} \binom{q}{q-i} B_{q-i} = 0$. By (2.6) and (2.13), one gets

$$d\sum_{i=0}^{q-2} \binom{q}{i} (lx+1)^{q-i} B_i \equiv d\left( qB_1 + \sum_{i=0}^{\frac{q-2}{2}} \binom{q}{2i} B_{2i} \right) \equiv 0 \pmod 4.$$

Then we deduce by (2.12) the following:

$$dP(x) \equiv -dqB_1x^{q-1} - d\sum_{i=0}^{\frac{q-2}{2}} \binom{q}{2i} B_{2i}x^{q-2i} \pmod 4. \tag{2.14}$$

Secondly, let $l \equiv 2 \pmod 4$. Then we have (2.8), and as above, the RHS of (2.8) becomes (2.9).

Since $2x+1$ is odd and $q \neq 2^\mu$ is even ($q \geq 10$) and $dq \equiv 0 \pmod 4$, considering (2.9) modulo 4 and using (2.6), (2.9) becomes

$$d\sum_{i=0}^{q-2} \binom{q}{i} (2x+1)^{q-i} B_i \equiv 0 \pmod 4.$$

So, in all cases, (2.12) reduces to (2.14). Then by (2.14) we have

$$dP(x) \equiv 2x^q - qx^{q-1} + \frac{1}{6}d\binom{q}{2}x^{q-2} + \cdots + dB_{q-2}\binom{q}{2}x^2 \pmod 4. \tag{2.15}$$

Write $dP(x) \equiv R^2(x)S(x)$, where $R, S \in \mathbb{Z}[x]$ and $S(x)$ only contains each factor of odd multiplicity of $P$ once. Then $\deg S(x) \geq 3$. The assertion easily follows by repeating the corresponding part of the proof of Lemma 5. Thus, the proof is completed for Case 2.

*Case 3.* Let $q \geq 3$ be odd. Then $d \equiv 2 \pmod 4$ and for $i = 1, 2, 4, \ldots, q-1$,

$$d\binom{q}{i}B_i \equiv \binom{q}{i} \pmod 2.$$

Now considering (2.2) modulo 2, we have

$$dP(x) \equiv d\sum_{i=0}^{q} \binom{q}{i}(1 - x^{q-i})B_i \pmod 2.$$

Since $\sum\limits_{\lambda=1}^{\frac{q-2}{2}} \binom{q}{2\lambda} = 2^{q-1} - 1 \equiv 1 \pmod 2$, we have

$$dP(x) \equiv x^{q-1} + \sum_{\lambda=1}^{\frac{q-1}{2}} \binom{q}{2\lambda} x^{q-2\lambda} \pmod 2. \tag{2.16}$$

From (2.2), we get

$$dP'(x) = d\left(\sum_{i=0}^{q}\binom{q}{i}[(lx+1)^{q-i} - x^{q-i}]B_i\right)' - dq(q-1)x^{q-2}, \tag{2.17}$$

and then

$$xdP'(x) \equiv \sum_{\lambda=1}^{\frac{q-1}{2}}\binom{q}{2\lambda}(q-2\lambda)x^{q-2\lambda} \pmod 2. \tag{2.18}$$

Hence, by using (2.16) and (2.18),

$$d(P(x) + xP'(x)) \equiv x^{q-1} \pmod 2.$$

Any common factor of $dP(x)$ and $dP'(x)$ must therefore be congruent to a power of $x \pmod 2$. Considering (2.17) modulo 2, $dP'(x) \equiv \binom{q}{q-1} = q \equiv 1 \pmod 2$. Since $dP'(0) \equiv 1 \pmod 2$, we find that $dP(x)$ and $dP'(x)$ are relatively prime (mod 2). So any common divisor of $dP(x)$ and $dP'(x)$ in $\mathbb{Z}[x]$ is of the shape $2R(x)+1$. Write $dP(x) = Q(x)S(x)$, where $Q(x) = \prod_i Q_i(x)^{k_i} \in \mathbb{Z}[x]$ contains the multiple factors of $dP$, and $S \in \mathbb{Z}[x]$ contains its simple factors, where $k_i$ denotes the multiplicity of the polynomial factor $Q_i(x)$. Then $Q(x)$ is of the shape $2R(x)+1$ with $R \in \mathbb{Z}[x]$, so

$$S(x) \equiv dP(x) \equiv x^{q-1} + \cdots \pmod 2.$$

Thus the degree of $S(x)$ is at least $q-1$, proving Case 3 whence $q > 3$.

If $q = 3$, then

$$dP(x) = (l-1)x(2(l^2+l+1)x^2 + 3(l+1)x + 1). \tag{2.19}$$

Considering (2.19), where $l \equiv 2 \pmod 4$, it follows that

$$dP(x) \equiv x(2x+1)(3x+1) \pmod 4.$$

So, $P(x)$ has three simple roots if $l \equiv 2 \pmod 4$. Now, we consider the case $l \equiv 0 \pmod 4$ in (2.18). Then we have

$$2P(x) \equiv 2x^3 + x^2 + 3x \pmod 4.$$

$P(x)$ has also three simple roots if $l \equiv 0 \pmod 4$. To prove this, suppose

$$2P(x) \equiv Q(x)T^2(x) \pmod 4, \tag{2.20}$$

with $\deg Q \leq 2$.

If $\deg Q = 0$, then $Q$ is an odd constant. So the quotient of $2P$ and $Q$ cannot be written as a square $\pmod 2$. If $\deg Q = 1$, then either $Q(x) \equiv x$ or $Q(x) \equiv x+1 \pmod 2$. In both case, the quotient of $2P$ and $Q$ cannot be written as a square $\pmod 2$. If $\deg Q = 2$, then either $Q(x) \equiv x^2$ or $Q(x) \equiv x^2 + x$ or $Q(x) \equiv x^2 + 1$ or $Q(x) \equiv x^2 + x + 1$. None of the $Q(x)$'s does divide $P \pmod 2$. We conclude that $2P$ cannot be written in the form (2.20) with $\deg Q < 3$, proving our claim. So, the proof of the lemma is completed. □

## 3. Exceptional values for $k$

Consider equation (1.3) for fixed $k \in \{1, 3\}$ and fixed $n = m > 1$. Then equation (1.3) is equivalent to the equation

$$(k+1)y^m = P(x), \tag{3.1}$$

where $P(x) = B_q(lx+1) - B_q(x+1)$, $q \in \{2, 4\}$, $q = k+1$.

If $q = 2$, then equation (3.1) becomes

$$2y^m = (l-1)x((l+1)x+1). \tag{3.2}$$

By using Lemma 2, we have $r_1 = r_2 = 1$, and so $t_1 = t_2 = 1$. From here, we get $m = 2$. In the case $m = 2$, equation (3.2) becomes

$$u^2 - 2(l-1)v^2 = 1, \tag{3.3}$$

where $u = 2x(l+1)+1$, $v = 2(l+1)y$, $l \equiv 0 \pmod 2$. By the theory of Pell's equation (see, e.g., [10, Ch. 8]), for infinitely many choices of $l$, (3.3) has infinitely many solutions.

If $q = 4$, then equation (3.1) becomes

$$4y^m = x^2(l-1)((l^2+1)x + l + 1)((l+1)x+1). \tag{3.4}$$

Similarly to the former case, by Lemma 2 we get $m = 2$. In this case, equation (3.4) becomes

$$u^2 - (l^4-1)v^2 = -l^2(l+1)(l^2+1)(l-1)^3, \tag{3.5}$$

where $u = (l^4-1)t$ $(t \in \mathbb{Z})$, $v = (l^4-1)x + (l^3-1)$, $l \equiv 0 \pmod 2$. So, (3.5) has infinitely many solutions.

*Remark 1.* Even if $l$ is odd, equations (3.3) and (3.5) are Pell's equation. In this work, however, we consider the title equation, where $l$ is even.

## 4. Proofs of the theorems

PROOF OF THEOREM 1. Let $x, y \geq 1$ and $n \geq 2$ be an arbitrary solution of (1.3) in integers. We know from Lemma 4 that $G(x)$ has at least two distinct zeroes. Hence, by applying Lemma 3, it follows from equation (1.3) that we get an effective bound for $n$. □

PROOF OF THEOREM 2. We know from Theorem 1 that $n$ is bounded, i.e. $n < C_1$ with an effectively computable $C_1$. So we may assume that $n$ is fixed. Then we get the following equation in integers $x, y \geq 1$

$$P(x) = y^n,$$

where $P$ is given by (2.1) with $q = k+1$. Write

$$P(x) = a_0 \prod_{i=1}^{n} (x - x_i)^{r_i},$$

where $a_0 \neq 0$, $x_i \neq x_j$ if $i \neq j$ and, for a fixed $n$, let $t_i = \frac{n}{(n, r_i)}$. If $n$ is even, then by Lemma 6 at least three zeroes have odd multiplicity, say $r_1, r_2, r_3$. Hence $t_1$, $t_2$ and $t_3$ are even. Consequently, the exceptional cases in Lemma 2 cannot occur. If $n$ is odd and $p|n$ for an odd prime $p$, then by Lemma 6 at least two zeroes of $P(x)$ have multiplicities prime to $p$. We may assume that $(r_1, p) = (r_2, p) = 1$, so $p|t_1$ and $p|t_2$. Using Lemma 2, we have $\max\{x, y\} < C_2(n)$ with an effectively computable $C_2(n)$. Finally, $n < C_1$ implies the required assertion. This proves the theorem. □

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## References

[1] B. BRINDZA, On some generalizations of the Diophantine equation $1^k + 2^k + \cdots + x^k = y^z$, *Acta Arith.* **44** (1984), 99–107.

[2] B. BRINDZA, On $S$-integral solutions of the equation $y^m = f(x)$, *Acta Math. Hungar.* **44** (1984), 133–139.

[3] B. Brindza and Á. Pintér, On equal values of power sums, *Acta Arith.* **77** (1996), 97–101.

[4] B. Brindza and Á. Pintér, On the number of solutions of the equation $1^k + 2^k + \cdots + (x-1)^k = y^z$, *Publ. Math. Debrecen* **56** (2000), 271–277.

[5] K. Dilcher, On a Diophantine equation involving quadratic characters, *Compositio Math.* **57** (1986), 383–403.

[6] K. Győry, R. Tijdeman and M. Voorhoeve, On the equation $1^k + 2^k + \cdots + x^k = y^z$, *Acta Arith.* **37** (1980), 233–240.

[7] K. Győry and Á. Pintér, On the equation $1^k + 2^k + \cdots + x^k = y^z$, *Publ. Math. Debrecen* **62** (2003), 403–414.

[8] H. Kano, On the equation $s(1^k + 2^k + \cdots + x^k) + r = by^z$, *Tokyo J. Math.* **13** (1990), 441–448.

[9] W. J. LeVeque, On the equation $y^m = f(x)$, *Acta Arith.* **9** (1964), 209–219.

[10] L. J. Mordell, Diophantine Equations, *Academic Press*, *London – New York*, 1969.

[11] Á. Pintér, A note on the equation $1^k + 2^k + \cdots + (x-1)^k = y^m$, *Indag. Math. (N.S.)* **8** (1997), 119–123.

[12] Cs. Rakaczki, On some generalizations of the Diophantine equation $s(1^k + 2^k + \cdots + x^k) + r = dy^n$, *Acta Arith.* **151** (2012), 201–216.

[13] H. Rademacher, Topics in Analytic Number Theory, *Springer-Verlag*, *New York – Heidelberg*, 1973.

[14] J. J. Schäffer, The equation $1^p + 2^p + \cdots + n^p = m^q$, *Acta Math.* **95** (1956), 155–189.

[15] A. Schinzel and R. Tijdeman, On the equation $y^m = P(x)$, *Acta Arith.* **31** (1976), 199–204.

[16] J. Urbanowicz, On the equation $f(1) \cdot 1^k + f(2) \cdot 2^k + \cdots + f(x) \cdot x^k + R(x) = by^z$, *Acta Arith.* **51** (1988), 349–368.

[17] M. Voorhoeve, K. Győry and R. Tijdeman, On the Diophantine equation $1^k + 2^k + \cdots + x^k + R(x) = y^z$, *Acta Math.* **143** (1979), 1–8; corrigendum *ibid.* **159** (1987), 151–152.

[18] Z. Zhang, On the Diophantine equation $(x-1)^k + x^k + (x+1)^k = y^n$, *Publ. Math. Debrecen* **85** (2014), 93–100.

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