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On weakly σ-quasinormal subgroups of finite groups

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Abstract. Let $\sigma = \{\sigma_i | i \in I\}$ be a partition of the set of all primes $\mathbb{P}$, and G be a finite group. A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set of G* if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $i \in I$, and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every i such that $\sigma_i \cap \pi(G) \neq \emptyset$. A group is said to be *σ-primary* if it is a finite σ_i-group for some i.

A subgroup A of G is said to be: *σ-quasinormal* in G if G possesses a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $H \in \mathcal{H}$ and all $x \in G$; *σ-subnormal* in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, t$; *weakly σ-quasinormal* in G if there are a σ-quasinormal subgroup S and a σ-subnormal subgroup T of G such that $G = AT$ and $A \cap T \leq S \leq A$.

We study G assuming that some subgroups of G are weakly σ-quasinormal in G.

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

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We use σ to denote some partition of $\mathbb{P}$. Thus $\sigma = \{\sigma_i | i \in I\}$, where $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. G is said to be: *σ-primary* [23] if G is a σ_i-group for some $i \in I$; *σ-decomposable* [21] or *σ-nilpotent* [10] if $G = G_1 \times \cdots \times G_n$ for some σ-primary groups $G_1, \dots, G_n$; *σ-soluble* [23] if every chief factor of G is σ-primary.

The symbol $\sigma(n)$ denotes the set $\{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$ [24]; $\sigma(G) = \sigma(|G|)$.

Recall that a *complete set of Sylow subgroups of* G [2] is a set $\mathcal{S}$ of Sylow subgroups of G containing exactly one Sylow p-subgroup for each prime p dividing the order $|G|$ of G. In general, a set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G (see [9], [25]) if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $i \in I$, and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every $\sigma_i \in \sigma(G)$.

Huppert proved [16] that if G is soluble and it has a complete set $\mathcal{S}$ of Sylow subgroups such that every maximal subgroup of every subgroup in $\mathcal{S}$ permutes with all other members of $\mathcal{S}$, then G is supersoluble. This result was improved by many other authors (see, in particular, the recent papers [2], [10], [13]–[14], and Chapters 2 and 3 in [8]). Here we only note that in the paper [10] Huppert's result and the main results in [2] were generalized in the terms of complete Hall σ-sets.

A subgroup A of G is said to be: *σ-permutable* [23] or *σ-quasinormal* [9] in G if G possesses a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $H \in \mathcal{H}$ and all $x \in G$; *σ-subnormal* [23] in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \dots, t$.

Note that in the classical case, when $\sigma = \{\{2\}, \{3\}, \dots\}$, σ-quasinormal subgroups are also called *S-quasinormal* or *S-permutable* ([4], [8]), and in this case, A is σ-subnormal in G if and only if it is subnormal in G.

The σ-quasinormal and σ-subnormal subgroups were studied in the papers [1], [6], [9], [11]–[12], [15], [23]–[24], [29]. In this paper, we consider some applications of the following generalization σ-quasinormality.

Definition 1.1. A subgroup A of G is said to be *weakly σ-quasinormal* in G if there are a σ-quasinormal subgroup S and a σ-subnormal subgroup T of G such that $G = AT$ and $A \cap T \leq S \leq A$.

Before continuing, consider some examples.

Recall that G is said to be: a D_π-group if G possesses a Hall π-subgroup E and every π-subgroup of G is contained in some conjugate of E; a *σ-full group of Sylow type* [23] if every subgroup E of G is a D_{σ_i}-group for each $\sigma_i \in \sigma(E)$.

Example 1.2. Let A be a subgroup of G, and $A_{\sigma G}$ the subgroup of A generated by all those subgroups of A which are σ-quasinormal in G.

(i) If A is σ-quasinormal in G, then for $T = G$ and $S = A$ we have $AT = G$ and $T \cap A = A$, so A is weakly σ-quasinormal in G.

(ii) In general, a weakly σ-quasinormal subgroup is not σ-quasinormal. Indeed, let $\sigma = \{\{2,3\},\{2,3\}'\}$ and $G = C_7 \rtimes \mathrm{Aut}(C_7)$, where C_7 is a group of order 7. Then $\mathrm{Aut}(C_7)$ is weakly σ-quasinormal in G and it is clearly not σ-quasinormal in G.

(iii) A subgroup A of G is said to be *weakly σ-permutable* in G [29] if there is a σ-subnormal subgroup T such that $G = AT$ and $A \cap T \leq A_{\sigma G}$. It is clear that every weakly σ-quasinormal subgroup is also weakly σ-permutable. On the other hand, if G is a σ-full group of Sylow type, then $A_{\sigma G}$ is σ-quasinormal in G by [7, A, 1.6], and so in this case every weakly σ-permutable subgroup of G is weakly σ-quasinormal in G.

(iv) In view of (iii), in the case when $\sigma = \{\{2\},\{3\},\dots\}$, Definition 1.1 is equivalent to the definition of weakly S-permutable subgroups in [26].

Some characterizations of σ-nilpotent groups were given in [23]. In particular, from [23, Proposition 2.3], see Lemma 2.5 below, it follows that G is σ-nilpotent if and only if either G possesses a complete Hall σ-set $\mathcal{H}$ such that every member of $\mathcal{H}$ is σ-quasinormal in G or every maximal subgroup of G is σ-quasinormal in G.

Our first result is the following fact.

Theorem A. *G is meta-σ-nilpotent (that is, G is an extension of a σ-nilpotent group by a σ-nilpotent group) if and only if G possesses a complete Hall σ-set $\mathcal{H}$ all of whose members are weakly σ-quasinormal in G.*

In view of Example 1.2(iii), we get from Theorem A the following:

Corollary 1.3 (see [29, Theorem 1.4])**.** *Let G be a σ-full group of Sylow type and every Hall σ_i-subgroup of G is weakly σ-permutable in G for all $\sigma_i \in \sigma(G)$. Then G is σ-soluble.*

A subgroup A of G is said to be *c-normal* in G [27] if for some normal subgroup T of G we have $AT = G$ and $A \cap T \leq A_G$.

In the case when $\sigma = \{\{2\},\{3\},\dots\}$, we get from Theorem A the following:

Corollary 1.4 (see [8, I, Theorem 3.49])**.** *G is metanilpotent if and only if every Sylow subgroup of G is c-normal.*

In view of Example 1.2(iv), we get from Theorem A the following as well:

Corollary 1.5. *G is metanilpotent if and only if every Sylow subgroup of G is weakly S-permutable.*

Theorem B. *If every non-nilpotent maximal subgroup of G is weakly σ-quasinormal, then G is σ-soluble.*

Corollary 1.6. *G is soluble if and only if every maximal subgroup of G is weakly σ-quasinormal in G, and G possesses a complete Hall σ-set $\mathcal{H}$ all of whose members are soluble groups.*

It is clear that every c-normal subgroup of G is also weakly σ-quasinormal in G for each partition σ of $\mathbb{P}$. Hence in the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$, we get from Corollary 1.6 the following known result.

Corollary 1.7 (see [27, Theorem 3.1])**.** *G is soluble if and only if every maximal subgroup of G is c-normal in G.*

Theorem C. *Suppose that G possesses a complete Hall σ-set $\mathcal{H}$ all of whose members are supersoluble. If the maximal subgroups of any non-cyclic subgroup $H \in \mathcal{H}$ are weakly σ-quasinormal in G, then G is supersoluble.*

Corollary 1.8 (see [29, Theorem 1.5])**.** *Let G be a σ-full group of Sylow type. Suppose that G possesses a complete Hall σ-set $\mathcal{H}$ such that all members of $\mathcal{H}$ are nilpotent. If the maximal subgroups of any non-cyclic subgroup $H \in \mathcal{H}$ are weakly σ-permutable in G, then G is supersoluble.*

Recall that a normal subgroup E of G is called hypercyclically embedded in G (see [20, p. 217]) if every chief factor of G below E is cyclic. Hypercyclically embedded subgroups play an important role in the theory of soluble groups (see [4], [8], [28]), and the conditions under which a normal subgroup is hypercyclically embedded in G were found by many authors (see the books [4], [8], [28] and, for example, the recent papers [3], [19], [22]). On the base of Theorem C, we prove the following result in this line of research.

Theorem D. *Let E be a normal subgroup of G. Suppose that G possesses a complete Hall σ-set $\{H_1, \ldots, H_t\}$ all of whose members are supersoluble. If the maximal subgroups of $H_i \cap E$ are weakly σ-quasinormal in G for all $i \in \{1, \ldots, t\}$ such that $H_i \cap E$ is non-cyclic, then every chief factor of G below E is cyclic.*

Theorem D covers many known results. In particular, we get from this result the next two known facts.

Corollary 1.9 (see [29, Theorem 1.5])**.** *Let G be a σ-full group of Sylow type, and E a normal subgroup of G. Suppose that G possesses a complete Hall*

σ-set $\mathcal{H} = \{H_1, \dots, H_t\}$ such that every subgroup $H \in \mathcal{H}$ is nilpotent. If the maximal subgroups of $H_i \cap E$ are weakly σ-permutable in G for all $i \in \{1, \dots, t\}$, then every chief factor of G below E is cyclic.

The class $1 \in \mathfrak{F}$ of groups is said to be a *formation* provided every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$. The formation $\mathfrak{F}$ is said to be *saturated*, provided $G \in \mathfrak{F}$ whenever $G^{\mathfrak{F}} \leq \Phi(G)$.

Corollary 1.10 (see [26, Theorem 1.4])**.** *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups, and E a normal subgroup of G with $G/E \in \mathfrak{F}$. If every maximal subgroup of every non-cyclic Sylow subgroup of E is weakly S-permutable in G, then $G \in \mathfrak{F}$.*

2. Preliminaries

We use $\mathfrak{N}_\sigma$ to denote the class of all σ-nilpotent groups.

Lemma 2.1 (see [23, Corollary 2.4 and Lemma 2.5])**.**

(i) *The class $\mathfrak{N}_\sigma$ is closed under taking products of normal subgroups, homomorphic images and subgroups.*

(ii) *If G/N and G/R are σ-nilpotent, then $G/(N \cap R)$ is σ-nilpotent.*

(iii) *If E is a normal subgroup of G and $E/(E \cap \Phi(G))$ is σ-nilpotent, then E is σ-nilpotent.*

Recall that $G^{\mathfrak{N}_\sigma}$ denotes the *σ-nilpotent residual* of G, that is, the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N. In view of in [5, Proposition 2.2.8], we get from Lemma 2.1 the following:

Lemma 2.2. *If N is a normal subgroup of G, then $(G/N)^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma}N/N$.*

Lemma 2.3.

(i) *G is meta-σ-nilpotent if and only if $G^{\mathfrak{N}_\sigma}$ is σ-nilpotent.*

(ii) *If G is meta-σ-nilpotent, then every quotient G/N of G is meta-σ-nilpotent.*

(iii) *If G/N and G/R are meta-σ-nilpotent, then $G/(N \cap R)$ is meta-σ-nilpotent.*

(iv) *If E is a normal subgroup of G and $E/(E \cap \Phi(G))$ is meta-σ-nilpotent, then E is meta-σ-nilpotent.*

Proof. Let $D = G^{\mathfrak{N}_\sigma}$. (i) First suppose that G is meta-σ-nilpotent. By definition, G has a normal σ-nilpotent subgroup N such that G/N is σ-nilpotent, and so $D \leq N$. But then D is σ-nilpotent by Lemma 2.1. On the other hand, if D is

σ-nilpotent, then G is meta-σ-nilpotent, since $G/D = G/G^{\mathfrak{N}_\sigma}$ is σ-nilpotent by Lemma 2.1(ii).

(ii) Part (i) and Lemmas 2.1 and 2.2 imply that $(G/N)^{\mathfrak{N}_\sigma} = DN/N \simeq D/D \cap N$ is σ-nilpotent, so G/N is meta-σ-nilpotent by Part (i).

(iii) $(G/R)^{\mathfrak{N}_\sigma} = DR/R \simeq D/(D \cap R)$ and $(G/N)^{\mathfrak{N}_\sigma} = DN/N \simeq D/(D \cap N)$ are σ-nilpotent by Part (i), so

$$D/((D \cap R) \cap (D \cap N)) = D/(D \cap (R \cap N)) \simeq D(R \cap N)/(R \cap N) = (G/(R \cap N))^{\mathfrak{N}_\sigma}$$

is σ-nilpotent by Lemma 2.1, and hence $G/(R \cap N)$ is meta-σ-nilpotent by Part (i).

(iv) Let $V/(E \cap \Phi(G))$ be a normal σ-nilpotent subgroup of $E/(E \cap \Phi(G))$ such that

$$(E/(E \cap \Phi(G)))/(V/(E \cap \Phi(G))) \simeq E/V$$

is σ-nilpotent. Then V is σ-nilpotent by Lemma 2.1(iii), and so E is meta-σ-nilpotent. The lemma is proved. $\square$

Let $\Pi \subseteq \sigma$ and $\Pi' = \sigma \setminus \Pi$. A natural number n is said to be a Π-*number* if $\sigma(n) \subseteq \Pi$. A subgroup A of G is said to be: a *Hall* Π-*subgroup* of G (see [23], [24]) if $|A|$ is a Π-number and $|G : A|$ is a Π'-number; a σ-*Hall subgroup* of G if A is a Hall Π-subgroup of G for some $\Pi \subseteq \sigma$.

Lemma 2.4. *Let A, K and N be subgroups of G. Suppose that A is σ-subnormal in G and N is normal in G.*

(1) *AN/N is σ-subnormal in G/N.*

(2) *If A is a σ-Hall subgroup of G, then A is normal in G.*

(3) *If A is a σ_i-group, then $A \leq O_{\sigma_i}(G)$.*

(4) *$A \cap K$ is σ-subnormal in K.*

(5) *If $H \neq 1$ is a Hall Π-subgroup of G and A is not a Π'-group, then $A \cap H \neq 1$ is a Hall Π-subgroup of A.*

(6) *If $A = A^{\mathfrak{N}_\sigma}$, that is, $O^{\sigma_i}(A) = A$ for all $\sigma_i \in \sigma(A)$, then A is subnormal in G.*

Proof. (1)–(5) See Lemma 2.6 in [23].

(6) Assume that this assertion is false, and let G be a counterexample of minimal order. By hypothesis, there is a chain $A = A_0 \leq A_1 \leq \cdots \leq A_r = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, r$. Let $M = A_{r-1}$. We can assume without loss of generality that $M \neq G$.

First we show that $A \leq M_G$. This is clear if M is normal in G. Now assume that G/M_G is a σ_i-group. Then from the isomorphism $AM_G/M_G \simeq A/A \cap M_G$ and $A = A^{\mathfrak{N}_\sigma}$, we get that $A = O^{\sigma_i}(A) \leq A \cap M_G$, so $A \leq M_G$.

The choice of G implies that A is subnormal in M, so A is subnormal in M_G by Assertion (4). Therefore, A is subnormal in G. The lemma is proved. □

Lemma 2.5 (see [23, Proposition 2.3]). *Any of the following conditions are equivalent:*

(i) *G is σ-nilpotent.*

(ii) *G has a complete Hall σ-set $\{H_1, \ldots, H_t\}$ such that $G = H_1 \times \cdots \times H_t$.*

(iii) *G has a complete Hall σ-set $\mathcal{H}$ such that every member of $\mathcal{H}$ is σ-subnormal in G.*

(iv) *Every subgroup of G is σ-subnormal in G.*

(v) *Every maximal subgroup of G is σ-subnormal in G.*

Lemma 2.6 (see [24, Theorems A and B]). *If G is σ-soluble, then G is a σ-full group of Sylow type.*

Suppose that G has a complete Hall σ-set $\mathcal{H} = \{H_1, \ldots, H_t\}$. For any subgroup H of G, we write $H \cap \mathcal{H}$ to denote the set $\{H \cap H_1, \ldots, H \cap H_t\}$. If $H \cap \mathcal{H}$ is a complete Hall σ-set of H, then we say that $\mathcal{H}$ *reduces into* H.

Lemma 2.7. *Let H, K and R be subgroups of G. Suppose that H is σ-quasinormal in G and R is normal in G. Then:*

(1) *H is σ-subnormal in G.*

(2) *If G is a σ-full group of Sylow type and $H \leq E \leq G$, then H is σ-quasinormal in E.*

(3) *The subgroup HR/R is σ-quasinormal in G/R.*

(4) *If G is a σ-full group of Sylow type, $R \leq K$ and K/R is σ-quasinormal in G/R, then K is σ-quasinormal in G.*

(5) *If H is a σ_i-group, then $O^{\sigma_i}(G) \leq N_G(H)$.*

(6) *H/H_G is σ-nilpotent.*

(7) *If K is a σ_i-group and $O^{\sigma_i}(G) \leq N_G(K)$, then K is σ-quasinormal in G.*

(8) *If G is a σ-full group of Sylow type and K is σ-quasinormal in G, then $H \cap K$ is σ-quasinormal in G.*

(9) *If $H \leq E \leq G$ and E is normal in G, then H is σ-quasinormal in E.*

Proof. (1)–(8) See Lemmas 2.8, 3.1 and Theorems B and C in [23].

(9) By hypothesis, there is a complete Hall σ-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ of G such that $H_i^x H = H H_i^x$ for all $i = 1, \ldots, t$ and all $x \in G$. Then, since E is normal

in G, $\{H_1 \cap E, \dots, H_t \cap E\}$ is a complete Hall σ-set of E by Lemma 2.4(5). Now let $x \in E$. Then

$$H_i^x H \cap E = (H_i^x \cap E)H = (H_i \cap E)^x H = H(H_i \cap E)^x.$$

Hence H is σ-quasinormal in E. The lemma is proved. □

Lemma 2.8. *Let A, K and N be subgroups of G. Suppose that A is weakly σ-quasinormal in G and N is normal in G.*

(1) *If either $N \leq A$ or $(|N|, |A|) = 1$, then AN/N is weakly σ-quasinormal in G/N.*

(2) *If G is a σ-full group of Sylow type, $N \leq K$, and K/N is weakly σ-quasinormal in G/N, then K is weakly σ-quasinormal in G.*

(3) *If G is a σ-full group of Sylow type and $A \leq E \leq G$, then A is weakly σ-quasinormal in E.*

(4) *If $A \leq E \leq G$ and E is normal in G, then A is weakly σ-quasinormal in E.*

Proof. By hypothesis, there are a σ-quasinormal subgroup S and a σ-subnormal subgroup T of G such that $G = AT$ and $A \cap T \leq S \leq A$.

(1) TN/N is a σ-subnormal subgroup of G/N by Lemma 2.4(1), and also we have

$$(AN/N)(TN/N) = ATN/N = G/N.$$

Suppose that $N \leq A$. Then

$$(A/N) \cap (TN/N) = (A \cap TN)/N = N(A \cap T)/N \leq SN/N,$$

where SN/N is σ-quasinormal in G/N by Lemma 2.7(3). Also we have $SN/N \leq A/N$. Hence A/N is weakly σ-quasinormal in G/N.

Now suppose that $(|N|, |A|) = 1$. Since $G = AT$, $|G : T|$ divides $|A|$. Hence $|NT : T|$ divides $|A|$. But $|NT : T| = |N : N \cap T|$ divides $|N|$. Therefore, $N \leq T$, so

$$(AN/N) \cap (T/N) = (AN \cap T)/N = N(A \cap T)/N \leq SN/N.$$

Hence AN/N is weakly σ-quasinormal in G/N.

(2), (3) In view of Example 1.2(iii), these two statements follow from [29, Lemma 2.5(1)(2)].

(4) First note that $E = E \cap AT = A(E \cap T)$ and $A \cap (E \cap T) = A \cap T \leq S \leq A$. Moreover, Lemma 2.4(4) implies that $E \cap T$ is σ-subnormal in E, and Lemma 2.7(9) implies that S is σ-quasinormal in E. Hence A is weakly σ-quasinormal in E. The lemma is proved. □

The following lemma is a corollary of [7, IV, (6.7)].

Lemma 2.9. *Let $N \leq E$ be normal subgroups of G such that $N \leq \Phi(E)$, and every chief factor of G between E and N is cyclic. Then every chief factor of G below E is cyclic.*

Lemma 2.10 (see [18, Knyagina and Monakhov])**.** *Let H, K and N be pairwise permutable subgroups of G, and let H be a Hall subgroup of G. Then $N \cap HK = (N \cap H)(N \cap K)$.*

3. Proofs of the results

Proof of Theorem A. Let $D = G^{\mathfrak{N}_\sigma}$ be the σ-nilpotent residual of G.

Necessity. The subgroup D is σ-nilpotent by Lemma 2.3(i). Moreover, G is σ-soluble, and so G is a σ-full group of Sylow type by Lemma 2.6. Hence G possesses a complete Hall σ-set $\mathcal{L} = \{L_1, \ldots, L_t\}$. Let $H = L_i$. We show that H is weakly σ-quasinormal in G. Suppose that $H_G \neq 1$. Then H/H_G is weakly σ-quasinormal in G by induction, since the hypothesis holds for G/H_G by Lemma 2.3(ii). Hence H is weakly σ-quasinormal in G by Lemma 2.8(2).

Now assume that $H_G = 1$. Then, since D is σ-nilpotent, it follows that $D \cap H = 1$. On the other hand, G/D is σ-nilpotent by Lemma 2.1. Thus $H \simeq HD/D$ is a Hall σ_i-subgroup of G/D, and so HD/D has a normal complement T/D in G/D by Lemma 2.5. Then T is a normal subgroup of G such that $HT = G$ and $T \cap H \leq T \cap HD \cap H \leq D \cap H = 1$. Hence H is weakly σ-quasinormal in G.

Sufficiency. Assume that this is false, and let G be a counterexample of minimal order. Then D is not σ-nilpotent by Lemma 2.3(i). Let $\mathcal{H} = \{H_1, \ldots, H_t\}$. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \ldots, t$. Let S_i be a σ-quasinormal and T_i a σ-subnormal subgroup of G such that $S_i \leq H_i$, $H_iT_i = G$ and $H_i \cap T_i \leq S_i$, for all $i = 1, \ldots, t$.

(1) *If R is a σ-soluble minimal normal subgroup of G, then G/R is meta-σ-nilpotent, and so G is σ-soluble. Moreover, R is the unique σ-soluble minimal normal subgroup of G.*

First we show that G/R is meta-σ-nilpotent. In view of the choice of G, it is enough to show that the hypothesis holds for G/R. First note that R is σ-primary. Indeed, since R is σ-soluble, for some i we have $O_{\sigma_i}(R) \neq 1$. But $O_{\sigma_i}(R)$ is characteristic in R and hence it is normal in G, so the minimality of R implies that $R = O_{\sigma_i}(R)$ is a σ_i-group. Hence $R \leq H_i$ and $(|R|, |H_j|) = 1$ for all $j \neq i$. Therefore, $\{H_1R/R, \ldots, H_tR/R\}$ is a complete Hall σ-set of G/R all of whose members are weakly σ-quasinormal in G/R by Lemma 2.8(1). Hence the hypothesis holds for G/R, so G/R is meta-σ-nilpotent and G is σ-soluble. Finally, Lemma 2.3(iii) implies that R is the unique σ-soluble minimal normal subgroup of G.

(2) *For some i, $i = 1$ say, we have $S_i = S_1 \neq 1$. Moreover, if for some k we have $S_k = 1$, then T_k is a normal complement to H_k in G.*

Assume that $S_i = 1$. Then $H_i \cap T_i = 1$, so T_i is a σ-subnormal Hall σ_i'-subgroup of G. Hence T_i is a normal complement to H_i in G by Lemma 2.4(2). Moreover, $G/T_i \simeq H_i$ is σ-nilpotent. Suppose that $S_i = 1$ for all $i = 1, \ldots, t$. Then $T_1 \cap \cdots \cap T_t = 1$ by [7, A, 1.6(b)], so

$$G \simeq G/1 = G/(T_1 \cap \cdots \cap T_t)$$

is σ-nilpotent by Lemma 2.1(ii), a contradiction. Hence for some i we have $S_i \neq 1$.

(3) *If $S_i \neq 1$, then $(H_i)_G \neq 1$.*

Indeed, S_i is σ-subnormal in G by Lemma 2.7(1), so $1 < S_i \leq O_{\sigma_i}(G) \leq H_i$ by Lemma 2.4(3).

(4) *G possesses a σ-soluble minimal normal subgroup, R say.*

Claims (2) and (3) imply that $(H_1)_G \neq 1$. Therefore, if R is a minimal normal subgroup of G contained in $(H_1)_G$, then R is σ-soluble.

The final contradiction for the sufficiency. Claims (1) and (4) imply that G is σ-soluble, so R is the unique minimal normal subgroup of G. Hence Claims (2) and (3) imply that $T_2, \ldots, T_t$ are normal subgroups of G and $G/T_k \simeq H_k$ for all $k = 2, \ldots, t$. Hence $G/(T_2 \cap \cdots \cap T_t)$ is σ-nilpotent by Lemma 2.1(ii). On the other hand, $T_2 \cap \cdots \cap T_t = H_1$ by [7, A, 1.6(b)], and so G is meta-σ-nilpotent, contrary to the choice of G. The theorem is proved. □

Proof of Theorem B. Assume that this theorem is false, and let G be a counterexample of minimal order. Let R be a minimal normal subgroup of G.

(1) *G/R is σ-soluble. Hence R is not σ-primary and it is the unique minimal normal subgroup of G.*

Note that if M/R is a non-nilpotent maximal subgroup of G/R, then M is a non-nilpotent maximal subgroup of G, and so it is weakly σ-quasinormal in G by hypothesis. Hence M/R is weakly σ-quasinormal in G/R by Lemma 2.8(1). Therefore, the hypothesis holds for G/R. Hence G/R is σ-soluble, and so R is not σ-primary by the choice of G. Now assume that G has a minimal normal subgroup $N \neq R$. Then G/N is σ-soluble and N is not σ-primary. But, in view of the G-isomorphism $RN/R \simeq N$, the σ-solubility of G/R implies that N is σ-primary. Hence we have (1).

In view of Claim (1), R is not abelian. Hence $|\pi(R)| > 1$. Let p be any odd prime dividing $|R|$, and R_p a Sylow p-subgroup of R.

(2) *If G_p is a Sylow p-subgroup of G with $R_p = G_p \cap R$, then there is a maximal subgroup M of G such that $RM = G$ and $G_p \leq N_G(R_p) \leq M$.*

It is clear that $G_p \leq N_G(R_p)$. The Frattini argument implies that $G = RN_G(R_p)$. On the other hand, Claim (1) implies that $N_G(R_p) \neq G$, so for some maximal subgroup M of G, we have $RM = G$ and $G_p \leq N_G(R_p) \leq M$.

(3) *M is not nilpotent and $M_G = 1$. Hence M is weakly σ-quasinormal in G.*

Assume that M is nilpotent, and let $D = M \cap R$. Then D is a normal subgroup of M, and R_p is a Sylow p-subgroup of D. Hence R_p is characteristic in D, and so it is normal in M. Therefore, $Z(J(R_p))$ is normal in M. Claims1(1) and (2) imply that $M_G = 1$. Hence $N_G(Z(J(R_p))) = M$, and so $N_R(Z(J(R_p))) = D$ is nilpotent. This implies that R is p-nilpotent by Glauberman–Thompson's theorem on the normal p-complements. But then R is a p-group, a contradiction. Hence we have (3).

(4) *There is a σ-subnormal subgroup T of G such that $MT = G$, $M \cap T = 1$, and p does not divide $|T|$.*

By Claim (3), there are a σ-quasinormal subgroup S and a σ-subnormal subgroup T of G such that $G = MT$ and $M \cap T \leq S \leq M$. Claim (3) and Lemma 2.7(6) imply that S is σ-nilpotent. Suppose that $S \neq 1$. Then for every $\sigma_i \in \sigma(S)$, we have $O_{\sigma_i}(S) \neq 1$. It is clear that $O_{\sigma_i}(S)$ is σ-subnormal in G, and hence $O_{\sigma_i}(S) \leq O_{\sigma_i}(G)$ by Lemma 2.4(3), which implies that $O_{\sigma_i}(G) \neq 1$. But then $R \leq O_{\sigma_i}(G)$ by Claim (1), and so R is σ-primary, which contradicts to Claim (1). Therefore $S = 1$, so $T \cap M = 1$. Hence $|T| = |G : M|$, so p does not divide $|T|$, since $G_p \leq M$ by Claim (2).

The final contradiction. Let L be a minimal σ-subnormal subgroup of G contained in T. Then L is a simple group. Lemma 2.4(3) and Claim (1) imply that L is not σ-primary. Hence L is non-abelian, so it is subnormal in G by Lemma 2.4(6). Suppose that $L \nleq R$. Then $L \cap R = 1$. On the other hand, $R \leq N_G(L)$ by [7, A, 14.3]. Hence $LR = L \times R$, so $L \leq C_G(R)$. But Claim (1) implies that $R \nleq C_G(R)$, and so $C_G(R) = 1$, a contradiction. Hence L is a minimal normal subgroup of R. It follows that p divides $|L|$, and hence p divides $|T|$, contrary to Claim (4). The theorem is proved. □

Proof of Corollary 1.6. In view of Theorem B, it is enough to show that if G is soluble, then every maximal subgroup M of G is weakly σ-quasinormal in G. If $M_G \neq 1$, then M/M_G is weakly σ-quasinormal in G/M_G by induction, so M is weakly σ-quasinormal in G by Lemma 2.8(2). On the other hand, if $M_G = 1$

and R is a minimal normal subgroup of G, then R is abelian, and so $G = R \rtimes M$. Hence M is weakly σ-quasinormal in G. The corollary is proved. □

PROOF OF THEOREM C. Assume that this theorem is false, and let G be a counterexample of minimal order. Then $|\sigma(G)| > 1$.

Let p be the smallest prime dividing $|G|$ and $\mathcal{H} = \{H_1, \ldots, H_t\}$. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \ldots, t$, and that $p \in \sigma_1$. Let R be a minimal normal subgroup of G.

(1) *If R is σ-soluble, then G is soluble and G/R is supersoluble.*

We show that the hypothesis holds for G/R. First, $\{H_1R/R, \ldots, H_tR/R\}$ is a complete Hall σ-set of G/R, where $H_iR/R \simeq H_i/H_i \cap R$ is supersoluble, since H_i is supersoluble by hypothesis.

Now let V/R be a maximal subgroup of H_iR/R, so $|(H_iR/R) : (V/R)| = p$ is a prime. Then $V = R(V \cap H_i)$. Hence

$$\begin{aligned} p &= |(H_iR/R) : (V/R)| = |(H_iR/R) : (R(V \cap H_i)/R)| = |H_iR : R(V \cap H_i)| \\ &= |H_i||R||R \cap (V \cap H_i)| : |V \cap H_i||R||H_i \cap R| = |H_i| : |V \cap H_i| = |H_i : (V \cap H_i)|, \end{aligned}$$

so $V \cap H_i$ is a maximal subgroup of H_i. Assume that H_iR/R is not cyclic. Then H_i is not cyclic, so $V \cap H_i$ is weakly σ-quasinormal in G by hypothesis. Moreover, since R is σ-soluble, it is a σ_k-group for some k, and hence $R \leq V \cap H_i$ in the case when $i = k$, and $(|R|, |V \cap H_i|) = 1$ in the case when $i \neq k$, since $R \leq H_k$. Then $V/R = R(V \cap H_i)/R$ is weakly σ-quasinormal in G/R by Lemma 2.8(1). Hence the hypothesis holds for G/R, so the choice of G implies that G/R is supersoluble and G is σ-soluble. Finally, since $R \leq H_k$, R is soluble, and so G is soluble.

(2) *G is σ-soluble. Hence G is soluble.*

Suppose that this is false. Then $(H_1)_G = 1$ and $O_{\sigma_k}(G) = 1$ for all $\sigma_k \in \sigma(G)$ by Claim (1). Moreover, H_1 is not cyclic. Indeed, if H_1 is cyclic, then G is p-nilpotent by [17, IV, 2.8], and so G is soluble by the Feit–Thompson theorem, since p is the smallest prime dividing $|G|$.

Since H_1 is supersoluble by hypothesis and $p \in \pi(H_1)$, H_1 has a maximal subgroup V such that $|H_1 : V| = p$. Then V is weakly σ-quasinormal in G by hypothesis, so there are a σ-quasinormal subgroup S and a σ-subnormal subgroup T of G such that $G = VT$ and $V \cap T \leq S \leq V$. Then $S_G \leq (H_1)_G = 1$, so S is σ-nilpotent by Lemma 2.7(6). Suppose that $S \neq 1$. Then for every $\sigma_i \in \sigma(S)$, we have $1 < O_{\sigma_i}(S) \leq O_{\sigma_i}(G)$ by Lemmas 2.4(3) and 2.7(1), a contradiction. Therefore $S = 1$, which implies that $T \cap V = 1$, and so for a Sylow p-subgroup T_p of T, we have $|T_p| = p$. Since p is the smallest prime dividing $|T|$, T is p-nilpotent by [17, IV, 2.8]. Let E be the p-complement of T. Then E is σ-subnormal in G, and

E is a Hall σ_1'-subgroup of G. Hence E is normal in G by Lemma 2.4(2), and E is soluble by the Feit–Thompson theorem. Therefore, $E = 1$ by Claim (1), since, by our assumption on G, it is not soluble. Therefore $G = H_1$ is supersoluble. This contradiction shows that we have (2).

(3) *R is the unique minimal normal subgroup of G, $R = O_q(G) \nleq \Phi(G)$ for some prime q and $|R| > q$.*

By Claim (2), G is soluble, and so R is a q-group for some prime q. Hence the choice of G and Claim (1) imply that R is the unique minimal normal subgroup of G and $|R| > q$. Moreover, $R \nleq \Phi(G)$ by [17, VI, 8.6], so $R = C_G(R) = O_q(G)$ by [7, A, 15.2].

The final contradiction. Let $q \in \sigma_i$. Then $R \leq H_i$, and for some maximal subgroup M of G, we have $G = R \rtimes M$ by Claim (3). Hence $H_i = R \rtimes (H_i \cap M)$. Since H_i is supersoluble, some maximal subgroup W of R is normal in H_i. Then $V = W(H_i \cap M)$ is a maximal subgroup of H_i. Hence there are a σ-quasinormal subgroup S, and a σ-subnormal subgroup T of G such that $G = VT$ and $V \cap T \leq S \leq V$. Then $V \cap T = S \cap T$. Since G is soluble by Claim (2), H_i has a complement E in G. Since $|\sigma(G)| > 1$, Lemma 2.4(5) and Claim (3) imply that $R \leq E^G \leq T$, since $|G : T| = |V : (V \cap T)|$ is a σ_i-number. Therefore,

$$W = W(R \cap H_i \cap M) = R \cap W(H_i \cap M) = R \cap V = R \cap V \cap T = R \cap S \cap T = R \cap S$$

is σ-quasinormal in G by Lemmas 2.6 and 2.7(8), since G is soluble by Claim (2). Hence for each $j \neq i$, we have $WH_j = H_jW$, which implies that $H_j \leq N_G(W)$, since $R \cap WH_j = W(R \cap H_j) = W$. Therefore, W is normal in G, so the minimality of R implies that $W = 1$, and hence $|R| = q$, which is impossible by Claim (3). This contradiction completes the proof of the result. $\square$

Proof of Theorem D. Assume that this theorem is false, and let G be a counterexample with $|G| + |E|$ minimal. Then $|\sigma(G)| > 1$. Let R be a minimal normal subgroup of G contained in E.

(1) *E is supersoluble. Hence R is a p-group for some prime p, so for some i, we have $R \leq H_i$.*

It is clear that $\{H_1 \cap E, \ldots, H_t \cap E\}$ is a complete Hall σ-set of E. Hence the hypothesis of Theorem C holds for E by Lemma 2.8(4), so E is supersoluble.

(2) *The hypothesis holds for $(G/R, E/R)$. Hence every chief factor of G between E and R is cyclic. Therefore R is not cyclic, R is the unique minimal normal subgroup of G contained in E, and $R \nleq \Phi(E)$.*

We show that the hypothesis holds for $(G/R, E/R)$. First, it is easy to see that $\{H_1R/R, \ldots, H_tR/R\}$ is a complete Hall σ-set of G/R all of whose members

$H_jR/R \simeq H_j/(H_j \cap R)$ are supersoluble. Now assume that

$$(H_jR/R) \cap (E/R) = R(H_j \cap E)/R \simeq (H_j \cap E)/((H_j \cap E) \cap R) = (H_j \cap E)/(H_j \cap R)$$

is not cyclic. Then $H_j \cap E$ is not cyclic. If V/R is a maximal subgroup of $(H_jR/R) \cap (E/R)$, then $V/R = (V \cap H_j)R/R$, where $V \cap H_j$ is a maximal subgroup of $H_j \cap E$ (see Claim (1) in the proof of Theorem C). Moreover, $R \leq V \cap H_j$ in the case when $j = i$, and $(|R|, |V \cap H_j|) = 1$ in the case when $j \neq i$, since $R \leq H_i$. Then $V/R = R(V \cap H_j)/R$ is weakly σ-quasinormal in G/R by hypothesis and Lemma 2.8(1). Therefore, the hypothesis holds for $(G/R, E/R)$, so the choice of G implies that every chief factor of G/R below E/R is cyclic. Hence every chief factor of G between E and R is cyclic. Suppose that G has a minimal normal subgroup $N \neq R$ contained in E. Then every chief factor of G between E and N is cyclic. On the other hand, from the G-isomorphism $RN/R \simeq N$, we get that N is cyclic, and so every chief factor of G below E is cyclic by the Jordan–Hölder theorem for the chief series, contrary to the choice of G. Thus R is the unique minimal normal subgroup of G contained in E, R is non-cyclic and, by Lemma 2.9, $R \nleq \Phi(E)$.

(3) *If T is a σ-subnormal subgroup of G such that $H_iT = G$, then $R \leq T$.*

Suppose that $R \nleq T$. Then $E \cap T_G = 1$ by Claim (2), so $T_G \leq C_G(E)$. Let $j \neq i$. Then Lemma 2.4(5) implies that $H_j^x \leq T$ for all $x \in G$, so $H_j \leq T_G$. On the other hand, some maximal subgroup W of R is normal in H_i, since H_i is supersoluble. Hence W is normal in G, so $W = 1$, and hence $|R| = p$, contrary to Claim (2).

(4) *A Sylow p-subgroup P of E is normal in G. Moreover, P is elementary.*

Since E is supersoluble by Claim (1), for some $q \in \pi(E)$, a Sylow q-subgroup Q of E is normal, and so it is characteristic in E. Hence Q is normal in G, so $q = p$ and $P = Q$ by Claim (2). Moreover, $\Phi(P)$ is normal in G, and $\Phi(P) \leq \Phi(E)$, so $\Phi(P) = 1$ by Claim (2). Hence P is elementary.

The final contradiction. Since H_i is supersoluble by hypothesis, some maximal subgroup W of R is normal in H_i, and P has a complement U in $E \cap H_i$. Maschke's theorem and Claim (4) imply that R has a complement K in P such that K is normal in $E \cap H_i$. Then $V = UKW$ is a maximal subgroup of $E \cap H_i$. Since R is not cyclic by Claim (2), V is weakly σ-quasinormal in G. Hence there are a σ-quasinormal subgroup S and a σ-subnormal subgroup T of G such that $G = VT$ and $V \cap T \leq S \leq V$. Claim (3) implies that $R \leq T$, so

$$R \cap S = R \cap V = R \cap UKW = W(R \cap UK) = W(R \cap U)(R \cap K) = W$$

by Lemma 2.10. Now let $j \neq i$. Then $H_j \leq O^{\sigma_i}(G) \leq N_G(S)$ by Lemma 2.7(5), so $H_j \leq N_G(R \cap S) = N_G(W)$. Thus W is normal in G, so $W = 1$, and hence $|R| = p$. But this contradicts to Claim (2). The theorem is proved. □

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