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(m, n)-Hom-Lie algebras

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Abstract. Let (H, β) be a monoidal Hom-Hopf algebra, and (A, α) an algebra in the (m, n)-Hom-Yetter–Drinfeld category $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$, where $m, n \in \mathcal{Z}$ (the set of integers). In this paper, we introduce the notion of (m, n)-Hom-Lie algebra (i.e., Lie algebras in the category $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$), and then prove that (A, α) can give rise to an (m, n)-Hom-Lie algebra with suitable Lie bracket when the braiding τ in $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$ is symmetric on (A, α). We also show that if also (A, α) is a sum of two (H, β)-commutative Hom-subalgebras, then $[A, A][A, A] = 0$.

1. Introduction

A Hom-type algebraic structure (e.g., algebra, coalgebra, Lie algebra, Hopf algebra, etc.) is a vector space, endowed with an endomorphism, such that the classical definition of this algebraic structure is "deformed" by this endomorphism. The theory of Hom-type algebraic structures has attracted many researchers in the past two decades, see [1], [3], [5], [7]–[10], [12]–[17], [19]–[20], [22], [24]–[29]. The origins of the study of Hom-algebras can be found in [5] by J. T. Hartwig, D. Larsson and S. D. Silvestrov, where the notion of Hom-Lie algebra in the context of q-deformation theory of Witt and Virasoro algebras (see [6]) was introduced, which plays an important role in physics, mainly in conformal field theory. The notion of a quasi-Hom-Lie algebra, which is a natural generalization of Hom-Lie algebras, can be found in [8], introduced

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by D. Larsson and S. D. Silvestrov. By the same authors, the concept of general quasi-Lie algebras was given in [7]. Γ-graded (color) quasi-Lie algebras were defined for the first time in [21] by G. Sigurðsson and S. D. Silvestrov, which include color Hom-Lie algebras and color Hom-Lie superalgebras as special cases. D. Yau defined the Hom-L-modules for Hom-Lie algebras and studied the corresponding homology in [26]. In [19], Y. H. Sheng studied the adjoint representation and the trivial representation of Hom-Lie algebras.

As we all know, each associative algebra can give rise to a Lie algebra via the commutator bracket. This result has been studied by some researchers, such as the case in the left comodule category on the cotriangular Hopf algebra by Y. Bahturin, D. Fishman and S. Montgomery (see [2]), the case in Yetter–Drinfeld category by S. H. Wang (see [23]), the Hom-case by A. Makhlouf and S. D. Silvestrov (see [14]), and by S. X. Wang and S. H. Wang (see [22]). In 2011, S. Caenepeel and I. Goyvaerts studied Hom-type algebras from the point of view of monoidal categories (see [3]). Recently, we introduced the notion of (m,n)-Hom-Yetter–Drinfeld category $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$, where $m,n \in \mathcal{Z}$ (the set of integers) via (m,n)-Radford biproduct monoidal Hom-Hopf algebra based on the module Hom-algebra defined by Y. Y. Chen, Z. W. Wang and L. Y. Zhang in [4], and also proved that the category $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$ is a braided monoidal category (see [11]). In this paper, we consider the above result about Lie algebra in the category $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$, and mainly give the following theorems.

Theorem A. *Let (H,β) be a monoidal Hom-Hopf algebra with bijective antipode S. Assume that (A,α) is an algebra in the Hom-Yetter–Drinfeld category $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$, and the braiding τ in $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$ is symmetric on (A,α). Then the quadruple $(A,[,],\alpha,\tau)$ is an (m,n)-Hom-Lie algebra, where the bracket product is defined by*

$$[,]: A\otimes A \to A, [a,b] = ab - (\beta^{m+n}(a_{-1}) \triangleright \alpha^{-1}(b))\alpha(a_0),$$

for all $a,b\in A$.

We also provide two examples for the above theorem.

Theorem B. *Let (H,β) be a monoidal Hom-Hopf algebra with bijective antipode S, and (A,α) be an algebra in the Hom-Yetter–Drinfeld category $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$, with Hom-subalgebras U and X in $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$ which are (H,β)-commutative such that $A = U + X$. Assume that the braiding τ is symmetric on (A,α). Then $[A,A][A,A] = 0$.*

2. Preliminaries

Throughout this paper, we follow the definitions and terminologies in [3]–[4], [11], with all algebraic systems supposed to be over the field K. Given a K-space U, we write id_U for the identity map on U.

We now recall some useful definitions.

Definition 2.1. A monoidal Hom-algebra is a quadruple $(A, \mu, 1_A, \alpha)$ (abbr. (A, α)), where A is a linear space, $\mu : A \otimes A \to A$ is a linear map, $1_A \in A$, and α is an automorphism of A, such that the conditions

$$\begin{aligned} &(\text{HA1}) \quad \alpha(aa') = \alpha(a)\alpha(a'); \quad \alpha(1_A) = 1_A, \\ &(\text{HA2}) \quad \alpha(a)(a'a'') = (aa')\alpha(a''); \quad a1_A = 1_A a = \alpha(a) \end{aligned}$$

are satisfied for $a, a', a'' \in A$. Here we use the notation $\mu(a \otimes a') = aa'$.

Definition 2.2. A monoidal Hom-coalgebra is a quadruple $(C, \Delta, \varepsilon, \beta)$ (abbr. (C, β)), where C is a linear space, $\Delta : C \to C \otimes C, \varepsilon : C \to K$ are linear maps, and β is an automorphism of C, such that

$$\begin{aligned} &(\text{HC1}) \quad \beta(c)_1 \otimes \beta(c)_2 = \beta(c_1) \otimes \beta(c_2); \quad \varepsilon \circ \beta = \varepsilon, \\ &(\text{HA2}) \quad \beta^{-1}(c_1) \otimes c_{21} \otimes c_{22} = c_{11} \otimes c_{12} \otimes \beta^{-1}(c_2); \quad \varepsilon(c_1)c_2 = c_1\varepsilon(c_2) = \beta^{-1}(c) \end{aligned}$$

are satisfied for $c \in C$. Here we use the notation $\Delta(c) = c_1 \otimes c_2$ (summation implicitly understood).

Definition 2.3. A monoidal Hom-bialgebra is a sextuple $(H, \mu, 1_H, \Delta, \varepsilon, \gamma)$ (abbr. (H, γ)), where $(H, \mu, 1_H, \gamma)$ is a monoidal Hom-algebra, and $(H, \Delta, \varepsilon, \gamma)$ is a monoidal Hom-coalgebra, such that Δ and ε are morphisms of Hom-algebras, i.e.,

$$\begin{aligned} \Delta(hh') = \Delta(h)\Delta(h'); \quad \Delta(1_H) = 1_H \otimes 1_H, \\ \varepsilon(hh') = \varepsilon(h)\varepsilon(h'); \quad \varepsilon(1_H) = 1. \end{aligned}$$

Furthermore, if there exists a linear map $S : H \to H$ such that

$$S(h_1)h_2 = h_1 S(h_2) = \varepsilon(h)1_H; \text{ and } S(\gamma(h)) = \gamma(S(h)),$$

then we call $(H, \mu, 1_H, \Delta, \varepsilon, \gamma, S)$ (abbr. (H, γ, S)) a monoidal Hom-Hopf algebra.

Definition 2.4. Let (A, β) be a monoidal Hom-algebra. A left (A, β)-Hom-module is a triple $(U, \triangleright, \alpha)$, where U is a linear space, $\triangleright : A \otimes U \to U$ is a linear map, and α is an automorphism of U, such that

$$\text{(HM1)} \quad \alpha(a \triangleright u) = \beta(a) \triangleright \alpha(u),$$
$$\text{(HM2)} \quad \beta(a) \triangleright (a' \triangleright u) = (aa') \triangleright \alpha(u); \quad 1_A \triangleright u = \alpha(u)$$

are satisfied for $a, a' \in A$ and $u \in U$.

Let $(U, \triangleright_U, \alpha_U)$ and $(V, \triangleright_V, \alpha_V)$ be two left (A, β)-Hom-modules. Then a linear morphism $f : U \to V$ is called a morphism of left (A, β)-Hom-modules if $f(a \triangleright_U u) = a \triangleright_V f(u)$ and $f \circ \alpha_U = \alpha_V \circ f$.

Definition 2.5. Let (H, β) be a monoidal Hom-bialgebra, and (A, α) a monoidal Hom-algebra. If $(A, \triangleright, \alpha)$ is a left (H, β)-Hom-module, and for all $h \in H$ and $a, a' \in A$,

$$\text{(HMA1)} \quad h \triangleright (aa') = (h_1 \triangleright a)(h_2 \triangleright a'),$$
$$\text{(HMA2)} \quad h \triangleright 1_A = \varepsilon(h) 1_A,$$

then $(A, \triangleright, \alpha)$ is called an (H, β)-module Hom-algebra.

Remark. When $\alpha = \mathrm{id}_A$ and $\beta = \mathrm{id}_H$, an (H, β)-module Hom-algebra is the usual H-module algebra.

Definition 2.6. Let (C, β) be a monoidal Hom-coalgebra. A left (C, β)-Hom-comodule is a triple (U, ρ, α), where U is a linear space, $\rho : U \to C \otimes U$ (where $\rho(u) = u_{-1} \otimes u_0, \forall\, u \in U$) is a linear map, and α is an automorphism of U, such that

$$\text{(HCM1)} \quad \alpha(u)_{-1} \otimes \alpha(u)_0 = \beta(u_{-1}) \otimes \alpha(u_0),$$
$$\text{(HCM2)} \quad \beta^{-1}(u_{-1}) \otimes u_{0-1} \otimes u_{00} = u_{-11} \otimes u_{-12} \otimes \alpha^{-1}(u_0); \quad \varepsilon(u_{-1}) u_0 = \alpha^{-1}(u)$$

are satisfied for all $u \in U$.

Let (U, ρ_U, α_U) and (V, ρ_V, α_V) be two left (C, β)-Hom-comodules. Then a linear map $f : U \to V$ is called a map of left (C, β)-Hom-comodules if $f(u)_{-1} \otimes f(u)_0 = u_{-1} \otimes f(u_0)$ and $f \circ \alpha_U = \alpha_V \circ f$.

Definition 2.7. Let (H, β) be a monoidal Hom-bialgebra and (A, α) a monoidal Hom-algebra. If (A, ρ, α) is a left (H, β)-Hom-comodule, and for all $a, a' \in A$, the conditions

$$\text{(HCMA1)} \quad \rho(aa') = a_{-1} a'_{-1} \otimes a_0 a'_0;$$
$$\text{(HCMA2)} \quad \rho(1_A) = 1_H \otimes 1_A$$

hold, then (A, ρ, α) is called an (H, β)-comodule Hom-algebra.

Remark. When $\alpha = \mathrm{id}_A$ and $\beta = \mathrm{id}_H$, an (H,β)-comodule Hom-algebra is the usual H-comodule algebra.

From now on, we always assume that (H,β) is a monoidal Hom-Hopf algebra with a bijective antipode S and let $\ell = m+n$.

Definition 2.8. Let (H,β) be a monoidal Hom-bialgebra, $(U, \triangleright_U, \alpha_U)$ a left (H,β)-module with action $\triangleright_U : H\otimes U \to U, h\otimes u \mapsto h \triangleright_U u$, and (U,ρ^U,α_U) a left (H,β)-comodule with coaction $\rho^U : U \to H\otimes U, u \mapsto u_{-1}\otimes u_0$. Then we call $(U,\triangleright_U,\rho^U,\alpha_U)$ an (m,n)-Hom-Yetter–Drinfeld module over (H,β) if the following condition holds:

$$\text{(HYD)}\ qh_1\beta^n(u_{-1})\otimes(\beta^m(h_2)\triangleright_U u_0)=\beta^{n-1}((\beta^{m+1}(h_1)\triangleright u)_{-1})h_2\otimes(\beta^{m+1}(h_1)\triangleright u)_0,$$

where $h\in H$ and $u\in U$.

Remark. When (H,β,S_H) is a monoidal Hom-Hopf algebra, the condition (HYD) is equivalent to

$$\begin{aligned}\text{(HYD)}'\quad &(\beta^m(h)\triangleright_U u)_{-1}\otimes(\beta^m(h)\triangleright_U u)_0\\ &\quad= (\beta^{-n}(h_{11})\beta^{-1}(u_{-1}))S(\beta^{-n}(h_2))\otimes(\beta^{m+1}(h_{12})\triangleright u_0).\end{aligned}$$

Definition 2.9. Let (H,β) be a monoidal Hom-Hopf algebra with a bijective antipode S. Denote by $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$ the category whose objects are (m,n)-Hom-Yetter–Drinfeld modules $(U,\triangleright_U,\rho^U,\alpha_U)$ over (H,β); the morphisms in the category are morphisms of left (H,β)-modules and left (H,β)-comodules. Then $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$ is a braided monoidal category, with tensor product defined by

$$\triangleright_{U\otimes V} : H\otimes(U\otimes V)\to U\otimes V,\quad h\triangleright_{U\otimes V}(u\otimes v) = (h_1\triangleright_U u)\otimes(h_2\triangleright_V v),$$

$$\rho^{U\otimes V} : U\otimes V\to H\otimes U\otimes V,\quad u\otimes v\mapsto u_{-1}v_{-1}\otimes(u_0\otimes v_0),$$

associativity constraints defined by the formula $a_{U,V,W} : (U\otimes V)\otimes W \to U\otimes(V\otimes W), (u\otimes v)\otimes w\mapsto \alpha_U(u)\otimes(v\otimes\alpha_W^{-1}(w))$, where $u\in U, v\in V$and $w\in W$, and braiding defined by

$$\tau_{U,V} : U\otimes V\to V\otimes U, u\otimes v\mapsto (\beta^\ell(u_{-1})\triangleright_V \alpha_V^{-1}(v))\otimes\alpha_U(u_0),$$

where $u\in U$ and $v\in V$.

Let (A,α) be an object in $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$. The braiding τ is called *symmetric* on (A,α) if the following condition holds:

$$\text{(SY1)}\quad (\beta^\ell((\beta^\ell(a_{-1})\triangleright\alpha^{-1}(b))_{-1})\triangleright a_0)\otimes\alpha((\beta^\ell(a_{-1})\triangleright\alpha^{-1}(b))_0) = a\otimes b,$$

which is equivalent to the condition

$$\text{(SY2)}\quad (\beta^{\ell}(a_{-1}) \triangleright \alpha^{-1}(b)) \otimes \alpha(a_0) = \alpha(b_0) \otimes (S^{-1}(\beta^{\ell}(b_{-1})) \triangleright \alpha^{-1}(a)),$$

for all $a, b \in A$.

Let (A, α) be an algebra in $\widetilde{\mathcal{H}}({}^{H}_{H}\mathcal{YD}(\mathcal{Z}))$. (A, α) is called (H, β)-commutative if

$$\text{(CO)}\ (\beta^{\ell}(a_{-1}) \triangleright \alpha^{-1}(b))\alpha(a_0) = ab,$$

for all $a, b \in A$.

Definition 2.10. A Hom-Lie algebra is a triple $(L, [,], \alpha)$, where L is a linear space, $[,] : L \otimes L \to L$ is a linear map, and α is an automorphism of L, such that

$$\begin{aligned} &\text{(L1)}\quad \alpha([x, y]) = [\alpha(x), \alpha(y)],\\ &\text{(L2)}\quad [x, y] = -[y, x] \quad \text{(Skew-symmetry)},\\ &\text{(L3)}\quad \circlearrowleft [\alpha(x), [y, z]] = 0 \quad \text{(Hom-Jacobi identity)} \end{aligned}$$

are satisfied for all $x, y, z \in L$, and $\circlearrowleft$ denotes the summation over the cyclic permutation on x, y, z.

3. (m, n)-Hom-Lie algebras

In this section, we introduce the notion of (m, n)-Hom-Lie algebra and prove that an algebra in the (m, n)-Hom-Yetter–Drinfeld category $\widetilde{\mathcal{H}}({}^{H}_{H}\mathcal{YD}(\mathcal{Z}))$ can give rise to an (m, n)-Hom-Lie algebra.

Definition 3.1. Let (H, β) be a monoidal Hom-Hopf algebra with a bijective antipode S. In the (m, n)-Hom-Yetter–Drinfeld category $\widetilde{\mathcal{H}}({}^{H}_{H}\mathcal{YD}(\mathcal{Z}))$, a Lie algebra is called an (m, n)-Hom-Lie algebra, i.e., an (m, n)-Hom-Lie algebra is a quadruple $(L, [,], \alpha, \tau)$, where L is a linear space, $[,] : L \otimes L \to L$, $\triangleright : H \otimes L \to L$, $\rho : L \to H \otimes L$ (write $\rho(x) = x_{-1} \otimes x_0$) are linear maps, and α is an automorphism of L such that the conditions

$$\begin{aligned} &\text{(GH1)}\quad \alpha([x, y]) = [\alpha(x), \alpha(y)],\\ &\text{(GH2)}\quad \alpha(h \triangleright x) = \beta(h) \triangleright \alpha(x),\\ &\text{(GH3)}\quad \beta(h) \triangleright (g \triangleright x) = (hg) \triangleright \alpha(x),\\ &\text{(GH4)}\quad h \triangleright [x, y] = [h_1 \triangleright x, h_2 \triangleright y], \end{aligned}$$

(GH5) $\alpha(x)_{-1} \otimes \alpha(x)_0 = \beta(x_{-1}) \otimes \alpha(x_0)$,

(GH6) $\beta^{-1}(x_{-1}) \otimes x_{0-1} \otimes x_{00} = x_{-11} \otimes x_{-12} \otimes \alpha^{-1}(x_0)$,

(GH7) $[x,y]_{-1} \otimes [x,y]_0 = x_{-1}y_{-1} \otimes [x_0, y_0]$,

(GH8) $[x,y] = -(([,] \circ \tau)(x \otimes y))$ (H-skew-symmetry),

(GH9) $\{x \otimes y \otimes z\} + \{(\tau \otimes \mathrm{id}_L)(\mathrm{id}_L \otimes \tau)(x \otimes y \otimes z)\}$
$+ \{(\mathrm{id}_L \otimes \tau)(\tau \otimes \mathrm{id}_L)(x \otimes y \otimes z)\} = 0$ (H-Jacobi identity),

and (HYD) hold for all $h, g \in H$, $x, y, z \in L$, and τ is the braiding in the category $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$, $\{x \otimes y \otimes z\}$ denotes $[\alpha(x), [y,z]]$.

In the following, we will prove Theorem A, firstly we need the lemmas below.

Lemma 3.2. *Let (H,β) be a monoidal Hom-Hopf algebra with bijective antipode S, (A,α) an (H,β)-module Hom-algebra and an (H,β)-comodule Hom-algebra. Assume that the braiding τ is symmetric on (A,α). Then we have*

$$\begin{aligned}(\beta^\ell(\alpha(\beta^\ell(a_{-1}) \triangleright \alpha^{-1}(\beta^\ell(b_{-1}) \triangleright \alpha^{-1}(c)))_{-1}) \triangleright (a_0b_0))\\ \times \alpha(\alpha(\beta^\ell(a_{-1}) \triangleright \alpha^{-1}(\beta^\ell(b_{-1}) \triangleright \alpha^{-1}(c)))_0) = \alpha(a)(bc),\end{aligned}$$

where $a, b, c \in A$.

Proof. For all $a, b, c \in A$, we compute as follows:

$$\begin{aligned}
&(\beta^\ell(\alpha(\beta^\ell(a_{-1}) \triangleright \alpha^{-1}(\beta^\ell(b_{-1}) \triangleright \alpha^{-1}(c)))_{-1}) \triangleright (a_0b_0))\\
&\quad \times \alpha(\alpha(\beta^\ell(a_{-1}) \triangleright \alpha^{-1}(\beta^\ell(b_{-1}) \triangleright \alpha^{-1}(c)))_0)\\
&\overset{(HM1)}{=} (\beta^\ell((\beta^{\ell+1}(a_{-1}) \triangleright (\beta^\ell(b_{-1}) \triangleright \alpha^{-1}(c)))_{-1}) \triangleright (a_0b_0))\\
&\quad \times \alpha((\beta^{\ell+1}(a_{-1}) \triangleright (\beta^\ell(b_{-1}) \triangleright \alpha^{-1}(c)))_0)\\
&\overset{(HM2)}{=} (\beta^\ell(((\beta^\ell(a_{-1})\beta^\ell(b_{-1})) \triangleright c)_{-1}) \triangleright (a_0b_0)) \times \alpha(((\beta^\ell(a_{-1})\beta^\ell(b_{-1})) \triangleright c)_0)\\
&\overset{(HA1)}{=} (\beta^\ell((\beta^\ell(a_{-1}b_{-1}) \triangleright c)_{-1}) \triangleright (a_0b_0))\alpha((\beta^\ell(a_{-1}b_{-1}) \triangleright c)_0)\\
&\overset{(HCMA1)}{=} (\beta^\ell((\beta^\ell((ab)_{-1}) \triangleright c)_{-1}) \triangleright (ab)_0)\alpha((\beta^\ell((ab)_{-1}) \triangleright c)_0)\\
&\overset{(SY1)}{=} (ab)\alpha(c)\\
&\overset{(HA2)}{=} \alpha(a)(bc),
\end{aligned}$$

finishing the proof. □

Lemma 3.3. *Let (H,β) be a monoidal Hom-Hopf algebra with bijective antipode S, (A,α) an (H,β)-module Hom-algebra and an (H,β)-comodule Hom-algebra. Then we have*

$$\begin{aligned}&\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright a^{-1}(c)))((\beta^{\ell}(\alpha(a_0)_{-1})\triangleright b_0)\alpha(\alpha(a_0)_0))\\&\quad=(\beta^{\ell}(\alpha(a)_{-1})\triangleright(\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c))b_0))\alpha(\alpha(a)_0),\end{aligned}$$

where $a,b,c\in A$.

Proof. For all $a,b,c\in A$, we compute as follows:

$$\begin{aligned}&\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c)))((\beta^{\ell}(\alpha(a_0)_{-1})\triangleright b_0)\alpha(\alpha(a_0)_0))\\
&\overset{\text{(HA2)}}{=}((\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c)))(\beta^{\ell}(\alpha(a_0)_{-1})\triangleright b_0))\alpha^2(\alpha(a_0)_0)\\
&\overset{\text{(HCM1)}}{=}((\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c)))(\beta^{\ell+1}(a_{0-1})\triangleright b_0))\alpha^2(\alpha(a_{00}))\\
&\overset{\text{(HCM2)}}{=}((\beta^{\ell+1}(a_{-11})\triangleright\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c)))(\beta^{\ell+1}(a_{-12})\triangleright b_0))\alpha^2(a_0)\\
&\overset{\text{(HMA1)}}{=}(\beta^{\ell+1}(a_{-1})\triangleright(\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c))b_0))\alpha^2(a_0)\\
&\overset{\text{(HCM1)}}{=}(\beta^{\ell}(\alpha(a)_{-1})\triangleright(\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c))b_0))\alpha(\alpha(a)_0),\end{aligned}$$

finishing the proof. □

Similarly, we have

Lemma 3.4. *Let (H,β) be a monoidal Hom-Hopf algebra with bijective antipode S, (A,α) an (H,β)-module Hom-algebra and an (H,β)-comodule Hom-algebra. Assume that the braiding τ is symmetric on (A,α). Then we have*

$$\begin{aligned}&(\beta^{\ell}(\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(b))_{-1})\triangleright\alpha^{-1}((\beta^{\ell}(\alpha(a_0)_{-1})\triangleright\alpha^{-1}(c))\alpha(\alpha(a_0)_0)))\\&\quad\times\alpha(\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(b))_0)=\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c)))(\alpha(a_0)\alpha(b_0)),\end{aligned}$$

where $a,b,c\in A$.

Lemma 3.5. *Let (H,β) be a monoidal Hom-Hopf algebra with bijective antipode S, (A,α) an (H,β)-module Hom-algebra and an (H,β)-comodule Hom-algebra. Then we have*

$$\begin{aligned}&\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(b))((\beta^{\ell}(\alpha(a_0)_{-1})\triangleright\alpha^{-1}(c))\alpha(\alpha(a_0)_0))\\&\quad=(\beta^{\ell}(\alpha(a)_{-1})\triangleright\alpha^{-1}(bc))\alpha(\alpha(a)_0),\end{aligned}$$

where $a,b,c\in A$.

Lemma 3.6. *Let (H,β) be a monoidal Hom-Hopf algebra with bijective antipode S, (A,α) an (H,β)-module Hom-algebra and an (H,β)-comodule Hom-algebra. Assume that the braiding τ is symmetric on (A,α). Then we have*

$$
\begin{aligned}
&(\beta^{\ell}(\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(b))_{-1})\triangleright\alpha^{-1}((\beta^{\ell}((\beta^{\ell}(\alpha(a_0)_{-1})\triangleright\alpha^{-1}(c))_{-1})\triangleright\alpha(a_0)_0)\\
&\quad\times\alpha((\beta^{\ell}(\alpha(a_0)_{-1})\triangleright\alpha^{-1}(c))_0)))\alpha(\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(b))_0)\\
&\quad=\alpha(a)((\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c))\alpha(b_0)),
\end{aligned}
$$

where $a,b,c\in A$.

Lemma 3.7. *Let (H,β) be a monoidal Hom-Hopf algebra with bijective antipode S, (A,α) an (H,β)-module Hom-algebra and an (H,β)-comodule Hom-algebra. Assume that the braiding τ is symmetric on (A,α). Then we have*

$$
\begin{aligned}
&(\beta^{\ell}(\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c)))_{-1})\triangleright\alpha^{-1}((\beta^{\ell}(\alpha(a_0)_{-1})\triangleright b_0)\\
&\times\alpha(\alpha(a_0)_0)))\alpha(\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c)))_0)=\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(b))\\
&\times((\beta^{\ell}((\beta^{\ell}(\alpha(a_0)_{-1})\triangleright\alpha^{-1}(c))_{-1})\triangleright\alpha(a_0)_0)\alpha((\beta^{\ell}(\alpha(a_0)_{-1})\triangleright\alpha^{-1}(c))_0)),
\end{aligned}
$$

where $a,b,c\in A$.

Next, we give the proof of Theorem A.

Proof of Theorem A. Firstly, we check the condition (GH1) as follows:

$$
\begin{aligned}
\alpha([a,b]) &= \alpha(ab-(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(b))\alpha(a_0))\\
&\overset{\text{(HM1)}}{=} \alpha(ab)-(\beta^{\ell+1}(a_{-1})\triangleright b)\alpha^2(a_0)\\
&\overset{\text{(HCM1)}}{=} \alpha(a)\alpha(b)-(\beta^{\ell}(\alpha(a)_{-1})\triangleright\alpha^{-1}(\alpha(b)))\alpha(\alpha(a)_0)=[\alpha(a),\alpha(b)].
\end{aligned}
$$

We find that the conditions (GH2), (GH3), (GH5) and (GH6) are exactly the conditions (HM1), (HM2), (HCM1) and (HCM2), respectively.

Secondly, we verify the condition (GH4). For all $a,b\in A$ and $h\in H$, we have:

$$
\begin{aligned}
&[h_1\triangleright a,h_2\triangleright b]\\
&\quad= (h_1\triangleright a)(h_2\triangleright b)-(\beta^{\ell}((h_1\triangleright a)_{-1})\triangleright\alpha^{-1}(h_2\triangleright b))\alpha((h_1\triangleright a)_0)\\
&\quad\overset{\text{(HM1)}}{=} (h_1\triangleright a)(h_2\triangleright b)-(\beta^{\ell}((h_1\triangleright a)_{-1})\triangleright(\beta^{-1}(h_2)\triangleright\alpha^{-1}(b)))\alpha((h_1\triangleright a)_0)\\
&\quad\overset{\text{(HM2)}}{=} (h_1\triangleright a)(h_2\triangleright b)-((\beta^{\ell-1}((h_1\triangleright a)_{-1})\beta^{-1}(h_2))\triangleright b)\alpha((h_1\triangleright a)_0)\\
&\quad\overset{\text{(HA1)}}{=} (h_1\triangleright a)(h_2\triangleright b)-(\beta^{\ell-1}((h_1\triangleright a)_{-1}\beta^{-\ell}(h_2))\triangleright b)\alpha((h_1\triangleright a)_0)\\
&\quad\overset{\text{(HYD)}}{=} (h_1\triangleright a)(h_2\triangleright b)-((\beta^{-1}(h_1)\beta^{\ell}(a_{-1}))\triangleright b)\alpha(\beta^{-1}(h_2)\triangleright a_0)
\end{aligned}
$$

$$\overset{\text{(HM1)}}{=} (h_1 \triangleright a)(h_2 \triangleright b) - ((\beta^{-1}(h_1)\beta^{\ell}(a_{-1})) \triangleright b)(h_2 \triangleright \alpha(a_0))$$
$$\overset{\text{(HM2)}}{=} (h_1 \triangleright a)(h_2 \triangleright b) - (h_1 \triangleright (\beta^{\ell}(a_{-1}) \triangleright \alpha^{-1}(b)))(h_2 \triangleright \alpha(a_0))$$
$$\overset{\text{(HMA1)}}{=} (h \triangleright (ab)) - h \triangleright ((\beta^{\ell}(a_{-1}) \triangleright \alpha^{-1}(b))\alpha(a_0)) = h \triangleright [a,b].$$

Thirdly, the condition (GH7) is satisfied, since

$$\begin{aligned}
&[a,b]_{-1} \otimes [a,b]_0 \\
&= (ab)_{-1} \otimes (ab)_0 - ((\beta^{\ell}(a_{-1}) \triangleright \alpha^{-1}(b))\alpha(a_0))_{-1} \\
&\quad \otimes ((\beta^{\ell}(a_{-1}) \triangleright \alpha^{-1}(b))\alpha(a_0))_0 \\
&\overset{\text{(HCMA1)}}{=} a_{-1}b_{-1} \otimes a_0b_0 - (\beta^{\ell}(a_{-1}) \triangleright \alpha^{-1}(b))_{-1}\alpha(a_0)_{-1} \\
&\quad \otimes (\beta^{\ell}(a_{-1}) \triangleright \alpha^{-1}(b))_0\alpha(a_0)_0 \\
&\overset{\text{(HCM1)}}{=} a_{-1}b_{-1} \otimes a_0b_0 - (\beta^{\ell}(a_{-1}) \triangleright \alpha^{-1}(b))_{-1}\beta(a_{0-1}) \\
&\quad \otimes (\beta^{\ell}(a_{-1}) \triangleright \alpha^{-1}(b))_0\alpha(a_{00}) \\
&\overset{\text{(HCM2)}}{=} a_{-1}b_{-1} \otimes a_0b_0 - (\beta^{\ell+1}(a_{-11}) \triangleright \alpha^{-1}(b))_{-1}\beta(a_{-12}) \\
&\quad \otimes (\beta^{\ell+1}(a_{-11}) \triangleright \alpha^{-1}(b))_0 a_0 \\
&\overset{\text{(HYD)}}{=} a_{-1}b_{-1} \otimes a_0b_0 - \beta(a_{-11})\beta(\alpha^{-1}(b)_{-1}) \otimes (\beta^{\ell}(a_{-12}) \triangleright \alpha^{-1}(b)_0)a_0 \\
&\overset{\text{(HCM1)}}{=} a_{-1}b_{-1} \otimes a_0b_0 - \beta(a_{-11})b_{-1} \otimes (\beta^{\ell}(a_{-12}) \triangleright \alpha^{-1}(b_0))a_0 \\
&\overset{\text{(HCM2)}}{=} a_{-1}b_{-1} \otimes a_0b_0 - a_{-1}b_{-1} \otimes (\beta^{\ell}(a_{0-1}) \triangleright \alpha^{-1}(b_0))\alpha(a_{00}) \\
&= a_{-1}b_{-1} \otimes [a_0, b_0].
\end{aligned}$$

Fourthly, for all $a, b \in A$, we have:

$$\begin{aligned}
&-(([,] \circ \tau)(a \otimes b)) \\
&= -[(\beta^{\ell}(a_{-1}) \triangleright \alpha^{-1}(b)), \alpha(a_0)] \\
&= -(\beta^{\ell}(a_{-1}) \triangleright \alpha^{-1}(b))\alpha(a_0) + (\beta^{\ell}((\beta^{\ell}(a_{-1}) \triangleright \alpha^{-1}(b))_{-1}) \triangleright a_0) \\
&\quad \times \alpha((\beta^{\ell}(a_{-1}) \triangleright \alpha^{-1}(b))_0) \\
&\overset{\text{(SY1)}}{=} -(\beta^{\ell}(a_{-1}) \triangleright \alpha^{-1}(b))\alpha(a_0) + ab = [a,b].
\end{aligned}$$

Thus (GH8) holds.

Finally, for all $a, b, c \in A$, we compute the condition (GH9) as follows:

$$
\begin{aligned}
&\{a\otimes b\otimes c\}\\
&=\alpha(a)(bc)-(\beta^{\ell}(\alpha(a)_{-1})\triangleright\alpha^{-1}(bc))\alpha(\alpha(a)_0)-\alpha(a)((\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c))\alpha(b_0))\\
&\quad+(\beta^{\ell}(\alpha(a)_{-1})\triangleright(\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c))b_0))\alpha(\alpha(a)_0)\\
&\triangleq(I)-(II)-(III)+(IV).
\end{aligned}
$$

$$
\begin{aligned}
&\{(\tau\otimes\mathrm{id}_A)(\mathrm{id}_A\otimes\tau)(a\otimes b\otimes c)\}\\
&\quad=\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c)))(\alpha(a_0)\alpha(b_0))\\
&\qquad-(\beta^{\ell}(\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c)))_{-1})\triangleright(a_0b_0))\\
&\qquad\times\alpha(\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c)))_0)\\
&\qquad-\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c)))((\beta^{\ell}(\alpha(a_0)_{-1})\triangleright b_0)\alpha(\alpha(a_0)_0))\\
&\qquad+(\beta^{\ell}(\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c)))_{-1})\triangleright\alpha^{-1}((\beta^{\ell}(\alpha(a_0)_{-1})\triangleright b_0)\\
&\qquad\times\alpha(\alpha(a_0)_0)))\alpha(\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(\beta^{\ell}(b_{-1})\triangleright\alpha^{-1}(c)))_0)\\
&\quad\triangleq(V)-(VI)-(VII)+(VIII).
\end{aligned}
$$

$$
\begin{aligned}
&\{(\mathrm{id}_A\otimes\tau)(\tau\otimes\mathrm{id}_A)(a\otimes b\otimes c)\}\\
&\quad=\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(b))((\beta^{\ell}(\alpha(a_0)_{-1})\triangleright\alpha^{-1}(c))\alpha(\alpha(a_0)_0))\\
&\qquad-(\beta^{\ell}(\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(b))_{-1})\triangleright\alpha^{-1}((\beta^{\ell}(\alpha(a_0)_{-1})\triangleright\alpha^{-1}(c))\alpha(\alpha(a_0)_0)))\\
&\qquad\times\alpha(\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(b))_0)\\
&\qquad-\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(b))((\beta^{\ell}((\beta^{\ell}(\alpha(a_0)_{-1})\triangleright\alpha^{-1}(c))_{-1})\triangleright\alpha(a_0)_0)\\
&\qquad\times\alpha((\beta^{\ell}(\alpha(a_0)_{-1})\triangleright\alpha^{-1}(c))_0))\\
&\qquad+(\beta^{\ell}(\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(b))_{-1})\triangleright\alpha^{-1}((\beta^{\ell}((\beta^{\ell}(\alpha(a_0)_{-1})\triangleright\alpha^{-1}(c))_{-1})\\
&\qquad\triangleright\alpha(a_0)_0)\alpha((\beta^{\ell}(\alpha(a_0)_{-1})\triangleright\alpha^{-1}(c))_0)))\alpha(\alpha(\beta^{\ell}(a_{-1})\triangleright\alpha^{-1}(b))_0)\\
&\quad\triangleq(IX)-(X)-(XI)+(XII).
\end{aligned}
$$

So,

$$
\begin{aligned}
&\{a\otimes b\otimes c\}+\{(\tau\otimes\mathrm{id}_A)(\mathrm{id}_A\otimes\tau)(a\otimes b\otimes c)\}+\{(\mathrm{id}_A\otimes\tau)(\tau\otimes\mathrm{id}_A)(a\otimes b\otimes c)\}\\
&\quad=(I)-(II)-(III)+(IV)+(V)-(VI)-(VII)+(VIII)\\
&\qquad+(IX)-(X)-(XI)+(XII).
\end{aligned}
$$

By Lemmas 3.2–3.7, we have $(VI) = (I)$, $(IV) = (VII)$, $(X) = (V)$, $(IX) = (II)$, $(XII) = (III)$, and $(VIII) = (XI)$. Therefore,

$$\{a \otimes b \otimes c\} + \{(\tau \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \tau)(a \otimes b \otimes c)\} + \{(\mathrm{id}_A \otimes \tau)(\tau \otimes \mathrm{id}_A)(a \otimes b \otimes c)\} = 0,$$

finishing the proof. □

In the following, two examples are provided.

Example 3.8. Let $H = K\{1, a\}$ be a Hopf group algebra with $a^2 = 1$ and $\Delta(a) = a \otimes a$ (see [18]). Then (H, id_H) is a monoidal Hom-Hopf algebra.

Let $A = K\{1, g, x, y\}$ be a K-linear space, and $0 \neq k \in K$. Define the linear map $\alpha : A \to A$ by

$$\alpha(1) = 1, \quad \alpha(g) = g, \quad \alpha(x) = kx, \quad \alpha(y) = ky.$$

Then α is an automorphism on A.

Define the multiplication μ on A by

μ	1	g	x	y
1	1	g	kx	ky
g	g	1	ky	kx
x	kx	$-ky$	0	0
y	ky	$-kx$	0	0

Then by a direct computation, we can get that (A, μ, α) is a monoidal Hom-algebra.

Define module action $\triangleright : H \otimes A \to A$ by

$$1_H \triangleright 1_A = 1_A, \quad 1_H \triangleright g = g, \quad 1_H \triangleright x = kx, \quad 1_H \triangleright y = ky,$$
$$a \triangleright 1_A = 1_A, \quad a \triangleright g = g, \quad a \triangleright x = kx, \quad a \triangleright y = ky.$$

Then by a routine computation, we can get $(A, \triangleright, \alpha)$ is a (H, id_H)-module Hom-algebra.

Define comodule action $\rho : A \to H \otimes A$ by

$$\rho : A \to H \otimes A, \quad 1_A \mapsto 1_H \otimes 1_A, \quad g \mapsto 1_H \otimes g, \quad x \mapsto k^{-1} a \otimes x, \quad y \mapsto k^{-1} a \otimes y.$$

Then (A, ρ, α) is an (H, id_H)-comodule Hom-algebra, and $(A, \triangleright, \rho, \alpha)$ is an (m, n)-Hom-Yetter–Drinfeld module over (H, id_H).

Moreover, we can check that the condition $(SY1)$ holds, i.e., the braiding τ is symmetric on (A, α).

Define the bracket product $[,] : A \otimes A \longrightarrow A$ by

[,]	1	g	x	y
1	0	0	0	0
g	0	0	$2ky$	$2kx$
x	0	-$2ky$	0	0
y	0	-$2kx$	0	0

Then by Theorem A, we can get $(A, [,], \alpha, \tau)$ is an (m,n)-Hom-Lie algebra.

Example 3.9. Let $H = K\{1, a\}$ a be Hopf group algebra with $a^2 = 1$ and $\Delta(a) = a \otimes a$ (see [18]). Then (H, id_H) is a monoidal Hom-Hopf algebra.

Let $A = K\{1, z\}$ be a K-linear space. Define the multiplication μ by

$$1z = z1 = lz, \quad z^2 = 0,$$

and the automorphism $\alpha : A \to A$ by

$$\alpha(1) = 1, \quad \alpha(z) = lz,$$

where $0 \neq \ell \in K$. Then (A, α) is a monoidal Hom-algebra.

Define module action $\triangleright : H \otimes A \to A$ by

$$1_H \triangleright 1_A = 1_A, \ 1_H \triangleright z = lz, \qquad a \triangleright 1_A = 1_A, \ a \triangleright z = -lz.$$

Then by a routine computation, we can get $(A, \triangleright, \alpha)$ is a (H, id_H)-module Hom-algebra.

Define comodule action $\psi : A \to H \otimes A$ by

$$\psi : A \to H \otimes A, \quad 1_A \mapsto 1_H \otimes 1_A, \quad z \mapsto \ell^{-1} a \otimes z.$$

Then we can get (A, ψ, α) is a (H, id_H)-comodule Hom-algebra, and $(A, \triangleright, \psi, \alpha)$ is an (m,n)-Hom-Yetter–Drinfeld module over (H, id_H).

Moreover, we can check that the condition (SY1) holds, i.e., the braiding τ is symmetric on (A, α).

Define the bracket product $[,] : A \otimes A \to A$ by

[,]	1	z
1	0	0
z	0	0

Then by Theorem A, we can get $(A, [,], \alpha, \tau)$ is an (m,n)-Hom-Lie algebra.

Remark. We note here that (A, α) in Example 3.9 gives rise to a trivial Lie bracket.

4. Hom-algebras which are sums of (H,β)-commutative Hom-subalgebras

In this section, we give Kegel's theorem for the (m,n)-Hom-Lie algebras, which generalizes the corresponding results in [2], [22]–[23].

The following result is obvious.

Lemma 4.1. *Let (H,β) be a monoidal Hom-Hopf algebra. (A,α) is an (H,β)-comodule Hom-algebra with Hom-subalgebras U and X such that $A = U+X$. Then we have*

$$\begin{aligned} & u_{-1}\otimes y_{-1}\otimes w'_{-1}\otimes w'_0 + u_{-1}\otimes y_{-1}\otimes z'_{-1}\otimes z'_0 \\ &\quad = \beta(u_{-11})\otimes\beta(y_{-11})\otimes u_{-12}y_{-12}\otimes\alpha^{-1}(w') \\ &\qquad + \beta(u_{-11})\otimes\beta(y_{-11})\otimes u_{-12}y_{-12}\otimes\alpha^{-1}(z'), \end{aligned}$$

for all $u\in U, y\in X$ and $u_0y_0 = w'+z' \in U+X$.

Lemma 4.2. *Let (H,β) be a monoidal Hom-Hopf algebra. (A,α) is an algebra in the Hom-Yetter–Drinfeld category $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$, with Hom-subalgebras U and X in $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$ such that $A = U+X$. Then we have*

$$\begin{aligned} & ((\beta^\ell(u_{-1})\triangleright\alpha^{-1}(x))\alpha^{-1}((\beta^\ell(\alpha(u_0)_{-1})\triangleright\alpha^{-1}(v))\alpha(\alpha(u_0)_0)))\alpha(y) \\ &\quad = ((\beta^\ell(u_{-1})\triangleright\alpha^{-2}(w))\alpha(u_0))\alpha(y) + (\beta^{\ell+1}(u_{-1})\triangleright\alpha^{-1}(z))\alpha(w')\varepsilon(y_{-1}) \\ &\qquad + (\beta^{\ell+1}(u_{-1})\triangleright\alpha^{-1}(z))\alpha(z')\varepsilon(y_{-1}), \end{aligned}$$

for all $u,v\in U$, $x,y\in X$, $xv = w+z\in U+X$ and $u_0y_0 = w'+z'\in U+X$.

Proof. We check the equality as follows:

$$\begin{aligned} & ((\beta^\ell(u_{-1})\triangleright\alpha^{-1}(x))\alpha^{-1}((\beta^\ell(\alpha(u_0)_{-1})\triangleright\alpha^{-1}(v))\alpha(\alpha(u_0)_0)))\alpha(y) \\ \overset{\text{(HCM1)}}{=} \; & ((\beta^\ell(u_{-1})\triangleright\alpha^{-1}(x))\alpha^{-1}((\beta^{\ell+1}(u_{0-1})\triangleright\alpha^{-1}(v))\alpha^2(u_{00})))\alpha(y) \\ \overset{\text{(HM1)}}{=} \; & ((\beta^\ell(u_{-1})\triangleright\alpha^{-1}(x))((\beta^\ell(u_{0-1})\triangleright\alpha^{-2}(v))\alpha(u_{00})))\alpha(y) \\ \overset{\text{(HCM2)}}{=} \; & ((\beta^{\ell+1}(u_{-11})\triangleright\alpha^{-1}(x))((\beta^\ell(u_{-12})\triangleright\alpha^{-2}(v))u_0))\alpha(y) \\ \overset{\text{(HA2)}}{=} \; & (((\beta^\ell(u_{-11})\triangleright\alpha^{-2}(x))(\beta^\ell(u_{-12})\triangleright\alpha^{-2}(v)))\alpha(u_0))\alpha(y) \\ \overset{\text{(HMA1)}}{=} \; & ((\beta^\ell(u_{-1})\triangleright\alpha^{-2}(xv))\alpha(u_0))\alpha(y) \\ = \; & ((\beta^\ell(u_{-1})\triangleright\alpha^{-2}(w))\alpha(u_0))\alpha(y) + ((\beta^\ell(u_{-1})\triangleright\alpha^{-2}(z))\alpha(u_0))\alpha(y) \\ & \text{(by } xv = w+z\text{)} \end{aligned}$$

$$\begin{aligned}
&\overset{(\mathrm{HCM2})}{=} ((\beta^\ell(u_{-1})\triangleright\alpha^{-2}(w))\alpha(u_0))\alpha(y) + ((\beta^\ell(u_{-1})\triangleright\alpha^{-2}(z))\alpha(u_0))\alpha^2(\varepsilon(y_{-1})y_0)\\
&\overset{(\mathrm{HA2})}{=} ((\beta^\ell(u_{-1})\triangleright\alpha^{-2}(w))\alpha(u_0))\alpha(y) + (\beta^{\ell+1}(u_{-1})\triangleright\alpha^{-1}(z))\alpha(u_0y_0)\varepsilon(y_{-1})\\
&= ((\beta^\ell(u_{-1})\triangleright\alpha^{-2}(w))\alpha(u_0))\alpha(y) + (\beta^{\ell+1}(u_{-1})\triangleright\alpha^{-1}(z))\alpha(w')\varepsilon(y_{-1})\\
&\quad \text{(by } u_0y_0 = w'+z'\text{)},
\end{aligned}$$

finishing the proof. □

Lemma 4.3. *Let (H,β) be a monoidal Hom-Hopf algebra with bijective antipode S, and (A,α) an algebra in the Hom-Yetter–Drinfeld category $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$, with Hom-subalgebras U and X in $\widetilde{\mathcal{H}}({}^H_H\mathcal{YD}(\mathcal{Z}))$ such that $A = U + X$. Assume that the braiding τ is symmetric on (A,α). Then we have*

$$\begin{aligned}
&(u\alpha^{-1}((\beta^\ell(x_{-1})\triangleright\alpha^{-1}(\beta^\ell(v_{-1})\triangleright\alpha^{-1}(y)))\alpha(x_0)))\alpha^2(v_0)\\
&\quad = \alpha(u)((\beta^{\ell-1}(z_{-1})\triangleright\alpha^{-1}(y))z_0) + \alpha(w')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(w))\varepsilon(u_{-1})\\
&\qquad + \alpha(z')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(w))\varepsilon(u_{-1}),
\end{aligned}$$

for all $u,v\in U$, $x,y\in X$, $xv = w+z\in U+X$ and $u_0y_0 = w'+z'\in U+X$.

Proof. We check the equality as follows:

$$\begin{aligned}
&(u\alpha^{-1}((\beta^\ell(x_{-1})\triangleright\alpha^{-1}(\beta^\ell(v_{-1})\triangleright\alpha^{-1}(y)))\alpha(x_0)))\alpha^2(v_0)\\
&\overset{(\mathrm{HM1})}{=} (u((\beta^{\ell-1}(x_{-1})\triangleright(\beta^{\ell-2}(v_{-1})\triangleright\alpha^{-3}(y)))x_0))\alpha^2(v_0)\\
&\overset{(\mathrm{HM2})}{=} (u((\beta^{\ell-2}(x_{-1}v_{-1})\triangleright\alpha^{-2}(y))x_0))\alpha^2(v_0)\\
&\overset{(\mathrm{HA2})}{=} \alpha(u)(((\beta^{\ell-2}(x_{-1}v_{-1})\triangleright\alpha^{-2}(y))x_0)\alpha(v_0))\\
&\overset{(\mathrm{HA2})}{=} \alpha(u)(\alpha(\beta^{\ell-2}(x_{-1}v_{-1})\triangleright\alpha^{-2}(y))(x_0v_0))\\
&\overset{(\mathrm{HCMA1})}{=} \alpha(u)(\alpha(\beta^{\ell-2}((xv)_{-1})\triangleright\alpha^{-2}(y))(xv)_0)\\
&= \alpha(u)(\alpha(\beta^{\ell-2}(w_{-1})\triangleright\alpha^{-2}(y))w_0) + \alpha(u)(\alpha(\beta^{\ell-2}(z_{-1})\triangleright\alpha^{-2}(y))z_0)\\
&\quad \text{(by } xv = w+z\text{)}\\
&\overset{(\mathrm{HM1})}{=} \alpha(u)((\beta^{\ell-1}(w_{-1})\triangleright\alpha^{-1}(y))w_0) + \alpha(u)(\alpha(\beta^{\ell-2}(z_{-1})\triangleright\alpha^{-2}(y))z_0)\\
&\overset{(\mathrm{HCM1})}{=} \alpha(u)((\beta^\ell(\alpha^{-1}(w)_{-1})\triangleright\alpha^{-1}(y))\alpha(\alpha^{-1}(w)_0)) + \alpha(u)(\alpha(\beta^{\ell-2}(z_{-1})\\
&\quad \triangleright\alpha^{-2}(y))z_0)\\
&\overset{(\mathrm{SY2})}{=} \alpha(u)(\alpha(y_0)(S^{-1}(\beta^\ell(y_{-1}))\triangleright\alpha^{-1}(\alpha^{-1}(w)))) + \alpha(u)(\alpha(\beta^{\ell-2}(z_{-1})\\
&\quad \triangleright\alpha^{-2}(y))z_0)
\end{aligned}$$

$$\begin{aligned}
&\overset{\text{(HCM2)}}{=} \alpha^2(u_0)(\alpha(y_0)(S^{-1}(\beta^\ell(y_{-1}))\triangleright\alpha^{-2}(w)))\varepsilon(u_{-1})+\alpha(u)(\alpha(\beta^{\ell-2}(z_{-1})\\
&\qquad \triangleright\alpha^{-2}(y))z_0)\\
&\overset{\text{(HA2)}}{=} \alpha(u_0y_0)\alpha(S^{-1}(\beta^\ell(y_{-1}))\triangleright\alpha^{-2}(w))\varepsilon(u_{-1})+\alpha(u)(\alpha(\beta^{\ell-2}(z_{-1})\\
&\qquad \triangleright\alpha^{-2}(y))z_0)\\
&\overset{\text{(HM1)}}{=} \alpha(u_0y_0)(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(w))\varepsilon(u_{-1})+\alpha(u)(\alpha(\beta^{\ell-2}(z_{-1})\\
&\qquad \triangleright\alpha^{-2}(y))z_0)\\
&= \alpha(u)((\beta^{\ell-1}(z_{-1})\triangleright\alpha^{-1}(y))z_0)+\alpha(w')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(w))\\
&\qquad \times\varepsilon(u_{-1})+\alpha(z')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(w))\varepsilon(u_{-1})\quad (\text{by } u_0y_0=w'+z'),
\end{aligned}$$

finishing the proof. □

Lemma 4.4. *Under the assumption of Lemma 4.3, we have*

$$\begin{aligned}
&((\beta^\ell(u_{-1})\triangleright\alpha^{-1}(x))\alpha(u_0))((\beta^\ell(v_{-1})\triangleright\alpha^{-1}(y))\alpha(v_0))\\
&\quad =(\beta^{\ell+1}(u_{-1})\triangleright\alpha^{-1}(w))\alpha(w')\varepsilon(y_{-1})+(\beta^{\ell+1}(u_{-1})\triangleright\alpha^{-1}(z))\alpha(w')\varepsilon(y_{-1})\\
&\qquad +\alpha(z')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(w))\varepsilon(u_{-1})\\
&\qquad +\alpha(z')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(z))\varepsilon(u_{-1})
\end{aligned}$$

for all $u,v\in U$, $x,y\in X$, $xv=w+z\in U+X$, and $u_0y_0=w'+z'\in U+X$.

Proof. We check the equality as follows:

$$\begin{aligned}
&((\beta^\ell(u_{-1})\triangleright\alpha^{-1}(x))\alpha(u_0))((\beta^\ell(v_{-1})\triangleright\alpha^{-1}(y))\alpha(v_0))\\
&\overset{\text{(SY2)}}{=} ((\beta^\ell(u_{-1})\triangleright\alpha^{-1}(x))\alpha(u_0))(\alpha(y_0)(S^{-1}(\beta^\ell(y_{-1}))\triangleright\alpha^{-1}(v)))\\
&\overset{\text{(HA2)}}{=} ((\beta^\ell(u_{-1})\triangleright\alpha^{-1}(x))(u_0y_0))(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright v)\\
&= ((\beta^\ell(u_{-1})\triangleright\alpha^{-1}(x))w')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright v)\\
&\qquad +((\beta^\ell(u_{-1})\triangleright\alpha^{-1}(x))z')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright v)\quad (\text{by } u_0y_0=w'+z')\\
&\overset{\text{(HA2,HM1)}}{=} (\beta^{\ell+1}(u_{-1})\triangleright x)(w'(\beta^\ell(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(v)))\\
&\qquad +((\beta^\ell(u_{-1})\triangleright\alpha^{-1}(x))z')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright v)\\
&\overset{\text{(CO)}}{=} (\beta^{\ell+1}(u_{-1})\triangleright x)((\beta^\ell(w'_{-1})\triangleright\alpha^{-1}(\beta^\ell(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(v)))\alpha(w'_0))\\
&\qquad +((\beta^\ell((\beta^\ell(u_{-1})\triangleright\alpha^{-1}(x))_{-1})\triangleright\alpha^{-1}(z'))\alpha((\beta^\ell(u_{-1})\triangleright\alpha^{-1}(x))_0))\\
&\qquad \times(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright v)
\end{aligned}$$

$$
\begin{aligned}
&\overset{\text{(HM2,SY2)}}{=}(\beta^{\ell+1}(u_{-1})\triangleright x)(((\beta^{\ell-1}(w'_{-1})\beta^{\ell-1}(S^{-1}(y_{-1})))\triangleright\alpha^{-1}(v))\alpha(w'_0))\\
&\qquad+(\alpha(z'_0)(S^{-1}(\beta^{\ell}(z'_{-1}))\triangleright\alpha^{-1}(\beta^{\ell}(u_{-1})\triangleright\alpha^{-1}(x))))\times(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright v)\\
&\overset{\text{(HM1,HM2)}}{=}(\beta^{\ell+1}(u_{-1})\triangleright x)(((\beta^{\ell-1}(w'_{-1})\beta^{\ell-1}(S^{-1}(y_{-1})))\triangleright\alpha^{-1}(v))\alpha(w'_0))\\
&\qquad+(\alpha(z'_0)((\beta^{\ell-1}(S^{-1}(z'_{-1}))\beta^{\ell-1}(u_{-1}))\triangleright\alpha^{-1}(x)))\times(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright v)\\
&=\quad(\beta^{\ell+2}(u_{-11})\triangleright x)(((\beta^{\ell-1}(u_{-12}y_{-12})\beta^{\ell}(S^{-1}(y_{-11})))\triangleright\alpha^{-1}(v))w')\\
&\qquad+(z'((\beta^{\ell-1}(S^{-1}(u_{-12}y_{-12}))\beta^{\ell}(u_{-11}))\triangleright\alpha^{-1}(x)))\\
&\qquad\times(\beta^{\ell+2}(S^{-1}(y_{-11}))\triangleright v)\quad\text{(by Lemma 4.1)}\\
&\overset{\text{(HA2)}}{=}\quad(\beta^{\ell+2}(u_{-11})\triangleright x)((\beta^{\ell+1}(u_{-12})\triangleright\alpha^{-1}(v))w')\varepsilon(y_{-1})\\
&\qquad+(z'(\beta^{\ell+1}(S^{-1}(y_{-12}))\triangleright\alpha^{-1}(x)))(\beta^{\ell+2}(S^{-1}(y_{-11}))\triangleright v)\varepsilon(u_{-1})\\
&\overset{\text{(HA2,HM1)}}{=}((\beta^{\ell+1}(u_{-11})\triangleright\alpha^{-1}(x))(\beta^{\ell+1}(u_{-12})\triangleright\alpha^{-1}(v)))\alpha(w')\varepsilon(y_{-1})\\
&\qquad+\alpha(z')((\beta^{\ell+1}(S^{-1}(y_{-12}))\triangleright\alpha^{-1}(x))(\beta^{\ell+1}(S^{-1}(y_{-11}))\triangleright\alpha^{-1}(v)))\times\varepsilon(u_{-1})\\
&\overset{\text{(HMA1)}}{=}\quad(\beta^{\ell+1}(u_{-1})\triangleright\alpha^{-1}(xv))\alpha(w')\varepsilon(y_{-1})\\
&\qquad+\alpha(z')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(xv))\varepsilon(u_{-1})\\
&=\quad(\beta^{\ell+1}(u_{-1})\triangleright\alpha^{-1}(w))\alpha(w')\varepsilon(y_{-1})+(\beta^{\ell+1}(u_{-1})\triangleright\alpha^{-1}(z))\alpha(w')\\
&\qquad\times\varepsilon(y_{-1})+\alpha(z')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(w))\varepsilon(u_{-1})\\
&\qquad+\alpha(z')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(z))\varepsilon(u_{-1})\quad(\text{by } xv=w+z),
\end{aligned}
$$

finishing the proof. □

Lemma 4.5. *Under the assumption of Lemma 4.3, we have*

$$
\begin{aligned}
&(\beta^{\ell+1}(u_{-1})\triangleright\alpha^{-1}(w))\alpha(w')\varepsilon(y_{-1})-(\beta^{\ell+1}(u_{-1})\triangleright\alpha^{-1}(z))\alpha(z')\varepsilon(y_{-1})\\
&\quad=\alpha(w')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(w))\varepsilon(u_{-1})\\
&\qquad-\alpha(z')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(z))\varepsilon(u_{-1}),
\end{aligned}
$$

for all $u,v\in U$, $x,y\in X$, $xv=w+z\in U+X$, and $u_0y_0=w'+z'\in U+X$.

PROOF. Similar to Lemma 4.4. □

Now we prove Theorem B.

PROOF OF THEOREM B. It is sufficient to prove that $[u,x][v,y]=0$, for all $u,v\in U$ and $x,y\in X$. We denote xv by $w+z$, where $w\in U$ and $z\in X$. In fact, we have

$$
\begin{aligned}
&[u,x][v,y]\\
&= (ux-(\beta^{\ell}(u_{-1})\triangleright\alpha^{-1}(x))\alpha(u_0))(vy-(\beta^{\ell}(v_{-1})\triangleright\alpha^{-1}(y))\alpha(v_0))\\
&= (ux)(vy)-((\beta^{\ell}(u_{-1})\triangleright\alpha^{-1}(x))\alpha(u_0))(vy)-(ux)((\beta^{\ell}(v_{-1})\triangleright\alpha^{-1}(y))\alpha(v_0))\\
&\quad+((\beta^{\ell}(u_{-1})\triangleright\alpha^{-1}(x))\alpha(u_0))((\beta^{\ell}(v_{-1})\triangleright\alpha^{-1}(y))\alpha(v_0))\\
&\overset{\text{(HA2)}}{=} (u\alpha^{-1}(xv))\alpha(y)-((\beta^{\ell}(u_{-1})\triangleright\alpha^{-1}(x))\alpha^{-1}(\alpha(u_0)v))\alpha(y)\\
&\quad-(u\alpha^{-1}(x(\beta^{\ell}(v_{-1})\triangleright\alpha^{-1}(y))))\alpha^{2}(v_0)\\
&\quad+((\beta^{\ell}(u_{-1})\triangleright\alpha^{-1}(x))\alpha(u_0))((\beta^{\ell}(v_{-1})\triangleright\alpha^{-1}(y))\alpha(v_0))\\
&= (u\alpha^{-1}(w))\alpha(y)+(u\alpha^{-1}(z))\alpha(y)-((\beta^{\ell}(u_{-1})\triangleright\alpha^{-1}(x))\alpha^{-1}(\alpha(u_0)v))\alpha(y)\\
&\quad-(u\alpha^{-1}(x(\beta^{\ell}(v_{-1})\triangleright\alpha^{-1}(y))))\alpha^{2}(v_0)\\
&\quad+((\beta^{\ell}(u_{-1})\triangleright\alpha^{-1}(x))\alpha(u_0))((\beta^{\ell}(v_{-1})\triangleright\alpha^{-1}(y))\alpha(v_0))\quad(\text{by } xv=w+z)\\
&\overset{\text{(HA2)}}{=} (u\alpha^{-1}(w))\alpha(y)+\alpha(u)(\alpha^{-1}(z)y)-((\beta^{\ell}(u_{-1})\triangleright\alpha^{-1}(x))\alpha^{-1}(\alpha(u_0)v))\alpha(y)\\
&\quad-(u\alpha^{-1}(x(\beta^{\ell}(v_{-1})\triangleright\alpha^{-1}(y))))\alpha^{2}(v_0)\\
&\quad+((\beta^{\ell}(u_{-1})\triangleright\alpha^{-1}(x))\alpha(u_0))((\beta^{\ell}(v_{-1})\triangleright\alpha^{-1}(y))\alpha(v_0))\\
&\overset{\text{(CO)}}{=} ((\beta^{\ell}(u_{-1})\triangleright\alpha^{-2}(w))\alpha(u_0))\alpha(y)+\alpha(u)((\beta^{\ell-1}(z_{-1})\triangleright\alpha^{-1}(y))z_0)\\
&\quad-((\beta^{\ell}(u_{-1})\triangleright\alpha^{-1}(x))\alpha^{-1}((\beta^{\ell}(\alpha(u_0)_{-1})\triangleright\alpha^{-1}(v))\alpha(\alpha(u_0)_0)))\alpha(y)\\
&\quad-(u\alpha^{-1}((\beta^{\ell}(x_{-1})\triangleright\alpha^{-1}(\beta^{\ell}(v_{-1})\triangleright\alpha^{-1}(y)))\alpha(x_0)))\alpha^{2}(v_0)\\
&\quad+((\beta^{\ell}(u_{-1})\triangleright\alpha^{-1}(x))\alpha(u_0))((\beta^{\ell}(v_{-1})\triangleright\alpha^{-1}(y))\alpha(v_0))\\
&= ((\beta^{\ell}(u_{-1})\triangleright\alpha^{-2}(w))\alpha(u_0))\alpha(y)+\alpha(u)((\beta^{\ell-1}(z_{-1})\triangleright\alpha^{-1}(y))z_0)\\
&\quad-(((\beta^{\ell}(u_{-1})\triangleright\alpha^{-2}(w))\alpha(u_0))\alpha(y)+(\beta^{\ell+1}(u_{-1})\triangleright\alpha^{-1}(z))\alpha(w')\varepsilon(y_{-1})\\
&\quad+(\beta^{\ell+1}(u_{-1})\triangleright\alpha^{-1}(z))\alpha(z')\varepsilon(y_{-1}))-(\alpha(u)((\beta^{\ell-1}(z_{-1})\triangleright\alpha^{-1}(y))z_0)\\
&\quad+\alpha(w')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(w))\varepsilon(u_{-1})\\
&\quad+\alpha(z')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(w))\varepsilon(u_{-1}))\\
&\quad+((\beta^{\ell}(u_{-1})\triangleright\alpha^{-1}(x))\alpha(u_0))((\beta^{\ell}(v_{-1})\triangleright\alpha^{-1}(y))\alpha(v_0))\\
&\quad\text{(by Lemmas 4.2 and 4.3)}\\
&= -(\beta^{\ell+1}(u_{-1})\triangleright\alpha^{-1}(z))\alpha(w')\varepsilon(y_{-1})-(\beta^{\ell+1}(u_{-1})\triangleright\alpha^{-1}(z))\alpha(z')\varepsilon(y_{-1})\\
&\quad-\alpha(w')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(w))\varepsilon(u_{-1})\\
&\quad-\alpha(z')(\beta^{\ell+1}(S^{-1}(y_{-1}))\triangleright\alpha^{-1}(w))\varepsilon(u_{-1}))\\
&\quad+((\beta^{\ell}(u_{-1})\triangleright\alpha^{-1}(x))\alpha(u_0))((\beta^{\ell}(v_{-1})\triangleright\alpha^{-1}(y))\alpha(v_0))
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -(\beta^{\ell+1}(u_{-1}) \triangleright \alpha^{-1}(z))\alpha(w')\varepsilon(y_{-1}) - (\beta^{\ell+1}(u_{-1}) \triangleright \alpha^{-1}(z))\alpha(z')\varepsilon(y_{-1}) \\
& - \alpha(w')(\beta^{\ell+1}(S^{-1}(y_{-1})) \triangleright \alpha^{-1}(w))\varepsilon(u_{-1}) \\
& - \alpha(z')(\beta^{\ell+1}(S^{-1}(y_{-1})) \triangleright \alpha^{-1}(w))\varepsilon(u_{-1})) \\
& + (\beta^{\ell+1}(u_{-1}) \triangleright \alpha^{-1}(w))\alpha(w')\varepsilon(y_{-1}) + (\beta^{\ell+1}(u_{-1}) \triangleright \alpha^{-1}(z))\alpha(w')\varepsilon(y_{-1}) \\
& + \alpha(z')(\beta^{\ell+1}(S^{-1}(y_{-1})) \triangleright \alpha^{-1}(w))\varepsilon(u_{-1}) \\
& + \alpha(z')(\beta^{\ell+1}(S^{-1}(y_{-1})) \triangleright \alpha^{-1}(z))\varepsilon(u_{-1}) \quad \text{(by Lemma 4.4)} \\
= \quad & -(\beta^{\ell+1}(u_{-1}) \triangleright \alpha^{-1}(z))\alpha(z')\varepsilon(y_{-1}) - \alpha(w')(\beta^{\ell+1}(S^{-1}(y_{-1})) \triangleright \alpha^{-1}(w)) \\
& \times \varepsilon(u_{-1}) + (\beta^{\ell+1}(u_{-1}) \triangleright \alpha^{-1}(w))\alpha(w')\varepsilon(y_{-1}) \\
& + \alpha(z')(\beta^{\ell+1}(S^{-1}(y_{-1})) \triangleright \alpha^{-1}(z))\varepsilon(u_{-1}) = 0 \quad \text{(by Lemma 4.5)},
\end{aligned}
$$

finishing the proof. □

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