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A note on trans-Sasakian manifolds

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Abstract. In this paper, we obtain a necessary and sufficient condition for a 3-dimensional compact connected positively curved trans-Sasakian manifold of type (α, β), to be homothetic to a Sasakian manifold. The necessary and sufficient condition is that the smooth function $\alpha \neq 0$ and satisfies the differential equation $H_\alpha(\xi, \xi) = 0$. Similarly, using the smooth function β, necessary and sufficient conditions are found for a trans-Sasakian manifold to be homothetic to a Sasakian manifold.

1. Introduction

Given a $(2n+1)$-dimensional almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ (cf. [1]), the product $\overline{M} = M \times R$ has natural almost complex structure J, which makes $(\overline{M}, G)$ an almost Hermitian manifold, where G is the product metric. There are sixteen different types of structures on the almost Hermitian manifold $(\overline{M}, J, G)$ (cf. [9]), and using the structure in the class $\mathcal{W}_4$ on $(\overline{M}, J, G)$, a structure $(\varphi, \xi, \eta, g, \alpha, \beta)$ on M called trans-Sasakian structure is introduced (cf. [17]), which generalizes a Sasakian structure, a Kenmotsu structure on a contact metric manifold (cf. [2], [8]), where α, β are smooth functions defined on M. Trans-Sasakian manifolds have been studied by Blair and Oubiña [2], and Marrero [10]. A trans-Sasakian manifold $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ is called a trans-Sasakian manifold of type (α, β), and trans-Sasakian manifolds of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ are called a cosymplectic, a α-Sasakian and a β-Kenmotsu manifold, respectively.

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A closely related subject, equidistant Sasakian manifolds and Kaehler manifolds, is found in [13], [14]. It is because of the work of Marrero [12], where it is proved that proper trans-Sasakian manifolds (that is, $\alpha \neq 0$, $\beta \neq 0$) exist only in 3-dimension, that there is an emphasis on studying geometry of 3-dimensional trans-Sasakian manifolds. In ([3]–[6], [11], [12], [17]), authors have studied 3-dimensional trans-Sasakian manifolds with some restrictions on the smooth functions α, β appearing in the definition of a trans-Sasakian manifold. There are several examples of trans-Sasakian manifolds constructed mostly on 3-dimensional Riemannian manifolds (cf. [2], [12], [17]). Moreover, as the geometry of Sasakian manifolds is rich and being derived from contact geometry, it is more important as compared to the geometry of trans-Sasakian manifolds. Therefore, the question of finding necessary and sufficient conditions for a 3-dimensional trans-Sasakian manifold to be homothetic to a Sasakian manifold is an important question. As a Sasakian manifold is characterized by α a constant and $\beta = 0$, and on such a manifold the Hessian $H_\alpha(\xi, \xi) = 0$, the question arises: 'Is a compact 3-dimensional trans-Sasakian manifold satisfying $H_\alpha(\xi, \xi) = 0$ necessarily homothetic to a Sasakian manifold?' In this paper, we address this question and other related ones (cf. Theorem 3.1). We also address the question raised in [5], namely, we study 3-dimensional trans-Sasakian manifolds satisfying the differential equation $\nabla\beta = \xi(\beta)\xi$. In dimension 3, the class of trans-Sasakian manifolds coincides with the class of normal almost contact metric manifolds. Indeed, it can be deduced from (cf. [16, Propositions 1 and 2]), one has only to exchange the role of the functions α and β.

2. Preliminaries

Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional contact metric manifold, where φ is a $(1, 1)$-tensor field, ξ a unit vector field, and η a smooth 1-form dual to ξ with respect to the Riemannian metric g satisfying

$$\begin{aligned} \varphi^2 = -I + \eta \otimes \xi, \quad \varphi(\xi) = 0, \, \eta \circ \varphi = 0, \\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \end{aligned} \tag{2.1}$$

$X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M (cf. [1]). If there are smooth functions α, β on an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ satisfying

$$(\nabla\varphi)(X, Y) = \alpha \left(g(X, Y)\xi - \eta(Y)X\right) + \beta \left(g(\varphi X, Y)\xi - \eta(Y)\varphi X\right), \tag{2.2}$$

then it is said to be a trans-Sasakian manifold, where $(\nabla\varphi)(X,Y) = \nabla_X \varphi Y - \varphi(\nabla_X Y)$, $X, Y \in \mathfrak{X}(M)$, and ∇ is the Levi–Civita connection with respect to the metric g (cf. [2]). We shall denote the trans-Sasakian manifold by $(M, \varphi, \xi, \eta, g, \alpha, \beta)$, and it is called a trans-Sasakian manifold of type (α, β). The equations (2.1) and (2.2) imply

$$\nabla_X \xi = -\alpha\varphi(X) + \beta(X - \eta(X)\xi), \quad X \in \mathfrak{X}(M). \tag{2.3}$$

Also, it is known that (cf. [3]) on a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$, we have

$$\xi(\alpha) = -2\alpha\beta. \tag{2.4}$$

We denote by 'Ric' the Ricci tensor of a Riemannian manifold (M, g), then the Ricci operator Q is a symmetric tensor field of type $(1,1)$, which satisfies

$$\mathrm{Ric}(X,Y) = g(QX,Y), X, Y \in \mathfrak{X}(M).$$

On a 3-dimensional trans-Sasakian manifold $(M, \varphi, \xi, \eta, g)$, the following is known (cf. [3]):

$$Q(\xi) = \varphi(\nabla\alpha) - \nabla\beta + 2(\alpha^2 - \beta^2)\xi - g(\nabla\beta, \xi)\xi, \tag{2.5}$$

where $\nabla\alpha$, $\nabla\beta$ are gradients of the smooth functions α, β. If $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ is a compact 3-dimensional trans-Sasakian manifold, then using equation (2.3), we have

$$\operatorname{div}\xi = 2\beta \quad \text{and} \quad \int_M \beta = 0. \tag{2.6}$$

Also, we have (cf. [5])

$$\int_M \alpha^k \beta = 0 \quad \text{for } k \neq 1. \tag{2.7}$$

Using a local orthonormal frame $\{e_1, e_2, e_3\}$ on M, equation (2.2) gives

$$\sum (\nabla\varphi)(e_i, e_i) = 2\alpha\xi. \tag{2.8}$$

On a Riemannian manifold (M, g), for a smooth function f on M, we denote by ∇f and A_f the gradient and Hessian operator, respectively, where $A_f X = \nabla_X \nabla f$, $X \in \mathfrak{X}(M)$. The Hessian H_f of the function f is defined by

$$H_f(X,Y) = g(A_f X, Y), \quad X, Y \in \mathfrak{X}(M), \tag{2.9}$$

and the Laplacian Δf of the function f has the relation with the Hessian

$$\Delta f = \sum H_f(e_i, e_i), \tag{2.10}$$

where $\{e_1, e_2, e_3\}$ is a local orthonormal frame on M. Moreover, on a Riemannian manifold (M, g), the Hessian operator A_f satisfies (cf. [7])

$$(\nabla A_f)(X, Y) - (\nabla A_f)(Y, X) = R(X, Y)\nabla f, \quad X, Y \in \mathfrak{X}(M), \tag{2.11}$$

where R is the curvature tensor field.

3. Trans-Sasakian manifolds homothetic to Sasakian manifolds

In this section, we study 3-dimensional trans-Sasakian manifolds and obtain necessary and sufficient conditions for trans-Sasakaian manifolds to be homothetic to Sasakian manifolds. Our first result uses a differential equation satisfied by the smooth functions α.

Theorem 3.1. *A 3-dimensional compact connected positively curved trans-Sasakian manifold $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ with $\alpha \neq 0$ (that is, α is nonzero at each point), satisfies the differential equation*

$$H_\alpha(\xi, \xi) = 0,$$

if and only if M is homothetic to a positively curved Sasakian manifold.

Proof. Suppose $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ is a 3-dimensional compact connected trans-Sasakian manifold and $H_\alpha(\xi, \xi) = 0$ holds. Then equation (2.4) leads to

$$\alpha\left(\xi(\beta) - 2\beta^2\right) = 0. \tag{3.1}$$

Since $\alpha \neq 0$, on connected M the above equation gives

$$\xi(\beta) = 2\beta^2.$$

Now, using equations (2.6) together with the above equation, we arrive at

$$\operatorname{div}(\beta\xi) = 4\beta^2,$$

which on using Stoke's theorem leads to $\beta = 0$, and consequently, by virtue of equation (2.4), $\xi(\alpha) = 0$. On using this information and equation (2.2), we get

$$\nabla_Y \varphi(\nabla\alpha) = \alpha Y(\alpha)\xi + \varphi A_\alpha Y, \quad Y \in \mathfrak{X}(M),$$

which, in view of equation (2.3), gives

$$\nabla_X\nabla_Y\varphi(\nabla\alpha) = X(\alpha)Y(\alpha)\xi + \alpha XY(\alpha)\xi - \alpha^2 Y(\alpha)\varphi X \\ + (\nabla\varphi)(X, A_\alpha Y) + \varphi\left((\nabla A_\alpha)(X,Y)\right) + \varphi A_\alpha(\nabla_X Y).$$

Now, using the above two equations, and equations (2.2) and (2.11), we compute

$$R(X,Y)\varphi(\nabla\alpha) = \alpha^2\left(X(\alpha)\varphi Y - Y(\alpha)\varphi X\right) + \alpha\left(H_\alpha(X,\xi)Y \\ - H_\alpha(Y,\xi)X\right) + \varphi\left(R(X,Y)\nabla\alpha\right). \tag{3.2}$$

We have $\xi(\alpha) = g(\nabla\alpha, \xi) = 0$, which, in view of equation (2.3), as $\beta = 0$, leads to

$$g(A_\alpha X, \xi) - g(\nabla\alpha, \alpha\varphi X) = 0, \quad \text{that is,} \quad A_\alpha\xi = -\alpha\varphi(\nabla\alpha). \tag{3.3}$$

Thus, equation (3.2) takes the form

$$R(X,Y)\varphi(\nabla\alpha) = \alpha^2\left(X(\alpha)\varphi Y - Y(\alpha)\varphi X\right) + \alpha^2\left(g(\varphi(\nabla\alpha), Y)X \\ - g(\varphi(\nabla\alpha), X)Y\right) + \varphi\left(R(X,Y)\nabla\alpha\right).$$

Replacing Y by $\nabla\alpha$ in the above equation, we arrive at

$$R(X,\nabla\alpha)\varphi(\nabla\alpha) = \alpha^2\Big(\left(X(\alpha) - g(\varphi(\nabla\alpha), X)\right)\nabla\alpha - \|\nabla\alpha\|^2 X\Big) \\ + \varphi\left(R(X,\nabla\alpha)\nabla\alpha\right). \tag{3.4}$$

Note that the operator $R(X,\nabla\alpha)\nabla\alpha$, $X \in \mathfrak{X}(M)$ is a symmetric operator, and therefore, we can find a local orthonormal set $\{X_1, X_2\}$ on M such that $X_i \perp \nabla\alpha$ and $R(X_i,\nabla\alpha)\nabla\alpha = \lambda_i X_i$, $i = 1, 2$, where λ_i are smooth functions. Taking $X = X_i$ in equation (3.4) and using $X_i(\alpha) = 0$, we arrive at

$$R(X_i,\nabla\alpha)\varphi(\nabla\alpha) = \alpha^2\Big(-g(\varphi(\nabla\alpha), X_i)\nabla\alpha - \|\nabla\alpha\|^2 X_i\Big) + \lambda_i\varphi(X_i).$$

Taking the inner product in the above equation by $\varphi(\nabla\alpha)$ leads to

$$-\alpha^2\|\nabla\alpha\|^2 g(X_i, \varphi(\nabla\alpha)) = 0, \tag{3.5}$$

where, we have used $g(\varphi(X_i), \varphi(\nabla\alpha)) = g(X_i, \nabla\alpha) = 0$, $i = 1, 2$.

Suppose that in equation (3.5) $g(X_i, \varphi(\nabla\alpha)) = 0$ holds, which will imply $\lambda_i g(X_i, \varphi(\nabla\alpha)) = 0$, that is,

$$R(X_i, \nabla\alpha; \nabla\alpha, \varphi(\nabla\alpha)) = 0.$$

This gives $g\left(R(\nabla\alpha,\varphi\left(\nabla\alpha\right))\nabla\alpha,X_i\right)=0$, $i=1,2$, that is,

$$R(\nabla\alpha,\varphi\left(\nabla\alpha\right))\nabla\alpha=\mu\nabla\alpha,$$

for a smooth function μ, as $\{X_1,X_2\}$ is a local orthonormal set orthogonal to $\nabla\alpha$. Thus, we have

$$R(\nabla\alpha,\varphi\left(\nabla\alpha\right);\nabla\alpha,\varphi\left(\nabla\alpha\right))=\mu g(\nabla\alpha,\varphi\left(\nabla\alpha\right))=0,$$

which, together with the hypothesis that M is positively curved, implies one of the following possibilities: (i) $\nabla\alpha=0$, (ii) $\varphi(\nabla\alpha)=0$. The second possibility also gives $\nabla\alpha=\xi(\alpha)\xi=0$. Thus, α is a nonzero constant. Using equations (2.2) and (2.3), we conclude

$$\alpha^{-2}\left(\nabla_X\nabla_Y\xi-\nabla_{\nabla_XY}\xi\right)=g(Y,\xi)X-g(X,Y)\xi,\quad X,Y\in\mathfrak{X}(M).$$

Hence M is homothetic to a Sasakian manifold (cf. [15, Theorem 1.1, p. 272], for English version).

The converse is trivial, as on a Sasakian manifold α is a nonzero constant, and therefore $H_\alpha(\xi,\xi)=0$. □

The requirement that $\alpha\neq 0$ in the hypothesis of Theorem 3.1 can be relaxed by enriching the topology of the trans-Sasakian manifold. For instance, if we assume that the de Rham cohomology group $H^1(M,R)=0$, then, as equation (3.1) implies either $\alpha=0$ or $\xi(\beta)=2\beta^2$, this additional topological requirement rules out the possibility $\alpha=0$ as follows: If $\alpha=0$, then equation (2.3) implies that η is closed, and therefore has to be exact. Hence there exists a smooth function f such that $\eta=df$ or, equivalently, $\xi=\nabla f$, and on compact M there exists a point $p\in M$ (critical point of f) such that $\xi_p=0$, which is a contradiction, as ξ is a unit vector field. Hence we have the following result:

Theorem 3.2. *A 3-dimensional compact connected positively curved trans-Sasakian manifold $(M,\varphi,\xi,\eta,g,\alpha,\beta)$, with cohomology group $H^1(M,R)=0$, satisfies the differential equation*

$$H_\alpha(\xi,\xi)=0,$$

if and only if M is homothetic to a positively curved Sasakian manifold.

In both these theorems, we seek to find whether the condition on curvature can be relaxed. The answer is yes, provided we make the condition $H_\alpha(\xi,\xi)=0$ stronger, namely, replace it by demanding that the vector field ξ annihilates the Hessian operator A_α. Indeed, we prove the following result:

Theorem 3.3. *A* 3*-dimensional compact connected trans-Sasakian manifold* $(M,\varphi,\xi,\eta,g,\alpha,\beta)$*, with* $\alpha \neq 0$*, is such that the vector field* ξ *annihilates the Hessian operator* A_α *if and only if* M *is homothetic to a Sasakian manifold.*

Proof. Suppose $(M,\varphi,\xi,\eta,g,\alpha,\beta)$ is a 3-dimensional compact connected trans-Sasakian manifold satisfying $\alpha\neq 0$, and $A_\alpha(\xi)=0$. Then we have $H_\alpha(\xi,\xi)=0$, and as in the proof of Theorem 3.1, we have equation (3.1), and the conclusion that $\beta=0$. Then equation (2.4) becomes $\xi(\alpha)=0$, and equation (2.3) leads to

$$\pounds_\xi g = 0,$$

where $\pounds_\xi$ is the Lie derivative with respect to ξ. Hence the vector field ξ is Killing, and consequently, the Ricci operator Q is invariant under the flow of ξ (being isometries of M), that is, we have

$$\left(\pounds_\xi Q\right)(X) = 0, \quad X \in \mathfrak{X}(M).$$

Thus, the above equation, in view of equation (2.3), gives

$$(\nabla Q)(\xi, X) = \alpha\left(Q\varphi(X) - \varphi Q(X)\right), \quad X \in \mathfrak{X}(M),$$

that is,

$$(\nabla Q)(\xi,\xi) = -\alpha\varphi Q(\xi). \tag{3.6}$$

Now, equation (2.5), together with $\beta=0$, reads

$$Q(\xi) = \varphi(\nabla\alpha) + 2\alpha^2\xi.$$

Using this equation and equations $\nabla_\xi\xi = 0$ (an outcome of equation (2.3)) and $\xi(\alpha)=0$, in equation (3.6), we arrive at

$$\nabla_\xi\left(\varphi(\nabla\alpha) + 2\alpha^2\xi\right) = \alpha\nabla\alpha,$$

that is,

$$(\nabla\varphi)(\xi,\nabla\alpha) + \varphi\left(A_\alpha\xi\right) = \alpha\nabla\alpha.$$

The above equation, in view of equation (2.2) and the hypothesis, leads to $\alpha\nabla\alpha = 0$, that is, α is a nonzero constant. Hence, as in the proof of Theorem 3.1, we get that M is homothetic to a Sasakian manifold.

The converse is trivial, as on a Sasakian manifold we have $A_\alpha = 0$ and α is a nonzero constant. □

Next, we use the smooth function β to find necessary and sufficient condition for a compact and connected 3-dimensional trans-Sasakian manifold $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ to be homothetic to a Sasakian manifold.

Theorem 3.4. *A 3-dimensional compact connected positively curved trans-Sasakian manifold $(M, \varphi, \xi, \eta, g, \alpha, \beta)$, with $\alpha \neq 0$, satisfies*

$$H_\beta(\xi, \xi) = 0 \quad \text{and} \quad \xi(\beta) \leq -2\beta^2,$$

if and only if M is homothetic to a positively curved Sasakian manifold.

Proof. Suppose $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ is a 3-dimensional compact connected positively curved trans-Sasakian manifold satisfying $\alpha \neq 0$ and the conditions in the hypothesis. We have

$$\begin{aligned} \operatorname{div}\left(\xi(\beta)^2\xi\right) &= 2\beta\xi(\beta)H_\beta(\xi,\xi) + \xi(\beta)^2 \operatorname{div}(\beta\xi) \\ &= \xi(\beta)^3 + 2\beta^2\xi(\beta)^2 = \xi(\beta)^2\left(\xi(\beta) + 2\beta^2\right), \end{aligned}$$

which on integrating and using the inequality $\xi(\beta) \leq -2\beta^2$ implies

$$\xi(\beta)^2\left(\xi(\beta) + 2\beta^2\right) = 0.$$

Hence, on connected M, we have either $\xi(\beta) = 0$ or $\xi(\beta) = -2\beta^2$. If $\xi(\beta) = 0$, then we have $\operatorname{div}(\beta\xi) = 2\beta^2$, and integrating the last equation gives $\beta = 0$. Now, if $\xi(\beta) = -2\beta^2$ holds, then we have $\operatorname{div}(\beta^3\xi) = -4\beta^4$, which on integration gives $\beta = 0$. Thus, we have $\beta = 0$, which, in view of equation (2.4) gives $\xi(\alpha) = 0$, and then the proof follows from the proof of Theorem 3.1.

The converse is trivial, as on a positively curved Sasakian manifold we have $\beta = 0$, and consequently, all the requirements of the hypothesis are met. $\square$

Finally, we partially answer the question raised in [5], namely, whether the differential equation $\nabla\beta = \xi(\beta)\xi$ on a compact and connected 3-dimensional trans-Sasakian manifold $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ renders it to be homothetic to a Sasakian manifold. We prove the following:

Theorem 3.5. *A 3-dimensional compact connected positively curved trans-Sasakian manifold $(M, \varphi, \xi, \eta, g, \alpha, \beta)$, with $\alpha \neq 0$ and sectional curvatures bounded below by α^2, satisfies*

$$\nabla\beta = \xi(\beta)\xi \quad \text{and} \quad \xi(\beta) \geq -\beta^2,$$

if and only if M is homothetic to a positively curved Sasakian manifold.

Proof. Suppose $(M,\varphi,\xi,\eta,g,\alpha,\beta)$ is a 3-dimensional compact connected positively curved trans-Sasakian manifold satisfying $\alpha\neq 0$ and the conditions in the hypothesis. Using $\nabla\beta=\xi(\beta)\xi$ and the equation (2.5), we get

$$\operatorname{Ric}(\nabla\beta,\xi)=\xi(\beta)\operatorname{Ric}(\xi,\xi)=2\xi(\beta)\left(\alpha^2-\beta^2-\xi(\beta)\right),$$

and

$$\operatorname{Ric}(\nabla\beta,\nabla\beta)=2\xi(\beta)^2\left(\alpha^2-\beta^2-\xi(\beta)\right).$$

Thus, for a constant c, we find

$$\operatorname{Ric}(\nabla\beta+c\xi,\nabla\beta+c\xi)=2\left(\alpha^2-\beta^2-\xi(\beta)\right)\left(\xi(\beta)+c\right)^2. \tag{3.7}$$

Also, we have

$$\|\nabla\beta+c\xi\|^2=\left(\|\nabla\beta\|^2+c^2+2c\xi(\beta)\right),$$

which, on using $\nabla\beta=\xi(\beta)\xi$, leads to

$$\|\nabla\beta+c\xi\|^2=\left(\xi(\beta)+c\right)^2. \tag{3.8}$$

Thus, equations (3.7) and (3.8) imply

$$\int_M\left(\operatorname{Ric}(\nabla\beta+c\xi,\nabla\beta+c\xi)-2\alpha^2\|\nabla\beta+c\xi\|^2\right)=-\int_M\left(\beta^2+\xi(\beta)\right)\left(\xi(\beta)+c\right)^2.$$

Since, the sectional curvatures are bounded below by α^2, the Ricci curvature is bounded below by $2\alpha^2$, using this in the above equation, together with the inequality $\xi(\beta)\geq-\beta^2$, we conclude that

$$\operatorname{Ric}(\nabla\beta+c\xi,\nabla\beta+c\xi)=2\alpha^2\|\nabla\beta+c\xi\|^2,$$

and

$$\left(\beta^2+\xi(\beta)\right)\left(\xi(\beta)+c\right)^2=0. \tag{3.9}$$

If $\xi(\beta)=-\beta^2$, then we get $\operatorname{div}\left(\beta^3\xi\right)=-\beta^4$, which by Stoke's theorem implies $\beta=0$. Now, suppose $\xi(\beta)+c=0$, then equation (3.8) implies $\nabla\beta=-c\xi$, which, in view of equation (2.3), gives

$$A_\beta X=c\alpha\varphi X-c\beta X+c\beta\eta(X)\xi,\quad X\in\mathfrak{X}(M),$$

that is,

$$H_\beta(X,Y)=c\alpha g(\varphi X,Y)-c\beta g(X,Y)+c\beta\eta(X)\eta(Y),\quad X,Y\in\mathfrak{X}(M).$$

Using the symmetry of the Hessian H_β in the above equation, we conclude that $2c\alpha g(\varphi X, Y) = 0$, $X, Y \in \mathfrak{X}(M)$, which, in view of $\alpha \neq 0$, gives $c = 0$, that is, $\nabla\beta = 0$, and we conclude that β is a constant. This constant β by virtue of equation (2.6) is zero. Hence on connected M, equation (3.9) implies that $\beta = 0$, and consequently $\xi(\alpha) = 0$. The rest of the proof follows on the similar lines as in Theorem 3.1. $\square$

Remark. Suppose $(M, \varphi, \xi, \eta, g)$ is a 3-dimensional Sasakian manifold, and f is a positive function on M, then the deformation of the metric g to define a new metric g' by

$$g^{'} = fg + (1-f)\eta \otimes \eta,$$

on M transforms it to a 3-dimensional trans-Sasakian manifold $(M$, φ, ξ, η, $g^{'}$, α, $\beta)$, where

$$\alpha = \frac{1}{f}, \quad \beta = \frac{1}{2}\xi(\ln f),$$

(cf. [2]), which we denote by $M_{\alpha^{-1}}$. It is worth finding conditions on a compact and connected 3-dimensional trans-Sasakian manifold $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ to be either isometric to a trans-Sasakian manifold $M_{\alpha^{-1}}$ or homothetic to a Sasakian manifold.

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