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# On the critical metrics of the total scalar curvature functional

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**Abstract.** The aim of this paper is to study the critical metrics of the total scalar curvature functional on compact manifolds with constant scalar curvature and unit volume, for simplicity, critical point equation (CPE) metrics. It has been conjectured that every CPE metric must be Einstein. We prove that the conjecture is true for CPE metrics under a suitable integral condition, and we also prove that it suffices the metric to be conformal to an Einstein metric.

## 1. Introduction

Let $(M^n, g)$ be a compact oriented manifold, and $\mathcal{M}$ the set of smooth Riemannian structures on $M^n$ of volume 1. Given a metric $g \in \mathcal{M}$, we define the total scalar curvature functional $\mathcal{S} : \mathcal{M} \to \mathbb{R}$ by

$$\mathcal{S}(g) = \int_{M^n} R_g dM_g, \tag{1.1}$$

where $R_g$ and $dM_g$ stand, respectively, for the scalar curvature and the volume form of the metric $g$. It is well-known that the critical metrics of the functional $\mathcal{S}$ restricted to $\mathcal{M}$ are Einstein, for more details, see Chapter 4 in [3].

We recall that there exists a constant scalar curvature metric in every conformal class of Riemannian metrics on a compact manifold $M^n$. From this, we may consider the set

$$\mathcal{C} = \{g \in \mathcal{M}; R_g \text{is constant}\}.$$

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In [7], Koiso showed that, under generic condition, $\mathcal{C}$ is an infinite dimensional manifold. Moreover, we recall that the linearization $\mathfrak{L}_g$ of the scalar curvature operator is given by

$$\mathfrak{L}_g(h) = -\Delta_g(tr_g(h)) + \operatorname{div}(\operatorname{div}(h)) - g(h, \operatorname{Ric}_g),$$

where $h$ is any symmetric bilinear form on $M^n$. Moreover, the formal $L^2$-adjoint $\mathfrak{L}_g^*$ of $\mathfrak{L}_g$ is given by

$$\mathfrak{L}_g^*(f) = -(\Delta_g f)g + \operatorname{Hess} f - f \operatorname{Ric}_g, \tag{1.2}$$

where $f$ is a smooth function on $M^n$.

It has been conjectured that the critical points of the total scalar curvature functional $\mathcal{S}$ restricted to $\mathcal{C}$ are Einstein. More precisely, in [3], the authors wrote (see [3, p. 128]):

*When restricting the total scalar curvature functional to $\mathcal{C}$, are there other critical points besides the Einstein metric?*

When restricting the total scalar curvature to a pointwise conformal class of metrics, a large critical set is obtained. Formally, the Euler–Lagrangian equation of Hilbert–Einstein action on the space of Riemannian metrics $g$ with unitary volume and constant Ricci scalar curvature is given by

$$\operatorname{Ric} - \frac{1}{n}Rg = \operatorname{Hess} f - f\left(\operatorname{Ric} - \frac{R}{n-1}g\right).$$

A *CPE metric* is a 3-tuple $(M^n, g, f)$, where $(M^n, g)$, $n \geq 3$, is an $n$-dimensional compact oriented Riemannian manifold with constant Ricci scalar curvature, and $f$ is a smooth potential function that satisfies the equation

$$\operatorname{Ric} - \frac{R}{n}g = \operatorname{Hess} f - f\left(\operatorname{Ric} - \frac{R}{n-1}g\right), \tag{1.3}$$

where Ric and $\operatorname{Hess} f$ stand, respectively, for the Ricci tensor and the Hessian of $f$. In order to proceed, we notice that computing the trace in (1.3), we obtain

$$\Delta f + \frac{R}{n-1}f = 0. \tag{1.4}$$

Therefore, $R$ lies on the spectrum of $M^n$, thus it must be positive.

The conjecture proposed in [3] in the middle of 1980's, can be announced in terms of CPE definition, see also [2], [6] and [10]. More precisely, the authors proposed the following conjecture.

**Conjecture 1.** *A CPE metric is always Einstein.*

It should be emphasized that Einstein metrics are recovered when $f = 0$. In the last years, many mathematicians have contributed to the proof of the CPE conjecture. However, none has obtained its complete proof. Among its partial answers, LAFONTAINE [8] proved that the CPE conjecture is true under locally conformally flat assumption. Recently, RIBEIRO JR and BARROS [2] showed that Conjecture 1 is also true for 4-dimensional half-conformally flat manifolds. While QING and YUAN [10] obtained a positive answer for Bach-flat manifolds in any dimension. In 2014, CHANG, HWANG and YUN [4] proved that the conjecture is true if the manifold has harmonic curvature. In [1], BARROS *et al.* showed that a 4-dimensional CPE metric with harmonic tensor $W^+$ must be isometric to a round sphere $\mathbb{S}^4$.

On the other hand, considering the function $h = |\nabla f|^2 + \frac{R}{n(n-1)} f^2$, recently, LEANDRO [9] was able to show that the CPE conjecture is true under the condition that $h$ is a constant. Whereas, BENJAMIN FILHO [5] improved this result requiring that $h$ is constant along of the flow of $\nabla f$.

Taking into account that height functions are eigenfunctions of the Laplacian on a sphere $\mathbb{S}^n$ with standard metric $g$, we may conclude that $(\mathbb{S}^n, g, h_v)$ is a CPE metric, where $h_v$ is a height function $h_v$ for an arbitrary fixed vector field $v \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$. Indeed, the existence of a non-constant solution is only known in the standard sphere for some height function.

Now we define $\rho_m(f, \nabla f)$, which, for simplicity, we denote by $\rho_m$, according to

$$\rho_m = (m-1)\int_M f^{m-2}|\nabla f|^4 dM_g - \frac{(n+2)R}{n(n-1)}\int_M f^m |\nabla f|^2 dM_g, \tag{1.5}$$

where $m \in \mathbb{N}$. It is easy to check that on $\mathbb{S}^n$ we have $\rho_m = 0$, for every $m = \{1, 2, 3, \dots\}$. We also recall that BENJAMIM FILHO proved in [5] that the CPE conjecture is true provided $\rho_1 \leq 0$ and $\rho_2 = 0$.

In this spirit, inspired by the historical development on the study of the CPE conjecture, we shall prove that the assumptions considered in [5] as well as [9] can be replaced by a weaker integral condition. We point out that this integral condition is satisfied in the standard sphere, hence it is a natural hypothesis to consider. In this sense, we have established the following result.

**Theorem 1.** *Conjecture 1 is true for CPE metrics $(M^n, g, f)$, provided that function* (1.5) satisfies

$$\rho_k + \rho_m \leq 0,$$

for $m > k$, where $m$ is even and $k$ is odd.

On the other hand, now we consider another approach. A conformal mapping between two Riemannian manifolds $(M,g)$ and $(N,h)$ is a smooth mapping $F:(M,g)\to(N,h)$, which satisfies the property $F^*h=\alpha^2 g$ for a smooth positive function $\alpha: M\to\mathbb{R}^+$. We look to conformal variations of $M$, that is, those variations of the form $e^{-2u}g$, for a smooth function $u: M\to\mathbb{R}$. We ask what happens if a CPE metric is conformal to an Einstein manifold? The answer is the following result:

**Theorem 2.** *Let $(M^n,g,f)$ be a CPE metric. If $g$ is conformal to an Einstein metric $\bar{g}$, then $M$ is isometric to the standard sphere.*

## 2. Preliminaries

In this section, we present a couple of results that are essential for our purpose. We prove some useful algebraic results.

**2.1. Algebraic tools.** First of all, we show a lemma which concerns to suitable polynomials. Letting $I^j(x)=x^j$, let us consider the polynomials $p_m,q_m,r_m,s_m:\mathbb{R}\to\mathbb{R}$, given by

$$(1)\ p_m=\sum_{k=1}^{m-1}(-1)^{k-1}\frac{k(2m+1-k)}{2}I^{m-1-k},\quad (2)\ q_m=\sum_{j=1}^{m-1}(-1)^{j+1}j(j+1)I^{j-1},$$

$$(3)\ r_m=mI^m+\sum_{i=2}^{m+1}(-1)^i I^{m+1-i},\qquad (4)\ \mathrm{s_m}=\mathrm{m}(\mathrm{I^m}+\mathrm{I^{m-1}})-\mathrm{r_m}.$$

We also set

$$(1)\ \upsilon_{m,k}=k(k+1)p_m+m(m+1)p_k,\qquad (2)\ \mu_m=r_m+\frac{m+1}{m-1}s_m,$$
$$(3)\ \lambda_{m,k}=k(k+1)r_m+m(m+1)r_k.$$

**Lemma 1.** *For $m>k$, where $m$ is even and $k$ is odd, the above polynomials satisfy the following:*

$$(1)\ \mu_m=\frac{1}{m-1}(I+1)^2q_m,\qquad (2)\ q_m>0,\qquad (3)\ \lambda_{m,k}=(I+1)^2\upsilon_{m,k},$$
$$(4)\ \upsilon_{m,k}>0.$$

Proof. Since $\mu_m(-1)=0$, we can decompose $\mu_m(x)=(x+1)\widetilde{\mu}_m(x)$, where

$$\widetilde{\mu}_m(x)=mx^{m-1}+\sum_{k=1}^{m-1}(-1)^{k+1}\frac{2(m-k)}{m-1}x^{m-1-k}.$$

In the same way, $\widetilde{\mu}_m(-1)=0$, which enables us to write

$$\begin{aligned}\widetilde{\mu}_m(x)&=\frac{1}{m-1}(x+1)\left(m(m-1)x^{m-2}+\sum_{k=1}^{m-2}(-1)^k(m-k)(m-k-1)x^{m-2-k}\right)\\&=\frac{1}{m-1}(x+1)\sum_{k=1}^{m-1}(-1)^{k+1}k(k+1)x^{k-1}=\frac{1}{m-1}(x+1)q_m(x),\end{aligned}$$

which corresponds to the first item.

We notice that $q_m(-x)>0$, for every $x\geq 0$, since $m$ is even. From now on, we suppose that $x>0$. Under this choice we can write

$$q_m(x)=x^{m-2}L_m(x^{-1}),$$

where $L_m(x)=\sum_{k=1}^{m-1}(-1)^{k+1}k(k+1)x^{m-k-1}$. Hence, it suffices to show that $L_m(x)$ is strictly positive for every $x>0$. Proceeding, it is easy to verify that $L_2(x)=2$, and then it is enough to prove that $L_{m+2}(x)>L_m(x)$ for every $x\in\mathbb{R}$ and $m$ even. Letting $T_{m+2}=L_{m+2}-L_m$, we have

$$\begin{aligned}T_{m+2}(x)&=\sum_{k=1}^{m+1}(-1)^{k+1}k(k+1)x^{m-k+1}-\sum_{k=1}^{m-1}(-1)^{k+1}k(k+1)x^{m-k-1}\\&=2x^m-6x^{m-1}+\sum_{k=1}^{m-1}(-1)^{k+1}2(2k+3)x^{m-k-1}.\end{aligned}$$

Whence we get

$$\frac{1}{2}T_{m+2}(x)(x+1)=x^{m+1}-2\big(x^m-x^{m-1}-\cdots+x^2-x+1\big)+2m+3. \quad (2.1)$$

Since $m$ is even, we have $x^m-x^{m-1}-\cdots+x^2-x+1=\frac{1}{x+1}\big(x^{m+1}+1\big)$. Hence, we deduce

$$\frac{1}{2}T_{m+2}(x)(x+1)^2=x^{m+2}-x^{m+1}+(2m+3)x+2m+1. \quad (2.2)$$

Since the right hand side of (2.2) is strictly positive for $x > 0$, we have the same conclusion for $T_{m+2}$, and we complete the proof of the second item.

Following the same argument, in the first item we write $\lambda_{m,k}(x) = (x+1)\,\widetilde{\upsilon}_{m,k}(x)$, where

$$\widetilde{\upsilon}_{m,k}(x)=k(k+1)\sum_{i=1}^{m}(-1)^{i+1}(m+1-i)x^{m-i}+m(m+1)\sum_{j=1}^{k}(-1)^{j+1}(k+1-j)x^{k-j},$$

and $\widetilde{\upsilon}_{m,k}(-1) = 0$ enables us to write

$$\widetilde{\upsilon}_{m,k}(x) = (x+1)(k(k+1)p_m(x) + m(m+1)p_k(x)) = (x+1)\upsilon_{m,k}(x),$$

which proves the third item. Now we aim to prove the last item. For this purpose, it suffices to prove that $\upsilon_{m+2,k}(x) - \upsilon_{m,k}(x) > 0$ , since $\upsilon_{k+1,k} = (k+1)q_{k+1}$ and by the second item we have $q_{k+1} > 0$. After a straightforward calculus, we obtain

$$\begin{aligned}\upsilon_{m,k}(x) = k(k+1)&\sum_{j=k-1}^{m-2}(-1)^j\frac{(m-1-j)(m+2+j)}{2}x^j\\ &+\sum_{j=0}^{k-2}(-1)^j\frac{(j+1)(j+2)(m-k)(m+k+1)}{2}x^j,\end{aligned}$$

which implies that

$$\begin{aligned}\upsilon_{m+2,k}(x) - \upsilon_{m,k}(x) &= k(k+1)(m+2)x^m + k(k+1)(2m+3)\\ &\quad\times\sum_{j=k-1}^{m-1}(-1)^j x^j + (2m+3)\sum_{j=0}^{k-2}(-1)^j(j+1)(j+2)x^j.\end{aligned}$$

By the above expression, $\upsilon_{m+2,k}(x) - \upsilon_{m,k}(x) > 0$, for every $x \le 0$. Now it remains to prove that $\upsilon_{m+2,k}(x) - \upsilon_{m,k}(x) > 0$, for every $x > 0$. Defining $Q_{m,k} = (I+1)(\upsilon_{m+2,k} - \upsilon_{m,k})$, a straightforward computation gives

$$Q_{m,k}(x)=k(k+1)(m+2)x^{m+1}-k(k+1)(m+1)x^m+2(2m+3)\sum_{j=0}^{k-1}(-1)^j(j+1)x^j.$$

Thus, for every $x \ge 1$, we have $\upsilon_{m+2,k}(x) - \upsilon_{m,k}(x) > 0$. Hence, we need to treat only the case $0 < x < 1$. If we define $\eta_{m,k}(x) = x^{m-2}\upsilon_{m,k}(x^{-1})$, we get

$$\begin{aligned}\eta_{m,k}(x) = k(k+1)&\sum_{i=0}^{m-k-1}(-1)^i\frac{(i+1)(2m-i)}{2}x^i\\ &+(m-k)(m+k+1)\sum_{i=m-k}^{m-2}(-1)^i\frac{(m-i-1)(m-i)}{2}x^i.\end{aligned}$$

Defining $V_{m,k}(x) = \eta_{m+2,k}(x) - \eta_{m,k}(x)$, after a straightforward calculus, we obtain that

$$V_{m,k}(x) = 2k(k+1)\sum_{i=0}^{m-k-1}(-1)^i(i+1)x^i + (2m+3)\sum_{i=m-k}^{m-2}(-1)^i(m-i-1)(m-i)x^i$$
$$+ (m-k+2)(m+k+3)\sum_{i=m-k}^{m}(-1)^i(2m-2i+1)x^i$$
$$+ ((k+1)(m+2)(m+3) + k(k+1)^2)x^{m-k}.$$

We aim to prove that $\eta_{m+2,k}(x) - \eta_{m,k}(x) > 0$, provided that $x > 1$. For this, we consider $P_{m,k}(x) = (x+1)^3(\eta_{m+2,k}(x) - \eta_{m,k}(x)) = (x+1)^3 V_{m,k}(x)$. Whence we get

$$P_{m,k}(x) = 2k(k+1)(x+1)^3\sum_{i=0}^{m-k-1}(-1)^i(i+1)x^i$$
$$+ (2m+3)(x+1)^3\sum_{i=m-k}^{m-2}(-1)^i(m-i-1)(m-i)x^i$$
$$+ (m-k+2)(m+k+3)(x+1)^3\sum_{i=m-k}^{m}(-1)^i(2m-2i+1)x^i$$
$$+ ((k+1)(m+2)(m+3)+k(k+1)^2)x^{m-k}(x+1)^3 = Z_1+Z_2+Z_3+Z_4.$$

Now, calculating separately $Z_1$, $Z_2$, and $Z_3$, we get

$$Z_1 = (x+1)^3\sum_{i=0}^{m-k-1}(-1)^i(i+1)x^i$$
$$= (m-k)x^{m-k+2} + (2m-2k+1)x^{m-k+1} + (m-k+1)x^{m-k} + x + 1,$$

$$Z_2 = (x+1)^3\sum_{i=m-k}^{m-2}(-1)^i(m-i-1)(m-i)x^i$$
$$= 2x^{m+1} - k(k+1)x^{m-k+2} - 2(k+1)(k-1)x^{m-k+1} - k(k-1)x^{m-k},$$

and

$$
\begin{aligned}
Z_3 &= (x+1)^3 \sum_{i=m-k}^{m} (-1)^i (2m-2i+1)x^i \\
&= x^{m+3} - x^{m+1} - (2k+3)x^{m-k+2} - 4(k+1)x^{m-k+1} - (2k+1)x^{m-k}.
\end{aligned}
$$

After a straightforward calculus, we obtain

$$
\begin{aligned}
&P_{m,k}(x) \\
&= 2k(k+1)(x+1) + 2(2m+3)x^{m+1} + (m-k+2)(m+k+3)(x^{m+3} - x^{m+1}) \\
&\quad + (k+1)(m^2+5m+k^2+k+6)x^{m-k+3} + k(m^2+5m+3k^2+6k+9)x^{m-k+2} \\
&\quad + (k+1)(-m^2-m+3k^2+3k)x^{m-k+1} + k(-m^2-m+k^2+2k+1)x^{m-k}.
\end{aligned}
$$

Then, by the above expression $P_{m,k}(x) > 0$, for every $x > 1$, which implies $\eta_{m+2,k}(x) \geq \eta_{m,k}(x)$ for all $x > 1$. Now we note that $\upsilon_{k+1,k}(x) = (k+1)q_{k+1}(x) > 0$ for every $x$. Since $\eta_{k+1,k}(x) = x^{k-1}\upsilon_{k+1,k}(x^{-1})$, we obtain $\eta_{k+1,k}(x) > 0$ for all $x$. Therefore, $\eta_{m,k}(x) > 0$ for every $x > 1$. Finally, since $\upsilon_{m,k}(x) = x^{m-2}\eta_{m,k}(x^{-1}) > 0$ for all $x^{-1} > 1$, we have $\upsilon_{m,k}(x) > 0$ for $0 < x < 1$. □

## 3. Integral formulae and proofs of the main results

Proceeding with the preliminaries, we focus our attention to a smooth function $f$ defined on a Riemannian manifold $M^n$ such that $\Delta f = -\frac{R}{n-1} f$, where $R$ is constant. Then we have

$$
\frac{1}{m}\Delta f^m = -\frac{R}{n-1} f^m + (m-1) f^{m-2}|\nabla f|^2. \tag{3.1}
$$

Whence, for $M^n$ compact, we immediately obtain from (3.1)

$$
\frac{R}{n-1}\int_M f^m dM_g = (m-1)\int_M f^{m-2}|\nabla f|^2 dM_g, \tag{3.2}
$$

as well as

$$
(m-1)\int_M f^{m-2}|\nabla f|^4 dM_g = \frac{R}{n-1}\int_M f^m |\nabla f|^2 dM_g + \frac{1}{m}\int_M |\nabla f|^2 \Delta f^m dM_g,
$$

which gives

$$
\begin{aligned}
(m-1)&\int_M f^{m-2}|\nabla f|^4 dM_g \\
&= \frac{R}{n-1}\int_M f^m|\nabla f|^2 dM_g - 2\int_M f^{m-1}\nabla^2 f(\nabla f,\nabla f)dM_g.
\end{aligned} \tag{3.3}
$$

On the other hand, we remember that for operators $S,T:\mathcal{H}\to\mathcal{H}$ defined over a finite dimensional Hilbert space $\mathcal{H}$, the Hilbert–Schmidt inner product is defined according to

$$
\langle S,T\rangle = \operatorname{tr}\big(\mathrm{ST}^*\big), \tag{3.4}
$$

where 'tr' and '$*$' denote, respectively, the trace and the adjoint operation. Moreover, if $I$ denotes the identity operator on $\mathcal{H}$ of dimension $n$, the traceless of an operator $T$ is given by

$$
\mathring{T} = T - \frac{\mathrm{trT}}{n}I. \tag{3.5}
$$

We notice that identity (1.3) becomes

$$
(f+1)\mathring{\mathrm{Ric}} = \mathring{\nabla}^2 f. \tag{3.6}
$$

Now we use (3.3) to write

$$
\rho_m = -2\int_M f^{m-1}\mathring{\nabla}^2 f(\nabla f,\nabla f)dM_g. \tag{3.7}
$$

Proceeding, given a $(0,2)$ symmetric tensor field $T$ and any vector field $X$ on a Riemannian manifold $M^n$, we have

$$
\operatorname{div}\big((\varphi T)(X)\big) = \varphi(\operatorname{div}T)(X) + \varphi\langle T,\nabla X\rangle + T(\nabla\varphi, X), \tag{3.8}
$$

where $\varphi$ is a smooth function on $M^n$. In particular, choosing $T=\mathring{\mathrm{Ric}}, X=\nabla u$ where $u$ is any smooth function on $M^n$ and using the second contracted Bianchi identity, we derive

$$
\operatorname{div}\big((\varphi\mathring{\mathrm{Ric}})(\nabla u)\big) = \frac{(n-2)}{2n}\varphi\langle\nabla R,\nabla u\rangle + \varphi\langle\mathring{\mathrm{Ric}},\mathring{\nabla}^2 u\rangle + \mathring{\mathrm{Ric}}(\nabla\varphi,\nabla u). \tag{3.9}
$$

On the other hand, for any smooth function $u$ on $M^n$, the Ricci–Bochner identity in tensorial language says $\operatorname{div}(\nabla^2 u) = \mathrm{Ric}(\nabla u,.) + \nabla\Delta u$. In particular, when $\Delta u = -\frac{R}{n-1}u$, we obtain $\operatorname{div}(\mathring{\nabla}^2 u) = \mathring{\mathrm{Ric}}(\nabla u,.)$. Using this in (3.8), we have

$$
\operatorname{div}\big((\varphi\mathring{\nabla}^2 f)(\nabla f)\big) = \varphi\big(\mathring{\mathrm{Ric}}(\nabla f,\nabla f) + |\mathring{\nabla}^2 f|^2\big) + \mathring{\nabla}^2 f(\nabla\varphi,\nabla f). \tag{3.10}
$$

Now we choose $\varphi = f^m$ in identity (3.10) to obtain

$$\int_M f^m\big(\mathring{\mathrm{Ric}}(\nabla f,\nabla f)dM_g + |\mathring{\nabla}^2 f|^2\big)dM_g = -m\int_M f^{m-1}\mathring{\nabla}^2 f(\nabla f,\nabla f)dM_g. \quad (3.11)$$

Using identity (3.7), we deduce

$$\rho_m = \frac{2}{m}\int_M f^m\big(\mathring{\mathrm{Ric}}(\nabla f,\nabla f) + |\mathring{\nabla}^2 f|^2\big)dM_g. \quad (3.12)$$

### 3.1. Proof of Theorem 1.

Proof. To begin with, taking $\varphi = f^{m+1}$ and $u = f$ in (3.9) (or see, for instance, item (2) of [5, Lemma 2]), we obtain from (3.12)

$$\rho_m = \frac{2}{m}\int_M f^m\big|\mathring{\nabla}^2 f|^2 dM_g - \frac{2}{m(m+1)}\int_M f^{m+1}\langle\mathring{\mathrm{Ric}},\mathring{\nabla}^2 f\rangle dM_g. \quad (3.13)$$

Next, we claim that

$$\rho_m = \frac{2}{m(m+1)}\int_M r_m(f)|\mathring{\nabla}^2 f|^2 dM_g, \quad (3.14)$$

where $r_m = mI^m + \sum_{i=2}^{m+1}(-1)^i I^{m+1-i}$ is one of the polynomials given before.

Indeed, we use item (3) of [5, Lemma 2] to deduce

$$\rho_m = \frac{2}{m}\int_M f^m\big|\mathring{\nabla}^2 f|^2 dM_g - \frac{2}{m(m+1)}\sum_{i=1}^{m+1}(-1)^{i+1}\int_M f^{m+1-i}|\mathring{\nabla}^2 f|^2 dM_g.$$

Jointing the first two terms of the above identity it becomes

$$\rho_m = \frac{2}{m(m+1)}\int_M\left(mf^m + \sum_{i=2}^{m+1}(-1)^i f^{m+1-i}\right)|\mathring{\nabla}^2 f|^2 dM_g, \quad (3.15)$$

which gives our claim.

Therefore, using the definition of $\lambda_{m,k}$ in (3.15), we can write

$$\rho_m + \rho_k = \frac{2}{m(m+1)k(k+1)}\int_M \lambda_{m,k}(f)|\mathring{\nabla}^2 f|^2 dM_g.$$

In particular, for $m$ even and $k$ odd, we deduce that $\int_M \lambda_{m,k}(f)|\mathring{\nabla}^2 f|^2 dM_g = 0$. In fact, we are supposing that $\rho_m + \rho_k \leq 0$, and according to Lemma 1, $\lambda_{m,k}(f) > 0$. On the other hand, since $f^{-1}(-1)$ has measure zero (see, for instance, Proposition 1 in [9]), we conclude that $\mathring{\nabla}^2 f = 0$. Now the result follows by [5, Lemma 2] or equation 3.6. □

Recalling that $h = |\nabla f|^2 + \frac{R}{n(n-1)} f^2$, and choosing $\varphi = 1$ in identity (3.10), we obtain $\Delta h = 2\mathring{\mathrm{Ric}}(\nabla f, \nabla f) + 2|\mathring{\nabla}^2 f|^2$ (see also [5, Lemma 1]). Thereby, (3.12) becomes

$$\rho_m = -\frac{1}{m} \int_M \langle \nabla f^m, \nabla h \rangle dM_g. \tag{3.16}$$

Therefore, (3.16) enables us to conclude that the hypotheses in [9] as well as in [5] imply that $\rho_m + \rho_k = 0$, for every $m$ and every $k$, this shows that the conditions in Theorem 1 are weaker than those of [9] and [5].

**3.2. The conformal case.** In the approach of conformal geometry, we have the following lemma, which is a well-known result of the conformal geometry theory, whose proof is standard, thus will be omitted.

**Lemma 2.** *Let $(M^n, g)$ be a Riemannian manifold, and $\bar{g} = \phi^{-2} g$ a metric conformal to $g$. Then the next relations occur:*

$$\overline{\mathrm{Ric}} = \mathrm{Ric} + \phi^{-1} \left( (n-2)\nabla^2 \phi - (n-1) \frac{|\nabla \phi|^2}{\phi} g + \Delta \phi g \right),$$

$$\bar{R} = \phi^2 \left( R + \phi^{-1} \left( 2(n-1)\Delta \phi - (n-1)n \frac{|\nabla \phi|^2}{\phi} \right) \right),$$

$$\overline{\mathring{\mathrm{Ric}}} = \mathring{\mathrm{Ric}} + (n-2)\phi^{-1} \mathring{\nabla}^2 \phi.$$

**3.3. Proof of Theorem 2.**

Proof. Considering the CPE metric $\bar{g}$ as "background" metric on $M$, we can write $\bar{g} = \phi^{-2} g$, where $\phi \in C^\infty(M)$ is strictly positive. Then, by Lemma 2, we have

$$\overline{\mathrm{Ric}} = \mathrm{Ric} + \phi^{-1} \left( (n-2)\nabla^2 \phi - (n-1) \frac{|\nabla \phi|^2}{\phi} g + \Delta \phi g \right),$$

in which the covariant derivatives and Laplacian are to be taken with respect to $g$, not with respect to $\bar{g}$. Since $\bar{g}$ is Einstein, we have

$$0 = \overline{\mathring{\mathrm{Ric}}} = \mathring{\mathrm{Ric}} + (n-2)\phi^{-1} \mathring{\nabla}^2 \phi.$$

Using the last equation, we get

$$\int_M \phi |\mathring{\mathrm{Ric}}|^2 dM_g = \int_M \phi \langle \mathring{\mathrm{Ric}}, \mathring{\mathrm{Ric}} \rangle dM_g = -(n-2) \int_M \langle \mathring{\mathrm{Ric}}, \mathring{\nabla}^2 \phi \rangle dM_g,$$

and taking $\varphi = 1$ and $u = \phi$ in (3.9), we obtain

$$\int_M \phi |\mathring{\mathrm{Ric}}|^2 dM_g \quad = \quad -(n-2) \int_M \langle \mathring{\mathrm{Ric}}, \mathring{\nabla}^2 \phi \rangle dM_g = 0.$$

Then, $\mathring{\mathrm{Ric}}$ is identically zero which implies that $g$ is Einstein, and this finishes the proof of the theorem. □

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