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# Composite rational functions and arithmetic progressions

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**Abstract.** In this paper, we deal with composite rational functions having zeros and poles forming consecutive elements of an arithmetic progression. We also correct a result published in [12] related to composite rational functions having a fixed number of zeros and poles.

## 1. Introduction

We consider a problem related to decompositions of polynomials and rational functions. In this subject, a classical result obtained by Ritt [13] says that if there is a polynomial $f \in \mathbb{C}[X]$ satisfying certain tameness properties and

$$f = g_1 \circ g_2 \circ \cdots \circ g_r = h_1 \circ h_2 \circ \cdots \circ h_s,$$

then $r = s$ and $\{\deg g_1, \ldots, \deg g_r\} = \{\deg h_1, \ldots, \deg h_r\}$. Ritt's fundamental result has been investigated, extended and applied in various wide-ranging contexts (see, e.g., [4], [6]–[7], [9]–[11], [14]–[15]). The above mentioned result is not valid for rational functions. Gutierrez and Sevilla [9] provided the following example:

$$f = \frac{x^3(x+6)^3(x^2-6x+36)^3}{(x-3)^3(x^2+3x+9)^3},$$
$$f = g_1 \circ g_2 \circ g_3 = x^3 \circ \frac{x(x-12)}{x-3} \circ \frac{x(x+6)}{x-3},$$

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$$f = h_1 \circ h_2 = \frac{x^3(x+24)}{x-3} \circ \frac{x(x^2-6x+36)}{x^2+3x+9}.$$

To determine decompositions of a given rational function, there were developed algorithms (see, e.g., [1]–[3]). In [2], Ayad and Fleischmann implemented a MAGMA [5] code to find decompositions, they provided the following example:

$$f = \frac{x^4-8x}{x^3+1},$$

and they obtained that $f(x) = g(h(x))$, where

$$g = \frac{x^2+4x}{x+1} \quad \text{and} \quad h = \frac{x^2-2x}{x+1}.$$

Fuchs and Pethő [8] proved the following theorem.

**Theorem A.** *Let $k$ be an algebraically closed field of characteristic zero. Let $n$ be a positive integer. Then there exists a positive integer $J$ and, for every $i \in \{1,\dots,J\}$, an affine algebraic variety $V_i$ defined over $\mathbb{Q}$ and with $V_i \subset \mathbb{A}^{n+t_i}$, for some $2 \le t_i \le n$, such that:*

*(i) If $f,g,h \in k(x)$ with $f(x) = g(h(x))$ and with $\deg g, \deg h \ge 2, g$ not of the shape $(\lambda(x))^m, m \in \mathbb{N}, \lambda \in PGL_2(k)$, and $f$ has at most $n$ zeros and poles altogether, then there exists, for some $i \in \{1,\dots,J\}$, a point $P = (\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_{t_i}) \in V_i(k)$, a vector $(k_1,\dots,k_{t_i}) \in \mathbb{Z}^{t_i}$ with $k_1+k_2+\dots+k_{t_i} = 0$ depending only [1] on $V_i$ , a partition of $\{1,\dots,n\}$ in $t_i+1$ disjoint sets $S_\infty, S_{\beta_1},\dots,S_{\beta_{t_i}}$ with $S_\infty = \emptyset$ if $k_1+k_2+\dots+k_{t_i} = 0$, and a vector $(l_1,\dots,l_n) \in \{0,1,\dots,n-1\}^n$, also both depending only on $V_i$, such that*

$$f(x) = \prod_{j=1}^{t_i} (\omega_j/\omega_\infty)^{k_j}, \quad g(x) = \prod_{j=1}^{t_i} (x-\beta_j)^{k_j},$$

*and*

$$h(x) = \begin{cases} \beta_j + \frac{\omega_j}{\omega_\infty} \quad (j = 1,\dots,t_i), & \text{if } k_1+k_2+\ldots+k_{t_i} \neq 0, \\ \frac{\beta_{j_1}\omega_{j_2} - \beta_{j_2}\omega_{j_1}}{\omega_{j_2}-\omega_{j_1}} \quad (1 \le j_1 < j_2 \le t_i), & \text{otherwise,} \end{cases}$$

*where*

$$\omega_j = \prod_{m \in S_{\beta_j}} (x-\alpha_m)^{l_m}, \quad j = 1,\dots,t_i,$$

[1] in [8], it is written as "or not depending", this typo is corrected here.

*and*

$$\omega_\infty = \prod_{m \in S_\infty} (x-\alpha_m)^{l_m}.$$

*Moreover, we have* $\deg h \le (n-1)/\max\{t_i-2,1\} \le n-1$.

(ii) *Conversely, for given data* $P \in V_i(k), (k_1,\dots,k_{t_i}),\ S_\infty, S_{\beta_1},\dots,S_{\beta_{t_i}}$, $(l_1,\dots,l_n)$ *as described in* (i), *one defines by the same equations rational functions* $f,g,h$ *with* $f$ *having at most* $n$ *zeros and poles altogether for which* $f(x)=g(h(x))$ *holds.*

(iii) *The integer* $J$ *and equations defining the varieties* $V_i$ *are effectively computable only in terms of* $n$.

Pethő and Tengely [12] provided some computational experiments that they obtained by using a MAGMA [5] implementation of the algorithm of Fuchs and Pethő [8].

If the zeros and poles of a composite rational function form an arithmetic progression, then we have the following result.

**Theorem 1.** *Let* $f,g,h$ *be rational functions as in Theorem A. Assume that the zeros and poles of* $f$ *form an arithmetic progression, that is*

$$\alpha_i = \alpha_0 + T_i d,$$

*for some* $\alpha_0, d \in k$ *and* $T_i \in \{0,1,\dots,n-1\}$. *If* $k_1+k_2+\dots+k_t \neq 0$, *then either the difference* $d$ *satisfies an equation of the form*

$$d^N = M$$

*for some* $N \in \mathbb{Z}, M \in \mathbb{Q}$ *or* $(l_1,\dots,l_n) \in \{0,1,\dots,n-1\}^n$ *satisfies a system of linear equations*

$$\sum_{r \in S_{\beta_i}} l_r = \sum_{s \in S_{\beta_j}} l_s, \qquad i,j \in \{1,\dots,t\}, i \neq j.$$

*If* $k_1+k_2+\dots+k_t = 0$ *and* $1 \le j_1 < j_2 < j_3 \le t$, *then*

$$d^{\sum_{m_1 \in S_{\beta_{j_1}}} l_{m_1}}, \quad d^{\sum_{m_2 \in S_{\beta_{j_2}}} l_{m_2}}, \quad d^{\sum_{m_3 \in S_{\beta_{j_3}}} l_{m_3}}$$

*satisfy a system of linear equations, and* $\beta_{j_1}, \beta_{j_2}, \beta_{j_3}$ *also satisfy a system of linear equations.*

We will apply the above theorem to determine composite rational functions having 4 zeros and poles. We define equivalence of rational functions. Two rational functions $f_1(x) = \prod_{u=1}^{n}(x-\alpha_u^{(1)})^{f_u^{(1)}}$ and $f_2(x) = \prod_{u=1}^{n}(x-\alpha_u^{(2)})^{f_u^{(2)}}$ are equivalent if there exist $a_{u,v} \in \mathbb{Q}, u \in \{1,2,\dots,n\}, v \in \{1,2,\dots,n+1\}$ such that

$$\alpha_u^{(1)} = a_{u,1}\alpha_1^{(2)} + a_{u,2}\alpha_2^{(2)} + \cdots + a_{u,n}\alpha_n^{(2)} + a_{u,n+1},$$

for all $u \in \{1,2,\dots,n\}$. We prove the following statement.

**Proposition 1.** *Let $k$ be an algebraically closed field of characteristic zero. If $f,g,h \in k(x)$ with $f(x) = g(h(x))$ and with $\deg g, \deg h \geq 2, g$ not of the shape $(\lambda(x))^m, m \in \mathbb{N}, \lambda \in PGL_2(k)$, and $f$ has 4 zeros and poles altogether forming an arithmetic progression, then $f$ is equivalent to the following rational function:*

$$(x-\alpha_0)^{k_1}(x-\alpha_0-d)^{k_2}(x-\alpha_0-2d)^{k_2}(x-\alpha_0-3d)^{k_1},$$

*for some $\alpha_0, d \in k$ and $k_1, k_2 \in \mathbb{Z}, k_1+k_2 \neq 0$.*

In this paper, we correct results obtained in [12], where the computations related to the case $k_1+k_2+\cdots+k_t \neq 0, S_\infty = \emptyset$ are missing. The following theorem is the corrected version of Theorem 1 from [12], where part (c) was missing.

**Theorem 2.** *Let $k$ be an algebraically closed field of characteristic zero. If $f,g,h \in k(x)$ with $f(x) = g(h(x))$ and with $\deg g, \deg h \geq 2, g$ not of the shape $(\lambda(x))^m, m \in \mathbb{N}, \lambda \in PGL_2(k)$, and $f$ has 3 zeros and poles altogether, then $f$ is equivalent to one of the following rational functions:*

(a) $\frac{(x-\alpha_1)^{k_1}(x+1/4-\alpha_1)^{2k_2}}{(x-1/4-\alpha_1)^{2k_1+2k_2}}$ *for some $\alpha_1 \in k$ and $k_1, k_2 \in \mathbb{Z}, k_1+k_2 \neq 0$,*

(b) $\frac{(x-\alpha_1)^{2k_1}(x+\alpha_1-2\alpha_2)^{2k_2}}{(x-\alpha_2)^{2k_1+2k_2}}$ *for some $\alpha_1, \alpha_2 \in k$ and $k_1, k_2 \in \mathbb{Z}, k_1+k_2 \neq 0$,*

(c) $\left(x-\frac{\alpha_1+\alpha_2}{2}\right)^{2k_1}(x-\alpha_1)^{k_2}(x-\alpha_2)^{k_2}$ *for some $\alpha_1, \alpha_2 \in k$ and $k_1, k_2 \in \mathbb{Z}, k_1+k_2 \neq 0$.*

*Remark.* The MAGMA procedure `CompRatFunc.m` can be downloaded from `http://shrek.unideb.hu/~tengely/CompRatFunc.m`. All systems in cases of $n \in \{3,4,5\}$ can be downloaded from `http://shrek.unideb.hu/~tengely/CFunc345.tar.gz`.

*Remark.* It is interesting to note that in the above formulas the zeros and poles form an arithmetic progression

$$
\begin{array}{lll}
\text{(a):} & \alpha_1 - \dfrac{1}{4}, \alpha_1, \alpha_1 + \dfrac{1}{4} & \text{difference: } \dfrac{1}{4}, \\
\text{(b):} & \alpha_1, \alpha_2, -\alpha_1 + 2\alpha_2 & \text{difference: } \alpha_2 - \alpha_1, \\
\text{(c):} & \alpha_1, \dfrac{\alpha_1 + \alpha_2}{2}, \alpha_2 & \text{difference: } \dfrac{\alpha_2 - \alpha_1}{2}.
\end{array}
$$

## 2. Auxiliary results

We repeat some parts of the proof of Theorem A from [8] that will be used here later on. Without loss of generality, we may assume that $f$ and $g$ are monic. Let

$$f(x) = \prod_{i=1}^{n} (x - \alpha_i)^{f_i},$$

with pairwise distinct $\alpha_i \in k$ and $f_i \in \mathbb{Z}$ for $i = 1, \ldots, n$. Similarly, let

$$g(x) = \prod_{j=1}^{t} (x - \beta_j)^{k_j},$$

with pairwise distinct $\beta_j \in k$ and $k_j \in \mathbb{Z}$ for $j = 1, \ldots, t$ and $t \in \mathbb{N}$. Hence we have

$$\prod_{i=1}^{n} (x - \alpha_i)^{f_i} = f(x) = g(h(x)) = \prod_{j=1}^{t} (h(x) - \beta_j)^{k_j}.$$

We shall write $h(x) = p(x)/q(x)$, with $p, q \in k[x], p, q$ coprime. Fuchs and Pethő [8] showed that if $k_1 + k_2 + \cdots + k_t \neq 0$, then there is a subset $S_\infty$ of the set $\{1, \ldots, n\}$ for which

$$q(x) = \prod_{m \in S_\infty} (x - \alpha_m)^{l_m},$$

and there is a partition of the set $\{1, \ldots, n\} \setminus S_\infty$ in $t$ disjoint non-empty subsets $S_{\beta_1}, \ldots, S_{\beta_t}$ such that

$$h(x) = \beta_j + \frac{1}{q(x)} \prod_{m \in S_{\beta_j}} (x - \alpha_m)^{l_m}, \tag{1}$$

where $l_m \in \mathbb{N}$ satisfies $l_m k_j = f_m$ for $m \in S_{\beta_j}$, and this holds true for every $j = 1, \dots, t$. We get at least two different representations of $h$, since we assumed that $g$ is not of the special shape $(\lambda(x))^m$. Therefore, we get at least one equation of the form

$$\beta_i + \frac{1}{q(x)} \prod_{r \in S_{\beta_i}} (x-\alpha_r)^{l_r} = \beta_j + \frac{1}{q(x)} \prod_{s \in S_{\beta_j}} (x-\alpha_s)^{l_s}. \tag{2}$$

If $k_1 + k_2 + \dots + k_t = 0$, then we have

$$(p(x) - \beta_j q(x))^{k_j} = \prod_{m \in S_{\beta_j}} (x-\alpha_m)^{f_m}.$$

Now we have that $t \geq 3$, otherwise $g$ is in the special form we excluded. Siegel's identity provides the equations in this case. That is if $1 \leq j_1 < j_2 < j_3 \leq t$, then we have

$$v_{j_1,j_2,j_3} + v_{j_3,j_1,j_2} + v_{j_2,j_3,j_1} = 0, \tag{3}$$

where

$$v_{j_1,j_2,j_3} = (\beta_{j_1} - \beta_{j_2}) \prod_{m \in S_{\beta_{j_3}}} (x-\alpha_m)^{l_m}.$$

## 3. Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. If $k_1+k_2+\dots+k_t \neq 0$ and there exist $r_1 \in S_{\beta_i}, s_1 \in S_{\beta_j}$ for some $i \neq j$ such that $l_{r_1} \neq 0$ and $l_{s_1} \neq 0$, then it follows from (2) that

$$\beta_i - \beta_j = \frac{\prod_{s \in S_{\beta_j}} (\alpha_{r_1} - \alpha_s)^{l_s}}{\prod_{m \in S_\infty} (\alpha_{r_1} - \alpha_m)^{l_m}}, \tag{4}$$

$$\beta_i - \beta_j = -\frac{\prod_{r \in S_{\beta_i}} (\alpha_{s_1} - \alpha_r)^{l_r}}{\prod_{m \in S_\infty} (\alpha_{s_1} - \alpha_m)^{l_m}}, \tag{5}$$

for any appropriate $\alpha_{r_1} \in S_{\beta_i}$ and $\alpha_{s_1} \in S_{\beta_j}$. Hence we obtain that

$$C_1(i,j,r_1,s_1) = d^{\sum_{r \in S_{\beta_i}} l_r - \sum_{s \in S_{\beta_j}} l_s},$$

where $C_1(i,j,r_1,s_1) \in \mathbb{Q}$. If there exist $S_{\beta_i}$ and $S_{\beta_j}$ for which $\sum_{r \in S_{\beta_i}} l_r - \sum_{s \in S_{\beta_j}} l_s \neq 0$, then the possible values of $d$ satisfy equations of the form $x^N = M$. Otherwise we get that

$$\sum_{r \in S_{\beta_i}} l_r = \sum_{s \in S_{\beta_j}} l_s, \qquad i,j \in \{1,\dots,t\}, i \neq j.$$

Let us consider the special case when $l_r = 0$ for all $r \in S_{\beta_i}$. If $l_s = 0$ for all $s \in S_{\beta_j}$, then we get that

$$h(x) = \beta_i + \frac{1}{q(x)} = \beta_j + \frac{1}{q(x)}.$$

Hence $\beta_i = \beta_j$ for some $i \neq j$, a contradiction. Thus we may assume that there exists $s_1 \in S_{\beta_j}$ for which $l_{s_1} \neq 0$. In a similar way as in the above case, it follows that

$$\beta_i - \beta_j = \frac{\prod_{s \in S_{\beta_j}} (\alpha_{r_1} - \alpha_s)^{l_s}}{\prod_{m \in S_\infty} (\alpha_{r_1} - \alpha_m)^{l_m}} - \frac{1}{\prod_{m \in S_\infty} (\alpha_{r_1} - \alpha_m)^{l_m}}, \tag{6}$$

$$\beta_i - \beta_j = -\frac{1}{\prod_{m \in S_\infty} (\alpha_{s_1} - \alpha_m)^{l_m}}. \tag{7}$$

Therefore

$$d^{\sum_{s \in S_{\beta_j}} l_s} = C_2(i, j, r_1, s_1),$$

where $C_2(i, j, r_1, s_1) \in \mathbb{Q}$. Since $s_1 > 0$, we have that $\sum_{s \in S_{\beta_j}} l_s \neq 0$, that is $d$ satisfies an appropriate polynomial equation.

If $k_1 + k_2 + \cdots + k_t = 0$, then there are at least 3 partitions, and for any appropriate $r_1 \in S_{\beta_{j_1}}, r_2 \in S_{\beta_{j_2}}, r_3 \in S_{\beta_{j_3}}$ (that is $l_{r_i} \neq 0, i = 1, 2, 3$) equation (3) implies that

$$\begin{aligned}
(\beta_{j_3} - \beta_{j_1}) \prod_{m_2 \in S_{\beta_{j_2}}} (\alpha_{r_3} - \alpha_{m_2})^{l_{m_2}} + (\beta_{j_2} - \beta_{j_3}) \prod_{m_1 \in S_{\beta_{j_1}}} (\alpha_{r_3} - \alpha_{m_1})^{l_{m_1}} &= 0 \\
(\beta_{j_1} - \beta_{j_2}) \prod_{m_3 \in S_{\beta_{j_3}}} (\alpha_{r_2} - \alpha_{m_3})^{l_{m_3}} + (\beta_{j_2} - \beta_{j_3}) \prod_{m_1 \in S_{\beta_{j_1}}} (\alpha_{r_2} - \alpha_{m_1})^{l_{m_1}} &= 0 \\
(\beta_{j_1} - \beta_{j_2}) \prod_{m_3 \in S_{\beta_{j_3}}} (\alpha_{r_1} - \alpha_{m_3})^{l_{m_3}} + (\beta_{j_3} - \beta_{j_1}) \prod_{m_2 \in S_{\beta_{j_2}}} (\alpha_{r_1} - \alpha_{m_2})^{l_{m_2}} &= 0,
\end{aligned}$$

which is a system of linear equations in $d_1, d_2, d_3$, where $d_i = d^{\sum_{m_i \in S_{\beta_{j_i}}} l_{m_i}}, i \in \{1, 2, 3\}$ and the statement follows. In a very similar way, we obtain a system of equations if $l_r = 0$ for all $r \in S_{\beta_{j_3}}$, the last two equations are as before, while on the left-hand side of the first one there is an additional term $\beta_{j_1} - \beta_{j_2}$. $\square$

Proof of Theorem 2. In [12], all cases are given with $k_1 + k_2 + \cdots + k_t = 0$ and also with $k_1 + k_2 + \cdots + k_t \neq 0, S_\infty \neq \emptyset$. Therefore, it remains to deal with those cases with $k_1 + k_2 + \cdots + k_t \neq 0, S_\infty = \emptyset$. First let $t = 2$. There are 18 systems of equations. Among these systems there are two types. The first one has only a single equation, e.g., when $S_{\beta_1} = \{1,2\}, S_{\beta_2} = \{3\}, (l_1, l_2, l_3) = (1,0,1)$, this equation is as follows:

$$\alpha_1 - \alpha_3 - \beta_1 + \beta_2 = 0.$$

Hence

$$h(x) = \beta_1 + (x - \alpha_1) = \beta_2 + (x - \alpha_3)$$

is a linear function. A system from the second type is given by $S_{\beta_1} = \{1,2\}, S_{\beta_2} = \{3\}, (l_1, l_2, l_3) = (1,1,2)$ and the equations as follows:

$$\alpha_1 + \alpha_2 - 2\alpha_3 = 0, \qquad (\alpha_2 - \alpha_3)^2 - \beta_1 + \beta_2 = 0.$$

That is we obtain that

$$h(x) = \beta_2 + \left(x - \frac{\alpha_1 + \alpha_2}{2}\right)^2, \quad g(x) = \left(x - \beta_2 - \left(\frac{\alpha_2 - \alpha_1}{2}\right)^2\right)^{k_1} (x - \beta_2)^{k_2},$$

$$f(x) = \left(x - \frac{\alpha_1 + \alpha_2}{2}\right)^{2k_1} (x - \alpha_1)^{k_2} (x - \alpha_2)^{k_2}.$$

It is a decomposition of type (c) in the theorem. Let $t = 3$. There are 6 systems of equations, all of the same type, e.g., $S_{\beta_1} = \{1\}, S_{\beta_2} = \{2\}, S_{\beta_3} = \{3\}, (l_1, l_2, l_3) = (1,1,1)$, and

$$\alpha_1 - \alpha_3 - \beta_1 + \beta_3 = 0, \qquad \alpha_2 - \alpha_3 - \beta_2 + \beta_3 = 0.$$

Hence the degree of $h$ is 1, which yields a trivial decomposition. □

## 4. Proof of Proposition 1

Proof of Proposition 1. In this section, we apply Theorem 1 to determine composite rational functions having zeros and poles as consecutive elements of certain arithmetic progressions. We need to handle the following cases:

$$\begin{aligned}
\text{(I)}: &\quad n = 4 \text{ and } t \in \{2,3,4\}, k_1 + k_2 + \cdots + k_t \neq 0, S_\infty = \emptyset,\\
\text{(II)}: &\quad n = 4 \text{ and } t \in \{2,3\}, k_1 + k_2 + \cdots + k_t \neq 0, S_\infty \neq \emptyset,\\
\text{(III)}: &\quad n = 4 \text{ and } t \in \{3,4\}, k_1 + k_2 + \cdots + k_t = 0, S_\infty = \emptyset.
\end{aligned}$$

In the proof, we use the notation of Theorem 1, that is we write

$$\alpha_i = \alpha_0 + T_i d,$$

where $\alpha_0, d \in k$ and $\{T_1, T_2, T_3, T_4\} = \{0, 1, 2, 3\}$. To make the presentation shorter, we also make use of the code `CompRatFunc.m`.

$\underline{\text{(I)} : t = 2, \{|S_{\beta_1}|, |S_{\beta_2}|\} = \{1, 3\}.}$ We may assume that $S_{\beta_1} = \{1\}, S_{\beta_2} = \{2, 3, 4\}$. We obtain that

$$h(x) = \beta_1 + (x - \alpha_1)^{l_1}, \qquad h(x) = \beta_2 + (x - \alpha_2)^{l_2}(x - \alpha_3)^{l_3}(x - \alpha_4)^{l_4}.$$

Substituting $x = \alpha_2, \alpha_3, \alpha_4$, yields (assuming $l_2 l_3 l_4 \neq 0$)

$$(\alpha_2 - \alpha_1)^{l_1} = (\alpha_3 - \alpha_1)^{l_1} = (\alpha_4 - \alpha_1)^{l_1}.$$

Since the zeros and poles form an arithmetic progression, one gets that either $d = 0$ or $l_1 = 0$. In the former case, the zeros and poles are not distinct, which is a contradiction. In the latter case, the degree of $h$ is less than 2, a contradiction again. If two out of $l_2, l_3, l_4$ are equal to zero, then it follows that $l_1 = 1$, hence the degree of $h$ is 1, a contradiction. If exactly one out of $l_2, l_3, l_4$ is zero, then $l_1 = 2$, and the corresponding $f$ has only 3 zeros and poles. As an example, we consider the case $l_4 = 0$. We obtain that

$$\alpha_1 = \frac{\alpha_2 + \alpha_3}{2} \quad \text{and} \quad \beta_2 = \beta_1 + \left(\frac{\alpha_2 - \alpha_3}{2}\right)^2.$$

It follows that $f(x) = \left(x - \frac{\alpha_2 + \alpha_3}{2}\right)^2 f_1(x)$, where $\deg f_1 = 2$.

$\underline{\text{(I)} : t = 2, \{|S_{\beta_1}|, |S_{\beta_2}|\} = \{2\}.}$ Here we may assume that $S_{\beta_1} = \{1, 2\}, S_{\beta_2} = \{3, 4\}$. We get that

$$h(x) = \beta_1 + (x - \alpha_1)^{l_1}(x - \alpha_2)^{l_2}, \qquad h(x) = \beta_2 + (x - \alpha_3)^{l_3}(x - \alpha_4)^{l_4}.$$

It follows that (assuming that $0 \notin \{l_1, l_2, l_3, l_4\}$)

$$(\alpha_1 - \alpha_3)^{l_3}(\alpha_1 - \alpha_4)^{l_4} = (\alpha_2 - \alpha_3)^{l_3}(\alpha_2 - \alpha_4)^{l_4}$$

and

$$(\alpha_3 - \alpha_1)^{l_1}(\alpha_3 - \alpha_2)^{l_2} = (\alpha_4 - \alpha_1)^{l_1}(\alpha_4 - \alpha_2)^{l_2}.$$

Using the fact that the zeros and poles form an arithmetic progression, it turns out that one has to deal with 80 cases.

- There are 8 cases with $(l_1,l_2,l_3,l_4)=(1,1,1,1)$. We obtain equivalent solutions, so we only consider one of these. Let $\alpha_1=\alpha_0, \alpha_2=\alpha_0+3d$. It follows that $\beta_2=\beta_1-2d^2$. That is we have

$$g(x)=(x-\beta_1)(x-\beta_1+2d^2), \qquad h(x)=\beta_1+(x-\alpha_0)(x-\alpha_0-3d),$$
$$f(x)=(x-\alpha_0)(x-\alpha_0-d)(x-\alpha_0-2d)(x-\alpha_0-3d).$$

- There are 16 equivalent cases with $(l_1,l_2,l_3,l_4)\in\{(1,1,2,2),(2,2,1,1)\}$. One obtains that $d^2=\pm\frac{1}{2}$ and $\beta_2=\beta_1\pm 1$. One example from this family is given by

$$g(x)=(x-\beta_1)(x-\beta_1-1), \qquad h(x)=\beta_1+(x-\alpha_0-\sqrt{2}/2)^2(x-\alpha_0-\sqrt{2})^2,$$
$$f(x)=(x-\alpha_0)\left(x-\alpha_0-\frac{\sqrt{2}}{2}\right)^2(x-\alpha_0-\sqrt{2})^2\left(x-\alpha_0-\frac{3\sqrt{2}}{2}\right)f_2(x),$$

where $f_2(x)$ is a quadratic polynomial such that $f$ has more than 4 zeros and poles. We remark that if we use the equations related to $\beta_2$, we have

$$g(x)=(x-\beta_2)(x-\beta_2+1), \qquad h(x)=\beta_2+(x-\alpha_0)(x-\alpha_0-3\sqrt{2}),$$
$$f(x)=(x-\alpha_0)\left(x-\alpha_0-\frac{\sqrt{2}}{2}\right)(x-\alpha_0-\sqrt{2})\left(x-\alpha_0-\frac{3\sqrt{2}}{2}\right),$$

that is we obtain a "solution" covered by the family given by the case $(l_1,l_2,l_3,l_4)=(1,1,1,1)$.

- There are 8 equivalent cases with $(l_1,l_2,l_3,l_4)=(2,2,2,2)$. All of these cases can be eliminated in the same way. From the equation

$$(\alpha_1-\alpha_3)^{l_3}(\alpha_1-\alpha_4)^{l_4}=-(\alpha_3-\alpha_1)^{l_1}(\alpha_3-\alpha_2)^{l_2}, \tag{8}$$

it follows that

$$d^{l_1+l_2-l_3-l_4}=\frac{(T_1-T_3)^{l_3}(T_1-T_4)^{l_4}}{-(T_3-T_1)^{l_1}(T_3-T_2)^{l_2}},$$

where $\{T_1,T_2,T_3,T_4\}=\{0,1,2,3\}$. The left-hand side is $d^0=1$ and the right-hand side is -1, a contradiction.

- There are 16 equivalent cases with $(l_1,l_2,l_3,l_4)\in\{(1,1,3,3),(3,3,1,1)\}$. As an example, we handle the one with $(l_1,l_2,l_3,l_4)=(3,3,1,1)$ and

$$\alpha_1=\alpha_0, \qquad \alpha_2=\alpha_0+3d, \qquad \alpha_3=\alpha_0+2d, \qquad \alpha_4=\alpha_0+d.$$

Equation (8) implies that either $d=0$ or $d^4=\frac{1}{4}$. If $d^2=\frac{1}{2}$, then we get

$$g(x)=(x-\beta_1)(x-\beta_1+1), \qquad h(x)=\beta_1+(x-\alpha_0)^3(x-\alpha_0-3\sqrt{2}/2)^3,$$
$$f(x)=(x-\alpha_0)^3\left(x-\alpha_0-\frac{\sqrt{2}}{2}\right)(x-\alpha_0-\sqrt{2})\left(x-\alpha_0-\frac{3\sqrt{2}}{2}\right)^3 f_3(x),$$

where $f_3(x)$ is a quartic polynomial resulting an $f$ having more than 4 zeros and poles. If $d^2=-\frac{1}{2}$, then we get

$$g(x)=(x-\beta_1)(x-\beta_1-1), \qquad h(x)=\beta_1+(x-\alpha_0)^3(x-\alpha_0-3\sqrt{-2}/2)^3,$$
$$f(x)=(x-\alpha_0)^3\left(x-\alpha_0-\frac{\sqrt{-2}}{2}\right)(x-\alpha_0-\sqrt{-2})\left(x-\alpha_0-\frac{3\sqrt{-2}}{2}\right)^3 f_4(x),$$

where $f_4$ is a quartic polynomial and we get a contradiction in the same way as before.

- There are 16 equivalent cases with $(l_1,l_2,l_3,l_4)\in\{(2,2,3,3),(3,3,2,2)\}$. We handle the case with $(l_1,l_2,l_3,l_4)=(3,3,2,2)$ and

$$\alpha_1=\alpha_0+3d, \qquad \alpha_2=\alpha_0, \qquad \alpha_3=\alpha_0+2d, \qquad \alpha_4=\alpha_0+d.$$

It follows from equation (8) that $d=0$ or $d^2=\frac{1}{2}$. Also we have that $\beta_2=\beta_1-1$. In a similar way as in the above cases, we obtain a composite function $f$ having 4 zeros and poles forming an arithmetic progression, but an additional quartic factor appears, a contradiction.

- There are 8 equivalent cases with $(l_1,l_2,l_3,l_4)=(3,3,3,3)$. Here we consider the case with

$$\alpha_1=\alpha_0, \qquad \alpha_2=\alpha_0+3d, \qquad \alpha_3=\alpha_0+d, \qquad \alpha_4=\alpha_0+2d.$$

It follows that $\beta_2=\beta_1-8d^6$. As in the previous cases, $g(h(x))$ has 4 zeros and poles coming from an arithmetic progression, but there is an additional quartic factor yielding a contradiction.

If $0\in\{l_1,l_2,l_3,l_4\}$, then we have three possibilities. Either $\{l_1,l_2\}=\{l_3,l_4\}=\{0,1\}$ or $\{l_1,l_2\}=\{1\},\{l_3,l_4\}=\{0,2\}$ or $\{l_1,l_2\}=\{0,2\},\{l_3,l_4\}=\{1\}$. In the first case, the degree of $h$ is 1, a contradiction. The last two cases can be handled in the same way, therefore, we only deal with the case $\{l_1,l_2\}=\{1\},\{l_3,l_4\}=\{0,2\}$. Without loss of generality, we may assume that $l_3=2, l_4=0$. It follows that $\alpha_1=2\alpha_3-\alpha_2$ and $\beta_2=\beta_1-(\alpha_2-\alpha_3)^2$. Thus

$$h(x)=\beta_1+(x-2\alpha_3+\alpha_2)(x-\alpha_2), \quad g(x)=(x-\beta_1)(x-\beta_1+(\alpha_2-\alpha_3)^2),$$

$$f(x) = (x-\alpha_2)(x-\alpha_3)^2(x-2\alpha_3+\alpha_2).$$

We conclude that $f(x)$ has only 3 zeros and poles, a contradiction.

(I) : $t = 3, |S_{\beta_1}| = |S_{\beta_2}| = 1, |S_{\beta_3}| = 2$. Here we may assume that $S_{\beta_1} = \{1\}$, $S_{\beta_2} = \{2\}, S_{\beta_3} = \{3,4\}$, that is one has

$$h(x)=\beta_1+(x-\alpha_1)^{l_1}, \qquad h(x)=\beta_2+(x-\alpha_2)^{l_2}, \qquad h(x)=\beta_3+(x-\alpha_3)^{l_3}(x-\alpha_4)^{l_4},$$

where $l_1, l_2 \in \{2,3\}$. Let us consider the case $l_3 \neq 0, l_4 \neq 0$. Substitute $\alpha_3, \alpha_4$ into the above system of equations to get

$$\beta_3 = \beta_1 + (\alpha_3-\alpha_1)^{l_1}, \qquad \beta_3 = \beta_2 + (\alpha_3-\alpha_2)^{l_2},$$
$$\beta_3 = \beta_1 + (\alpha_4-\alpha_1)^{l_1}, \qquad \beta_3 = \beta_2 + (\alpha_4-\alpha_2)^{l_2}.$$

These equations imply that $\alpha_i = \alpha_j$ for some $i \neq j$, a contradiction. Now assume that $l_4 = 0$, hence $l_3 = 2$ or 3. We can reduce the system as follows

$$(\alpha_1-\alpha_2)^{l_2} + (\alpha_2-\alpha_1)^{l_1} = 0, \quad (\alpha_1-\alpha_3)^{l_3} + (\alpha_3-\alpha_1)^{l_1} = 0,$$

$$(\alpha_2-\alpha_3)^{l_3} + (\alpha_3-\alpha_2)^{l_2} = 0,$$

where $l_1, l_2, l_3 \in \{2,3\}$. We get a contradiction in all these cases.

(I) : $t = 4, S_{\beta_1} = \{1\}, S_{\beta_2} = \{2\}, S_{\beta_3} = \{3\}, S_{\beta_4} = \{4\}$. We obtain the system of equations

$$h(x) = \beta_1 + (x-\alpha_1)^{l_1}, \qquad h(x) = \beta_2 + (x-\alpha_2)^{l_2},$$
$$h(x) = \beta_3 + (x-\alpha_3)^{l_3}, \qquad h(x) = \beta_4 + (x-\alpha_4)^{l_4},$$

where $l_i \geq 2$ (since $\deg h \geq 2$.) Here we prove that this type of composite rational function cannot exist. One has that for any different $i, j$,

$$(\alpha_i - \alpha_j)^{l_j - l_i} = (-1)^{l_i+1}.$$

If $l_i = l_j = 2$, then we have a contradiction. Assume that $l_i = 2$. There exist $l_j = l_k = 3$. Hence $\alpha_i = \alpha_j - 1$ and $\alpha_i = \alpha_k - 1$, a contradiction. Let us deal with the case $(l_1, l_2, l_3, l_4) = (3,3,3,3)$. Substituting $\alpha_1 + \alpha_2$ into the system of equations yields $\beta_1 = \beta_2 + \alpha_1^3 - \alpha_2^3$. We also have that $\beta_1 = \beta_2 + (\alpha_1 - \alpha_2)^3$. By combining these equations, we get that

$$-3\alpha_1\alpha_2(\alpha_1-\alpha_2) = 0.$$

In a similar way, we obtain

$$-3\alpha_3\alpha_4(\alpha_3-\alpha_4)=0.$$

It follows that for some different $i,j$, one has $\alpha_i=\alpha_j$, a contradiction.

(II) : $t=2, |S_\infty|=2, |S_{\beta_1}|=|S_{\beta_2}|=1$. We may assume that $S_\infty=\{1,2\}, S_{\beta_1}=\{3\}, S_{\beta_2}=\{4\}$. The system of equations in this case is as follows:

$$h(x)=\beta_1+\frac{(x-\alpha_3)^{l_3}}{(x-\alpha_1)^{l_1}(x-\alpha_2)^{l_2}}, \qquad h(x)=\beta_2+\frac{(x-\alpha_4)^{l_4}}{(x-\alpha_1)^{l_1}(x-\alpha_2)^{l_2}}.$$

If $l_3=l_4=0$, then it follows that $\beta_1=\beta_2$, a contradiction. Let us deal with the case $l_3=0, l_4\neq 0$ (in a similar way, one can handle the case $l_3\neq 0, l_4=0$). There are only three systems to consider. If $(l_1,l_2,l_3,l_4)=(0,1,0,1)$ or $(1,0,0,1)$, then $\beta_1-1=\beta_2$, and the composite function $f$ has only 2 zeros and poles, a contradiction. If $(l_1,l_2,l_3,l_4)=(1,1,0,2)$, then $\beta_1-1=\beta_2$ and $\alpha_4=\alpha_2\pm 1, \alpha_1=\alpha_2\pm 2$. In all these cases, we obtain a composite function $f$ having only 3 zeros and poles, a contradiction. Let us consider the cases with $l_3\neq 0, l_4\neq 0$. There are 18 systems to deal with. It turns out that $d$ satisfies the equation

$$d^{l_4-l_3}=-\frac{(T_4-T_3)^{l_3}(T_3-T_1)^{l_1}(T_3-T_2)^{l_2}}{(T_4-T_1)^{l_1}(T_4-T_2)^{l_2}(T_3-T_4)^{l_4}},$$

where $\alpha_i=\alpha_0+T_id$, for some $T_i\in\{0,1,2,3\}$. If $(l_1,l_2,l_3,l_4)=(1,0,2,2)$, then

$$(T_1,T_2,T_3,T_4)\in\{(1,3,0,2),(1,3,2,0),(2,0,1,3),(2,0,3,1)\}.$$

In all these cases, we obtain a composite function $f$ having only 3 zeros and poles, a contradiction. As an example, we compute $f$ when $(T_1,T_2,T_3,T_4)=(1,3,0,2)$. We get that $\beta_2=\beta_1+4d$, and

$$h(x)=\beta_1+\frac{(x-\alpha_0)^2}{(x-\alpha_0-d)}, \quad g(x)=(x-\beta_1)(x-\beta_1-4d),$$

$$f(x)=\frac{(x-\alpha_0-2d)^2(x-\alpha_0)^2}{(x-\alpha_0-d)^2}.$$

We exclude the tuple $(l_1,l_2,l_3,l_4)=(0,1,2,2)$ following the same lines. If $(l_1,l_2,l_3,l_4)=(1,1,1,2)$, then we also have that $d=\frac{1}{T_1+T_2-2T_4}$ and $d=\frac{T_2-T_3}{(T_2-T_4)^2}$, it is easy to check that such tuple $(T_1,T_2,T_3,T_4)$ does not exist. In a very similar way, if $(l_1,l_2,l_3,l_4)=(1,1,2,1)$, we obtain that

$$d=\frac{1}{T_1+T_2-2T_3}=\frac{T_2-T_4}{(T_2-T_3)^2},$$

and such tuple $(T_1, T_2, T_3, T_4)$ does not exist. If $(l_1, l_2, l_3, l_4) = (2, 1, 2, 3)$, then

$$\frac{(T_3 - T_4)^3}{(T_3 - T_1)^2(T_3 - T_2)} = 1, \qquad -\frac{(T_4 - T_3)^2}{(T_4 - T_1)^2(T_4 - T_2)} = \frac{4}{27(T_3 - T_4)}.$$

There is no solution in $T_i \in \{0, 1, 2, 3\}, T_i \neq T_j, i \neq j$. We obtain a very similar system of equations in case of $(l_1, l_2, l_3, l_4) = (1, 2, 3, 2), (1, 2, 2, 3), (2, 1, 3, 2)$. If $(l_1, l_2, l_3, l_4) = (1, 1, 3, 3)$, then we get

$$T_1 + T_2 = T_3 + T_4, \qquad (T_4 - T_1)(T_4 - T_2) = (T_3 - T_1)(T_3 - T_2),$$

$$27(T_2 - T_4)^4(T_4 - T_1)^2 = 9(T_4 - T_3)^3(T_2 - T_4)^2(T_4 - T_1) - (T_4 - T_3)^6.$$

The above system has no solution in $(T_1, T_2, T_3, T_4)$. If $(l_1, l_2, l_3, l_4) = (1, 2, 3, 1)$, then

$$T_1 - 4T_3 + 3T_4 = 0, \qquad 2T_2 + T_3 - 3T_4 = 0, \qquad (T_4 - T_3)^3 = (T_4 - T_1)(T_4 - T_2)^2.$$

The system has no solution. The same argument works in case of $(l_1, l_2, l_3, l_4) = (1, 2, 1, 3), (2, 1, 1, 3), (2, 1, 3, 1)$. If $(l_1, l_2, l_3, l_4) = (0, 2, 2, 1)$, then we have

$$\alpha_2 = \alpha_4 + \frac{1}{4}, \qquad \alpha_3 = \alpha_4 - \frac{1}{4},$$

hence

$$h(x) = \beta_1 + \frac{(x - \alpha_4 + \frac{1}{4})^2}{(x - \alpha_4 - \frac{1}{4})^2}, \qquad g(x) = (x - \beta_1)(x - \beta_1 - 1),$$

$$f(x) = \frac{(x - \alpha_4)(x - \alpha_4 + \frac{1}{4})^2}{(x - \alpha_4 - \frac{1}{4})^4}.$$

That is $f$ has only 3 zeros and poles, a contradiction. We handle in the same way the tuples $(l_1, l_2, l_3, l_4) = (2, 0, 2, 1), (2, 0, 1, 2), (0, 2, 1, 2)$. If $(l_1, l_2, l_3, l_4) = (0, 0, 1, 1)$, then $\deg h(x) = 1$, a contradiction.

$\underline{(\mathrm{II}) : t = 3, |S_\infty| = |S_{\beta_1}| = |S_{\beta_2}| = |S_{\beta_3}| = 1.}$ We may assume that $S_\infty = \{1\}$, $S_{\beta_1} = \{2\}, S_{\beta_2} = \{3\}, S_{\beta_3} = \{4\}$. In this case, $h(x)$ can be written as follows:

$$h(x) = \beta_1 + \frac{(x - \alpha_2)^{l_2}}{(x - \alpha_1)^{l_1}}, \qquad h(x) = \beta_2 + \frac{(x - \alpha_3)^{l_3}}{(x - \alpha_1)^{l_1}}, \qquad h(x) = \beta_3 + \frac{(x - \alpha_4)^{l_4}}{(x - \alpha_1)^{l_1}}.$$

The only possible exponent tuple $(l_1, l_2, l_3, l_4)$ is $(0, 1, 1, 1)$. Thus $\deg h(x) = 1$, a contradiction.

(III) : $t = 3, |S_{\beta_1}| = 2, |S_{\beta_2}| = |S_{\beta_3}| = 1$. We may assume that $S_{\beta_1} = \{1,2\}$, $S_{\beta_2} = \{3\}, S_{\beta_3} = \{4\}$. The only exponent tuple for which $\deg h(x) > 1$ is given by $(l_1,l_2,l_3,l_4)$ is $(1,1,2,2)$. We obtain the following system of equations if $d \neq 0$:

$$\begin{aligned}(\beta_3-\beta_1)(T_4-T_3)^2+(\beta_2-\beta_3)(T_4-T_1)(T_4-T_2)&=0\\(\beta_1-\beta_2)(T_3-T_4)^2+(\beta_2-\beta_3)(T_3-T_1)(T_3-T_2)&=0\\(\beta_1-\beta_2)(T_1-T_4)^2+(\beta_3-\beta_1)(T_1-T_3)^2&=0\\(\beta_1-\beta_2)(T_2-T_4)^2+(\beta_3-\beta_1)(T_2-T_3)^2&=0,\end{aligned}$$

where $\{T_1,T_2,T_3,T_4\} = \{0,1,2,3\}$. Solving the above system of equations for all possible tuples $(T_1,T_2,T_3,T_4)$, one gets that $\beta_i = \beta_j$ for some $i \neq j$, a contradiction.

(III) : $t = 3, |S_{\beta_1}| = |S_{\beta_2}| = |S_{\beta_3}| = |S_{\beta_4}| = 1$. We may assume that $S_{\beta_1} = \{1\}$, $S_{\beta_2} = \{2\}, S_{\beta_3} = \{3\}, S_{\beta_4} = \{4\}$. The only possible exponent tuple is $(l_1,l_2,l_3,l_4)$ $= (1,1,1,1)$. Thus the corresponding $h(x)$ has degree 1, a contradiction. As an example, we consider the case

$$\alpha_1=\alpha_0+d,\qquad \alpha_2=\alpha_0,\qquad \alpha_3=\alpha_0+3d,\qquad \alpha_4=\alpha_0+2d.$$

We use equation (3) here with $(j_1,j_2,j_3) = (1,2,3)$ and $(j_1,j_2,j_3) = (1,2,4)$. If $d \neq 0$, then we have

$$\beta_3=3\beta_1-2\beta_2,\qquad \beta_4=2\beta_1-\beta_2.$$

Let $k_1,k_2,k_3,k_4 \in \mathbb{Z}$ such that $k_1+k_2+k_3+k_4 = 0$. Theorem A implies that

$$\begin{aligned}g(x)&=(x-\beta_1)^{k_1}(x-\beta_2)^{k_2}(x-3\beta_1+2\beta_2)^{k_3}(x-2\beta_1+\beta_2)^{k_4},\\h(x)&=\frac{1}{d}(\beta_1(x-\alpha_0)-\beta_2(x-\alpha_0-d)),\\f(x)&=(x-\alpha_0-d)^{k_1}(x-\alpha_0)^{k_2}(x-\alpha_0-3d)^{k_3}(x-\alpha_0-2d)^{k_4}.\end{aligned}$$ □

## 5. Cases with $n = 4$

In this section, we provide some details of the computation corresponding to cases with $n = 4, t \in \{2,3,4\}, k_1+k_2+\cdots+k_t \neq 0, S_\infty = \emptyset$. These are the cases which are not mentioned in [12, Section 5].

The case $n = 4, t = 2$ and $S_\infty = \emptyset$. There are 134 systems to deal with. We treat only a few representative examples.

If $S_{\beta_1} = \{1,2\}, S_{\beta_2} = \{3,4\}$ and $(l_1,l_2,l_3,l_4) = (2,1,2,1)$, then we have

$$\begin{aligned}
\alpha_1 + 1/2\alpha_2 - \alpha_3 - 1/2\alpha_4 &= 0\\
\alpha_2 - 4/3\alpha_3 + 1/3\alpha_4 &= 0\\
\alpha_2\alpha_3^2 - 2\alpha_2\alpha_3\alpha_4 + \alpha_2\alpha_4^2 - \alpha_3^2\alpha_4 + 2\alpha_3\alpha_4^2 - \alpha_4^3 - 9\beta_1 + 9\beta_2 &= 0\\
\alpha_2 - 4/3\alpha_3 + 1/3\alpha_4 &= 0\\
\alpha_3^3 - 3\alpha_3^2\alpha_4 + 3\alpha_3\alpha_4^2 - \alpha_4^3 - 27/4\beta_1 + 27/4\beta_2 &= 0.
\end{aligned}$$

The corresponding rational functions are as follows:

$$f(x) = (x-\alpha_1)^{2k_1}(x-\alpha_2)^{k_1}(x - \frac{1}{3}\alpha_1 - \frac{2}{3}\alpha_2)^{2k_2}(x - \frac{4}{3}\alpha_1 + \frac{1}{3}\alpha_2)^{k_2},$$
$$g(x) = (x-\beta_1)^{k_1}(x-\beta_1 - \frac{4}{27}(\alpha_1-\alpha_2)^3)^{k_2}, \qquad h(x) = \beta_1 + (x-\alpha_1)^2(x-\alpha_2),$$

where $k_1+k_2 \neq 0$. We note that the zeros and poles of $f$ do not form an arithmetic progression for all values of the parameters as the choice $\alpha_1 = 0, \alpha_2 = 3$ shows.

If $S_{\beta_1} = \{1,2\}, S_{\beta_2} = \{3,4\}$ and $(l_1,l_2,l_3,l_4) = (1,1,0,2)$, then we get the system of equations

$$\alpha_1 + \alpha_2 - 2\alpha_4 = 0, \qquad (\alpha_2 - \alpha_4)^2 - \beta_1 + \beta_2 = 0.$$

It yields a decomposable rational function $f$ having only 3 zeros and poles altogether.

If $S_{\beta_1} = \{1,2\}, S_{\beta_2} = \{3,4\}$ and $(l_1,l_2,l_3,l_4) = (1,1,1,1)$, then we obtain

$$\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = 0, \qquad \alpha_2^2 - \alpha_2\alpha_3 - \alpha_2\alpha_4 + \alpha_3\alpha_4 - \beta_1 + \beta_2 = 0.$$

It yields the following solution:

$$\begin{aligned}
f(x) &= (x + \alpha_2 - \alpha_3 - \alpha_4)^{k_1}(x-\alpha_2)^{k_1}(x-\alpha_3)^{k_2}(x-\alpha_4)^{k_2},\\
g(x) &= (x-\beta_1)^{k_1}(x - \beta_1 + \alpha_2^2 - \alpha_2\alpha_3 - \alpha_2\alpha_4 + \alpha_3\alpha_4)^{k_2},\\
h(x) &= \beta_1 + (x - \alpha_3 - \alpha_4 + \alpha_2)(x-\alpha_2),
\end{aligned}$$

where $k_1 + k_2 \neq 0$.

If $S_{\beta_1} = \{1,2,3\}, S_{\beta_2} = \{4\}$ and $(l_1,l_2,l_3,l_4) = (1,1,1,3)$, then we have

$$\alpha_1 + \alpha_2 + \alpha_3 - 3\alpha_4 = 0, \qquad \alpha_2^2 + \alpha_2\alpha_3 - 3\alpha_2\alpha_4 + \alpha_3^2 - 3\alpha_3\alpha_4 + 3\alpha_4^2 = 0,$$

$$\alpha_3^3 - 3\alpha_3^2\alpha_4 + 3\alpha_3\alpha_4^2 - \alpha_4^3 - \beta_1 + \beta_2 = 0.$$

We obtain the following rational functions:

$$f(x) = (x-\alpha_1)^{k_1}(x-\alpha_2)^{k_1}(x-\alpha_3)^{k_1}(x-\alpha_4)^{3k_2},$$

$$g(x) = (x-\beta_2-(\alpha_3-\alpha_4)^3)^{k_1}(x-\beta_2)^{k_2}, \qquad h(x) = \beta_2+(x-\alpha_4)^3,$$

where $k_1+k_2 \neq 0$ and

$$\alpha_1 = \frac{1}{2}\alpha_4\Big(-i\sqrt{3}+3\Big)-\frac{1}{2}\alpha_3\Big(-i\sqrt{3}+1\Big), \qquad \alpha_2 = \frac{1}{2}\alpha_4\Big(i\sqrt{3}+3\Big)+\frac{1}{2}\alpha_3\Big(-i\sqrt{3}-1\Big).$$

The case $n=4, t=3$ and $S_\infty = \emptyset$. There are 48 systems to handle in this case. We consider one of these. Let $S_{\beta_1} = \{1\}, S_{\beta_2} = \{2,3\}, S_{\beta_3} = \{4\}$ and $(l_1,l_2,l_3,l_4) = (1,1,0,1)$. We obtain the system of equations

$$\alpha_1-\alpha_4-\beta_1+\beta_3 = 0, \qquad \alpha_2-\alpha_4-\beta_2+\beta_3 = 0.$$

It follows that $h$ is a linear function, which only provides trivial decomposition. In the remaining cases, we have the same conclusion.

The case $n=4, t=4$ and $S_\infty = \emptyset$. Here we get 24 systems to consider. In all cases, we have that

$$\{S_{\beta_1}, S_{\beta_2}, S_{\beta_3}, S_{\beta_4}\} = \{\{1\},\{2\},\{3\},\{4\}\},$$

and $(l_1,l_2,l_3,l_4) = (1,1,1,1)$. Therefore $h$ is linear, a contradiction.

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