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The augmentation terminals of groups

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1. Introduction

Let R be a commutative ring with unity, G a group and RG its group ring and let $A(RG)$ denote the *augmentation ideal* of RG, that is the kernel of the ring homomorphism $\phi : RG \to R$ which maps the group elements to 1. It is easy to see that an R-module $A(RG)$ is a free module with the elements $g - 1\,(g \in G)$ as a basis. It is clear that $A(RG)$ is the ideal genereted by all elements of the form $g - 1, g \in G$.

The powers $A^\lambda(RG)$ of $A(RG)$ are defined inductively: $A(RG) = A^1(RG)$, $A^{\lambda+1}(RG) = A^\lambda(RG) \cdot A(RG)$, if λ is not a limit ordinal, and $A^\lambda(RG) = \bigcap\limits_{\nu<\lambda} A^\nu(RG)$ otherwise.

It is easy to see that the right ideal $A^\lambda(RG)$ is a two-sided ideal of RG for all ordinals $\lambda \geq 1$.

Evidently there exists a least ordinal $\tau = \tau_{_R}(G)$ such that $A^\tau(RG) = A^{\tau+1}(RG)$. In [2] τ was called the *augmentation terminal* (or *terminal* for simple when it is obvious from the context what ring R we are working with) of G with respect to R. We shall use this terminilogy, and also we shall write

$$A^\omega(RG) = \bigcap_{n=1}^{\infty} A^n(RG)$$

for the first limit ordinal ω. If $G = \langle 1 \rangle$ we put $\tau_{_R}(G) = 1$.

In general, the question of the classification of groups in regarding to values of the terminals and also of the computation of these terminals, is far from being simple (see [2]).

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We are interested in the finitenes of the terminals of groups. The groups with finite terminals with respect to integers are well known and easily described (see [1], §4.3). In this case the terminals of groups are 1 or 2.

We are primarily concerned with finding all groups whose terminals with respect to commutative rings with unity are finite and with describing the terminals of such groups. In this paper we give necessary and sufficient conditions for groups which have finite terminals with respect to a commutative ring with unity (Theorems 3.3 and 3.6). In Theorems 3.4 and 3.7 we give the qualitatyve characterisation of $\tau_R(G)$ useing the ring-theoretical terminology.

2. Notations and some known facts

If H is a normal subgroup of G, then $I(RH)$ (or $I(H)$ for short when it is obvious from the context what ring R we are working with) denotes the ideal of RG generated by all elements of the form $h-1, (h \in H)$. It is well known that $I(RH)$ is the kernel of the natural epimorphism $\overline{\phi} : RG \to RG/H$ induced by the group homomorphism ϕ of G onto G/H. We notice that if $H = G$ then $I(RG) = A(RG)$.

If $\mathcal{K}$ denotes a class of groups (by which we understand that $\mathcal{K}$ comtains all groups of order 1 and, with each $H \in \mathcal{K}$, all isomorphic copies of H), we define the class $\mathbf{R}\mathcal{K}$ of *residually-*$\mathcal{K}$ groups by letting $G \in \mathbf{R}\mathcal{K}$ if and only if : whenewer $1 \neq g \in G$, there exists a normal subgroup H_g of the group G such that $G/H_g \in \mathcal{K}$ and $g \notin H_g$

We use the following notations for standard group classes: $\mathcal{N}_o$ – *torsion-free nilpotent groups,* $\overline{\mathcal{N}}_p$ – *nilpotent p-groups of finite exponent,* that is, nilpotent group in which every element g satisfies the equation $g^{p^n} = 1$ for some $n = n(G)$.

Let $\mathcal{K}$ be a class of groups. A group G is said to be *discriminated* by $\mathcal{K}$ if for every finite subset $g_1, g_2, \ldots, g_n$ of distinct elements of G , there exists a group $H \in \mathcal{K}$ and a homomorphism ϕ of G into H such that $\phi(g_i) \neq \phi(g_j)$ for all $g_i \neq g_j$, $(1 \leq i, j \leq n)$.

Lemma 2.1. *Let a class $\mathcal{K}$ of groups be closed under the taking of subgroups (that is all subgroups of any member of the class $\mathcal{K}$ are again in the class $\mathcal{K}$) and also finite direct products and let G be a residually-$\mathcal{K}$ group. Then G is discriminated by $\mathcal{K}$.*

The proof can be obtained easily.

It is easy to show that if G is discriminated by a class of groups $\mathcal{K}$ and if x is a non-zero element of RG, then there exists a group $H \in \mathcal{K}$ and a homomorphism ϕ of RG into RH such that $\phi(x) \neq 0$.

From this fact we have

Lemma 2.2. *If G is discriminated by a class of groups $\mathcal{K}$ and for each $H \in \mathcal{K}$ the equation $A^{\omega}(RH) = 0$ holds, then $A^{\omega}(RG) = 0$.*

If K, L are two subgroups of G, then we shall denote by $[K, L]$ the subgroup generated by all commutators $[g, h] = g^{-1}h^{-1}gh,\ \ g \in K, h \in L$.

A series

$$G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_n \supseteq \ldots$$

of normal subgroups of a group G is called an $N-$series if $[G_i, G_j] \subseteq G_{i+j}$ for all $i, j \geq 1$ and also each of the Abelian groups G_i/G_j is a direct product of (possibly infinitely many) cyclic groups which are either infinite or of order p^k , where p is a fixed prime and k is bounded by some integer depending only on G.

It is easy to see that the lower central series of a nilpotent p-group of finite exponent is an N-series.

In this paper we shall use the following theorems:

Theorem 2.1 ([6] Lemma 2.21, page 27)**.** *The augmentation ideal $A(RG)$ is nilpotent if and only if G is a finite p-group and R has characteristic p^n for some prime p.*

The ideal $J_p(R)$ of a ring R is defined by

$$J_p(R) = \bigcap_{n=1}^{\infty} p^n R.$$

Theorem 2.2 ([3], Theorem E)**.** *Let G be a group having a finite $N-$series and R be a commutative ring with unity satisfying $J_p(R) = 0$. Then $A^{\omega}(RG) = 0$.*

In this paper we apply Theorem 2.2 for residually-$\mathcal{N}_p$ groups.

Theorem 2.3. *Let R be a commutative ring with unity satisfying $J_p(R) = 0$. If G is a residually-$\mathcal{N}_p$ group, then $A^{\omega}(RG) = 0$.*

The proof of this theorem follows from Lemmas 2.1 and 2.2 and Theorem 2.2 because the class $\mathcal{N}_p$ is closed under the taking of subgroups and also finite direct products.

Theorem 2.4 ([5], VI.,Theorem 2.15)**.** *If G is a residually torsion-free nilpotent group and R is a commutative ring with unity such that its additive group is torsion-free, then $A^{\omega}(RG) = 0$.*

The n-th term of the lower central series of G is defined inductively: $\gamma_1(G) = G, \gamma_2(G) = G'$ is the derived subgroup $[G, G]$ of G, and $\gamma_n(G) = [\gamma_{n-1}(G), G]$.

We shall also use the following well known fact:

$$I(\gamma_n(G)) \subseteq A^n(RG)$$

for all $n \geq 1$.

3. The augmentation terminals.

3.1. *Throughout this section R will denote a commutative ring with unity of non-zero characteristic and also p will denote a prime number.*

Let p be a prime and n a natural number. Then we shall denote by G^{p^n} the subgroup generated by all elements of the form $g^{p^n}, g \in G$.

The normal subgroups $G_{p,k}$ is defined by

$$G_{p,k} = \bigcap_{n=1}^{\infty} G^{p^n}\gamma_k(G),$$

where $\gamma_k(G)$ is the k-th term of the lower central series of G. It is clear, that the factor-group $G/G_{p,k}$ is residually-$\mathcal{N}_p$ group for every k.

We have the following sequence

$$G = G_{p,1} \supseteq G_{p,2} \supseteq \ldots \supseteq G_p \tag{1}$$

of normal subgroups $G_{p,k}$ of a group G, where $G_p = \bigcap_{k=1}^{\infty} G_{p,k}$.

Lemma 3.1. *Let R be a commutative ring of characteristic p^s. Then $I(G_{p,k}) \subseteq A^k(RG)$ for all $k \geq 1$.*

Proof. Let the element $h-1$ be in $I(G_{p,k})$. It will be sufficient to show that $h-1 \in A^k(RG)$. Writing the element h as $h = h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} y_k$ ($h_i \in G$, $y_k \in \gamma_k(G)$) and using the identity

$$ab - 1 = (a-1)(b-1) + (a-1) + (b-1) \tag{2}$$

we have that

$$h - 1 = (h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} y_k - 1)(y_k - 1) + (h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} - 1) + (y_k - 1).$$

Since $I(\gamma_k(G)) \subseteq A^k(RG)$we have $y_k - 1 \in A^k(RG)$. Therefore

$$h - 1 \equiv (h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} - 1) \pmod{A^k(RG)}.$$

Applying (2) repeatedly to $(h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} - 1)$ from the preceding expression it follows that

$$h - 1 \equiv \sum_{i=1}^{m} (h_i^{p^n} - 1) b_i \equiv \sum_{i=1}^{m} \sum_{j=1}^{p^n} \binom{p^n}{j} (h_i - 1)^j b_i \pmod{A^k(RG)},$$

where $b_i \in RG$. The elements $(h_i - 1)^j$ are in $A^k(RG)$ for all i and $j \geq k$. If $n \geq s + k$, then p^s divides $\binom{p^n}{j}$ for $j = 1, 2, \ldots, k-1$. Therefore

$$(3) \quad h - 1 \equiv \sum_{i=1}^{m} (h_i^{p^n} - 1) b_i \equiv p^s \sum_{i=1}^{m} \sum_{j=1}^{k-1} d_j (h_i - 1)^j b_i \equiv$$

$$\equiv p^s F_k(h) \pmod{A^k(RG)},$$

where $F_k(h) = \sum_{i=1}^{m} \sum_{j=1}^{k-1} d_j (h_i - 1)^j b_i$ and $p^s d_j = \binom{p^n}{j}$. Since p^s is zero in R we have that $h - 1 \in A^k(RG)$ which implies the inclusion $I(G_{p,k}) \subseteq A^k(RG))$ and completes the proof of the lemma.

Lemma 3.2. *Let R be a commutative ring of characteristic p^s. Then*

$$A^\omega(RG) = I(G_p).$$

Proof. From Lemma 3.1 the inclusion $I(G_p) \subseteq A^\omega(RG)$ follows. We can readily verify that G/G_p is residually-$\mathcal{N}_p$ group and so

$$A^\omega(RG/G_p) = 0,$$

by Theorem 2.3. Hence we have the inclusion $A^\omega(RG) \subseteq I(G_p)$ which completes the proof of the lemma.

If G is a finite p-group and R a commutative ring of characteristic p^s, then the ideal $A(RG)$ is nilpotent (see Theorem 2.1). Denote by $\tau^\circ(A(RG))$ the *nilpotency index* of $A(RG)$ i.e. the natural number $k = \tau^\circ(A(RG))$ for which $A^{k-1}(RG) \neq A^k(RG) = 0$. If $G = \langle 1 \rangle$ we put $\tau^\circ(A(RG)) = 1$

Let $\tau_p(G)$ denote the smallest natural number k (if it exists) such that $G_{p,k-1} \neq G_{p,k} = \ldots = G_p$.

Theorem 3.1. *Let R be a commutative ring of characteristic p^s. Then the augmentation terminal of G with respect to R is finite if and only if G/G_p is a finite p-group.*

Proof. By Lemma 3.2, $\tau_R(G) = 1$ if and only if $G = G_p$.

Now we suppose that $\tau_R(G) = k > 1$. Then

$$\ldots \supset A^{k-1}(RG) \supset A^k(RG) = A^{k+1}(RG) = \ldots = A^\omega(RG)$$

and hence

$$\ldots \supseteq A^{k-1}(RG/G_{p,i}) \supseteq A^k(RG/G_{p,i}) = \\ = A^{k+1}(RG/G_{p,i}) = \ldots = A^\omega(RG/G_{p,i}),$$

that is $\tau_R(G/G_{p,i})$ is finite and not greater than $\tau_R(G)$ for all $i > 1$. It is very easy to see that $G/G_{p,i}$ are residually-$\mathcal{N}_p$ groups and consequently, by Theorem 2.3, $A^\omega(RG/G_{p,i}) = 0$ for all $i > 1$. Because $\tau_R(G/G_{p,i}) \leq k$,

$$A^k(RG/G_{p,i}) = 0 \tag{4}$$

for every i. So from the isomorphism

$$A^k(RG/G_{p,i}) \cong (A^k(RG) + I(G_{p,i}))/I(G_{p,i})$$

the inclusion $A^k(RG) \subseteq I(G_{p,i})$ follows for all $i > 1$. If $i = k$ then from Lemma 3.1 it follows that $A^k(RG) = I(G_{p,k})$. Hence $I(G_{p,k}) \subseteq I(G_{p,i})$ and, therefore,$G_{p,k} \subseteq G_{p.i}$ for all $i > 1$. This implies that

$$\ldots \supseteq G_{p,k} = G_{p,k+1} = \ldots = G_p$$

and from (4) we have that $A^k(RG/G_p) = 0$. So, by Theorem 2.1, G/G_p is a finite p-group.

Conversely, let G/G_p be a finite p-group. Then, by Theorem 2.1, $A^k(RG/G_p) = 0$ for the nilpotency index $\tau^\circ(A(RG/G_p)) = k$. It follows that $A^k(RG) \subseteq I(G_p)$. Hence, by Lemma 3.2, we obtain that $A^k(RG) \subseteq A^\omega(RG)$. The inverse inclusion, of course, is trivial. Therefore $A^k(RG) = A^\omega(RG)$. Consequently

$$A^k(RG) = A^{k+1}(RG) = \ldots$$

which was to be proved.

Theorem 3.2. *Let R be a commutative ring of characteristic p^s and let the augmentation terminal of G with respect to R be finite. Then*

$$\tau_R(G) = \tau^\circ(A(RG/G_p)) \geq \tau_p(G).$$

Proof. Let $\tau_R(G) = k$. It is obvious that $\tau_R(G/G_p)$ is finite and also the inequality $\tau_R(G) \geq \tau_R(G/G_p)$ holds. By Theorem 3.1 G/G_p is finite

p-group. Keeping in mind the previovs inequalities, by Theorem 2.1, we have that

$$A^k(RG/G_p) = 0.$$

Consequently, $\tau^\circ(A(RG/G_p)) \leq \tau_R(G)$.

Now we show that $\tau^\circ(A(RG/G_p)) = \tau_R(G)$. If this equation is not true we can choose a non-negative integer $i < k$ such that $A^i(RG/G_p) = 0$. Hence we have that $A^i(RG) \subseteq I(G_p)$. Then by Lemma 3.2, $I(G_p) = A^\omega(RG)$. Therefore $A^i(RG) \subseteq A^\omega(RG)$ and $A^i(RG) = A^{i+1}(RG)$ which contradicts to the equation $\tau_R(G) = k$. Consequently,

$$\tau_R(G) = \tau^\circ(A(RG/G_p)).$$

From $\tau_R(G) = k$ it follows that $A^k(RG) = A^\omega(RG)$ and, by Lemmas 3.1 and 3.2, $I(G_{p,k}) \subseteq A^k(RG) = A^\omega(RG) = I(G_p)$. Then $G_{p,k} \subseteq G_p$ and by (1) we obtain that $G_{p,k} = G_p$, that is $\tau_p(G) \leq \tau_R(G)$. This completes the proof of the theorem.

Let $\Pi(n)$ denote the set of prime divisors of a natural number n.

Theorem 3.3. *Let R be a commutative ring of non-zero characteristic n. Then the augmentation terminal of G with respect to R is finite if and only if G/G_p is finite p-group for all $p \in \Pi(n)$.*

Proof. Let $n = p_1^{m_1} p_2^{m_2} \cdots p_t^{m_t}$ be the prime power decomposition of the natural number n. We shall write $R_{p_i} = R/n_i R$ for $n_i = p_i^{m_i}$, where $p_i \in \Pi(n) = \{p_1, p_2, \ldots, p_t\}$.

Let $\tau_R(G)$ be finite. It follows that $\tau_{R_{p_i}}(G)$ is also finite and

$$\tau_R(G) \geq \tau_{R_{p_i}}(G) \tag{5}$$

for all $p_i \in \Pi(n)$. Then, by Threorem 3.1, G/G_{p_i} a finite p_i-group for all $p_i \in \Pi(n)$.

We notice that from (5) the inequality

$$\tau_R(G) \geq \max_{p_i \in \Pi(n)} \left\{\tau_{R_{p_i}}(G)\right\} \tag{6}$$

follows.

Conversely. Let G/G_{p_i} be finite p_i-group for all $p_i \in \Pi(n)$. Then by Theorem 3.1, the augmentation terminal $\tau_{R_{p_i}}(G)$ is finite for all $p_i \in \Pi(n)$. Let

$$k = \max_{p_i \in \Pi(n)} \left\{\tau_{R_{p_i}}(G)\right\}.$$

Then

$$\ldots \supseteq A^{k-1}(R_{p_i}G) \supseteq A^k(R_{p_i}G) = A^{k+1}(R_{p_i}G) = \ldots = A^\omega(R_{p_i}G) \tag{7}$$

for all $p_i \in \Pi(n)$. From the isomorphism

$$A^k(R_{p_i}G) \cong (A^k(RG) + n_iR \cdot RG)/n_iR \cdot RG$$

and from (7) it follows that for every $p_i \in \Pi(n)$, an arbitrary element x of $A^k(RG)$ can be written as

$$x = x_i + n_i a_i, \tag{8}$$

where $x_i \in A^{k+1}(RG), a_i \in RG$ and $i = 1, 2, \dots, t$.

If $\overline{n}_i = n/n_i$, then $\overline{n}_i$ is non-zero and $\overline{n}_i n_i = 0$ in R. Then $\overline{n}_i x = \overline{n}_i x_i$ for all $i = 1, 2, \dots, t$ and from (8) we have that

$$\left(\sum_{i=1}^{t} \overline{n}_i\right) x = \sum_{i=1}^{t} \overline{n}_i x_i.$$

It is easy to see that $\overline{n}_i$ and n_i are coprimes and also n_j divides $\overline{n}_i$ for all $i \neq j$. Therefore the numbers $\sum\limits_{i=1}^{t} \overline{n}_i$ and n are coprimes. Hence $\sum\limits_{i=1}^{t} \overline{n}_i$ is invertible in R, because the characteristic of R equals to n. Then from the previous equation we obtain that

$$x = \alpha \sum_{i=1}^{t} \overline{n}_i x_i,$$

where $\alpha \sum_{i=1}^{t} \overline{n}_i = 1$, $\alpha \in R^*$ and R^* the unit group of R. Therefore $x \in A^{k+1}(RG)$ and hence we conclude that $A^k(RG) \subseteq A^{k+1}(RG)$. The inverse inclusion is trivial. Consequently, $A^k(RG) = A^{k+1}(RG)$ and

$$\tau_R(G) \leq \max_{p_i \in \Pi(n)} \left\{\tau_{R_{p_i}}(G)\right\}, \tag{9}$$

that is the augmentation terminal $\tau_R(G)$ of G in regarding to R is finite which was to be proved.

Theorem 3.4. *Let R be a commutative ring of non-zero characteristic n and let the augmentation terminal of G with respect to R be finite. Then*

$$\tau_R(G) = \max_{p_i \in \Pi(n)} \left\{\tau_{R_{p_i}}(G)\right\} = \max_{p_i \in \Pi(n)} \{\tau^\circ(A(R_{p_i}G/G_{p_i}))\} \geq$$
$$\geq \max_{p_i \in \Pi(n)} \{\tau_{p_i}(G)\}.$$

Proof. By Theorem 3.2 we have that $\max\limits_{p_i \in \Pi(n)} \left\{\tau_{R_{p_i}}(G)\right\} =$ $= \max\limits_{p_i \in \Pi(n)} \{\tau^\circ(A(R_{p_i}G/G_{p_i}))\} \geq \max\limits_{p_i \in \Pi(n)} \{\tau_{p_i}(G)\}$. From (6) and (9) we conclude that $\tau_R(G) = \max\limits_{p_i \in \Pi(n)} \left\{\tau_{R_{p_i}}(G)\right\}$ which was to be proved.

3.2. *In this section R will denote a commutative ring with unity of characteristic zero.*

An element g of an Abelian group G is called an element of *infinite p-height* (in G), if the equation $x^{p^n} = g$ has a solution in G for every natural number n.

Lemma 3.3. *Let $g_1, g_2 \in G$ and suppose that at least one of the following statements*

1) *$g_i \in G'$ or g_iG' is a p-element of G/G' and also p is invertible in R ($i = 1$ or 2)*
2) *g_iG' is an element of infinite p-height in G/G' and g_jG' is a p-element of $G/G', i \neq j$*
3) *g_iG' is a p-element and g_jG' is a q-element of G/G' for $p \neq q$*
4) *g_iG' is a p-element of G/G' and $G = G_p$*

hold. Then $(g_1 - 1)(g_2 - 1) \in A^3(RG)$.

Proof. Let gG' be a p-element of G/G' . Then $g^{p^n} \in G'$ for some n. Clearly, $g^{p^n} - 1 \in A^2(RG)$ and from the identity

$$g^{p^n} - 1 = p^n(g-1) + \binom{p^n}{2}(g-1)^2 + \cdots + (g-1)^{p^n}$$

it follows that

$$p^n(g-1) \in A^2(RG). \tag{10}$$

1) It is clear, that if $g_i \in G'$ ($i = 1$ or 2), then $g_i - 1 \in A^2(RG)$ and consequently, $(g_1 - 1)(g_2 - 1) \in A^3(RG)$.

Let g_iG' be a p-element of G/G' and let p be invertible in R. So from (9) we have that $(g_i - 1) \in A^2(RG)$. Therefore $(g_1 - 1)(g_2 - 1) \in A^3(RG)$.

2.) Let g_iG' be an element of infinite p-height in G/G' and p^n the order of the element $g_jG'(i \neq j)$. Then for g_i we can write $g_i = x^{p^n}c$ for some $x \in G$ and a suitable $c \in G'$. Applying the identity (2) to $g_i - 1$ we have that $(g_i - 1) \equiv (x^{p^n} - 1) \equiv p^n(x - 1) \pmod{A^2(RG)}$ because $c - 1 \in A^2(RG)$. By (10) $p^n(g_j - 1) \in A^2(RG)$ and the statement follows.

3.) The proof of this statement follows from the fact that a q-element of G is an element of infinite p-height in G for every prime $p \neq q$.

4.) Let $G = G_p$ and let p^n be the order of the element g_iG'. The elements g_1, g_2 are in G_p and hence, as in Lemma 3.1 we obtain $g_j - 1 \equiv p^nF(g_j) \pmod{A^2(RG)}$, where $F(g_j) \in A(RG)$. Then by (10) $p^n(g_i - 1) \in A^2(RG)$ and consequently $(g_1 - 1)(g_2 - 1) \in A^3(RG)$.

We shall use the following notation: $\mathcal{P}$ denotes the set of primes and R^* denotes the unit group of R, that is, an element $a \in R$ belongs to R^* if and only if there exists $b \in R$ such that $ab = 1$.

There are four subsets of primes which are important for us: $E(R) = \{p \in \mathcal{P} \mid p^i R = p^{i+1} R \text{ for some integer } i \geq 0\}$, $E_1(R) = E(R) \cap R^*$, $E_2(R) = E(R) \setminus E_1(R)$ and $E_3(R) = \mathcal{P} \setminus E(R)$.

It is easy to see that $E_i(R) \cap E_j(R) = \emptyset$ for $i \neq j (i, j = 1, 2, 3)$ and also $E(R) = E_1(R) \cup E_2(R)$, $P = E(R) \cup E_3(R)$ for every commutative ring R with unity.

For every p of $E(R)$ let us denote by $e = e(p)$ the smallest non-negative integer satisfying the equation $p^e R = p^{e+1} R$.

Let $M = \{p_1, p_2, \dots, p_n\}$ be a nonempty finite subset of $E(R)$ and $k_M = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$, where $e_i = e(p_i)$ for $p_i \in M$.

The next two lemmas are modifications of Lemmas 1.2 and 1.3 from [6] (page 65).

Lemma 3.4.. *Let R be a commutative ring with unity and M a nonempty finite subset of $E_2(R)$. If $C = \{c \in R \mid k_M c = 0\}$, then C is an ideal of R, furthermore $E(R) = E(R/C), M \subseteq E_1(R/C)$ and $E_3(R) = E_3(R/C)$.*

Proof. It is evident that C is an ideal of R and also that the inclusion $E(R) \subseteq E(R/C)$ holds. We show that this is a proper inclusion. Really, if $E(R) \neq E(R/C)$, then $E(R/C) \setminus E(R) \neq \emptyset$ and there exists an element $p \in E(R/C) \setminus E(R)$ such that $p^i(R/C) = p^{i+1}(R/C)$ for some non-negative integer i. From this equation we have that $p^i = p^{i+1} a + c$ for some $a \in R$ and a suitable $c \in C$. Because $M \subseteq E_2(R) \subseteq E(R)$ and $p \notin E(R)$ we have that p and k_M are coprimes. Let us choose integers t and m such that $pt + mk_M = 1$. Therefore $p^i m k_M = p^{i+1} a m k_M$, because $c k_M = 0$. Then

$$p^i(1 - pt) = p^{i+1} a m k_M$$

and so $p^i \in p^{i+1} R$, i.e. $p \in E(R)$ which is a contradiction. Consequently, $E(R) = E(R/C)$.

Let p be an arbitrary element of M. Then $p^e = p^{e+1} a \quad (e = e(p))$ for a suitable $a \in R$ and hence $p^e(1 - pa) = 0$. Since p^e divides k_M we have $k_M(1 - pa) = 0$. Therefore $1 - pa \in C$ and for this reason $p + C$ is an unit of R/C. Consequently, $M \subseteq E_1(R/C)$.

Clearly, the sets $E(R) \cap E_3(R)$ and $E(R/C) \cap E_3(R/C)$ are empty. Then from the equations $\mathcal{P} = E(R) \cup E_3(R) = E(R/C) \cup E_3(R/C)$ which hold for every commutative ring with unity, and from the equality $E(R) = E(R/C)$ which we have proved above, we obtain that $E_3(R) = E_3(R/C)$. So the proof is complete.

Lemma 3.5. *Let R be a commutative ring with unity, $C = \{c \in R \mid k_M c = 0\}$, and let M be a nonempty finite subset of $E_2(R)$. Then*

$$R \cong R/k_M R \oplus R/C.$$

PROOF. First we show that $R \cong k_M R \oplus C$. Let $x \in R$. Since $k_M R = k_M^2 R$ then we have the equality $k_M x = k_M^2 y$ for a suitable $y \in R$. Then $k_M(x - k_M y) = 0$ and so $(x - k_M y) \in C$ which follows directly from the definition of C. From the decomposition of x, as $x = k_M y + (x - k_M y)$, we have the inclusion $R \subseteq k_M R + C$. The inverse inclusion is evident. Therefore

$$R = k_M R + C.$$

Let $a \in k_M R \cap C$. Then $a = k_M z \quad (z \in R)$ and $k_M a = 0$ because $a \in C$. Since $k_M = k_M^2 v$, $a = k_M z = k_M^2 vz = 0$ for a suitable $v \in R$. This means that $k_M R \cap C = 0$ and therefore

$$R \cong k_M R \oplus C$$

Let us define the homomorphism

$$\phi : R \to R/k_M R \oplus R/C$$

by letting $\phi(a) = (a + k_M R, a + C)$ for $a \in R$. From the above isomorphism it follows that ϕ is a homomorphism of R onto $R/k_M R \oplus R/C$ with zero kernel. So ϕ is an isomorphism which proves the lemma.

Let $O(G, p)$ be the set of those primes for which $G \neq G_p$.

An Abelian group G is called a *p-divisible* if all elements of G are elements of infinite p-height in G, that is $G = G_p$.

Theorem 3.5. *Let R be a commutative ring with unity of characteristic zero. Let G/G' be a torsion group and $E_2(R) \cap O(G, p) = \emptyset$. If G/G' is p-divisible for every $p \in E_3(R)$, then $A^2(RG) = A^3(RG)$.*

PROOF. For an Abelian torsion group G/G' we can write

$$G/G' = P_1 \otimes P_2 \otimes \cdots , \tag{11}$$

where P_i are the p_i-components of G/G' for $i = 1, 2, \ldots$.

Let g_1 and g_2 be arbitrary elements of G. It will be sufficient to show that $(g_1 - 1)(g_2 - 1) \in A^3(RG)$. From (11) it follows that these elements can be written as $g_1 = x_{i_1} x_{i_2} \cdots x_{i_n} c_1$ and $g_2 = y_{j_1} y_{j_2} \cdots y_{j_m} c_2$ where $x_{i_k} G', y_{i_k} G' \in P_{i_k}$ and c_1, c_2 are suitable elements of G'. Let us use the identity (2) repeatedly to $g_1 - 1$ and $g_2 - 1$. Then we have that

$(g_1-1)(g_2-1) \equiv \sum_{i,j}(x_i-1)(y_j-1) \pmod{A^3(RG)}$ because c_1-1, c_2-1 are in $A^2(RG)$. Hence by case 3 of Lemma 3.3 we obtain that

$$(g_1-1)(g_2-1) \equiv \sum_{i}(x_i-1)(y_i-1) \pmod{A^3(RG)}.$$

From Lemma 3.3 (cases 1 and 4) it follows that $(g_1-1)(g_2-1) \in A^3(RG)$ because $E_2(R) \cap O(G,p) = \emptyset$ that is for every $p \in E_2(R)$ we have $G = G_p$. Therefore $A^2(RG) \supseteq A^3(RG)$. The inverse inclusion $A^2(RG) \subseteq A^3(RG)$ is trivial. Consequently $A^2(RG) = A^3(RG)$.

We note that $J_p(R) = \bigcap_{n=1}^{\infty} p^n R$ and $T(R^+)$ is the additive group of the ring R.

Theorem 3.6. *Let R be a commutative ring with unity of characteristic zero. Then the augmentation terminal of G with respect to R is finite if and only if either $G = G'$ or G/G' is a torsion group and*

1) *G/G' p-divisible for every $p \in E_3(R)$*
2) *$E_2(R) \cap O(G,p)$ is a finite set and G/G_p is a finite p-group for every $p \in E_2(R) \cap O(G,p)$.*

Proof. It is known that $A(RG) = A^2(RG)$ if $G = G'$ and we may assume that $G \neq G'$ and

$$\ldots \supset A^{k-1}(RG) \supset A^k(RG) = A^{k+1}(RG) = \ldots = A^{\omega}(RG).$$

If $\overline{R} = R/T(R^+)$ then from the above expression it follows that

$$\ldots \supseteq A^{k-1}(\overline{R}G) \supseteq A^k(\overline{R}G) = A^{k+1}(\overline{R}G) = \ldots = A^{\omega}(\overline{R}G)$$

and therefore $\tau_{\overline{R}}(G)$ is finite and not greater than $\tau_R(G)$.

Let H/G' denote the torsion subgroup of G/G' . Then G/H is a torsion-free Abelian group and by Theorem 2.4 $A^{\omega}(\overline{R}G/H) = 0$, because $\overline{R}^+$ is torsion-free. The inequality $\tau_{\overline{R}}(G) \leq k$ implies that $\tau_{\overline{R}}(G/H) \leq k$ and therefore the ideal $A(\overline{R}G/H)$ is nilpotent. Because the characteristic of $\overline{R}$ is zero, by Theorem 2.1, we have that $G/H = \langle 1 \rangle$ Consequently, $G = H$ and G/G' is a torsion group.

1) Let p be an arbitrary element of $E_3(R)$. Then $p^i R \neq p^{i+1}R$ for all $i \geq 0$ and therefore $J_p(R) \neq R$.

Let $R_p = R/J_p(R)$ and let H/G' be the set of those elements of G/G' which have infinite p-height in G/G' . Then G/H is an Abelian group with no elements of infinite p-height, that is G/H is residually by the class of Abelian p-groups of finite exponent. It is evident that $J_p(R_p) = 0$ and

hence, by Theorem 2.3, $A^{\omega}(R_pG/H) = 0$. From the equation $\tau_{R}(G) = k$ it follows that $\tau_{R_p}(G) \leq \tau_{R}(G) = k$ and so $A^{k}(R_pG/H) = 0$. By Theorem 2.1,the last equation holds only if $H = G$ because the characteristic of R is zero. Consequently, G/G' is p-divisible for all $p \in E_3(R)$ and the proof of statement 1) is complete.

2) Now we suppose that $p \in E_2(R) \cap O(G,p)$. Then $e = e(p) > 0$ and $p^eR \neq R$, which follows directly from the definition on the set $E_2(R)$. If $R_p = R/p^eR$, then from the equalition $\tau_{R}(G) = k$ it follows that the augmentation terminal of G with respect to R_p is finite and not greater than k, i.e.

$$\tau_{R_p}(G) \leq \tau_{R}(G) = k. \tag{12}$$

The characteristic of the ring $R_p = R/p^eR$ is p^e and so, by Theorem 3.1 G/G_p is a finite p-group.

Now we show that $E_2(R) \cap O(G,p)$ is a finite set. Let $p \in E_2(R) \cap O(G,p)$. By Theorem 3.2 $\tau_{R_p}(G) = \tau^{\circ}_{R_p}(A(R_pG/G_p))$. It is clear that $p \leq \tau^{\circ}_{R_p}(A(R_pG/G_p))$. Keeping in mind (12) we obtain

$$p \leq \tau_{R_p}(G) \leq \tau_{R}(G) = k$$

which hold for all $p \in E_2(R) \cap O(G,p)$. It is obvious that this implies the finitenes of the set $E_2(R) \cap O(G,p)$ and the proof of the "if" part our theorem is complete.

Conversely, let G/G' be a torsion group and suppose that 1) and 2) hold. If the set $E_2(R) \cap O(G,p)$ is empty, then by Theorem 3.5 $A^2(RG) = A^3(RG)$ and in this case the proof is complete.

Now suppose that $M = E_2(R) \cap O(G,p) = \{p_1, p_2, \ldots, p_n\}$ is a finite nonempty set and let $k_{M} = p_1^{e_1}p_2^{e_2}\cdots p_n^{e_n}$, where $e_i = e(p_i), p_i \in M$. By Lemma 3.5

$$R \cong R/k_{M}R \oplus R/C,$$

where $C = \{c \in R \,|\, k_{M}c = 0\}$.

Let us define the homomorphism

$$\Theta : RG \to KG \oplus SG$$

by letting

$$\Theta\left(\sum_i (\alpha_i, \beta_i)g_i\right) = \left(\sum_i \alpha_i g_i, \sum_i \beta_i g_i\right)$$

for $a_i \in K, \beta_i \in S, g_i \in G$, where $K = R/k_{M}R, S = R/C$. It is easy to check that Θ is an isomorphism, i.e.

$$RG \cong KG \oplus SG.$$

Clearly

$$A^i(RG) = A^i(KG) \oplus A^i(SG)$$

for all i. Therefore it is enough to show that for some n the following equations

$$A^n(KG) = A^{n+1}(KG) \quad \text{and} \quad A^n(SG) = A^{n+1}(SG)$$

hold. The characteristic of the ring K is $k_M > 0$, so from statement 2) of our theorem and from Theorem 3.3 we have that the augmentation terminal $\tau_K(G)$ of G with respect to K is finite, that is

$$A^n(KG) = A^{n+1}(KG). \tag{13}$$

Theorem 3.4 gives the following correlations

(14)
$$n = \tau_K(G) = \max_{p\in M}\left\{\tau_{K_p}(G)\right\} = \max_{p\in M}\left\{\tau^\circ(A(K_pG/G_p))\right\} \geq \max_{p\in M}\left\{\tau_p(G)\right\},$$

where $K_p = K/p^eK$. Since $p \in M \subseteq O(G,p)$,we have $G \neq G_p$. Therefore

$$n = \tau_K(G) \geq \tau_p(G) \geq 2$$

for all $p \in M$.

It remains to show that $A^n(KG) = A^{n+1}(SG)$. At first we prove that the set $E_2(S) \cap O(G,p)$ is empty. If this is not true then there exists an element p of the set $E_2(S) \cap O(G,p)$. Because the set $E_1(S) \cap E_2(S)$ is empty, $p \notin E_1(S)$. It is obvious that $E_1(R) \subseteq E_1(S) = E_1(R/C)$ and, by Lemma 3.4, $E(R) = E(R/C)$. Then

$$E_2(R) \supseteq E_2(R/C) \quad \text{and } p \in E_2(R) \cap O(G,p) = M.$$

By Lemma 3.4, $M \subseteq E_1(R/C)$ and so $p \in E_1(R/C) = E_1(S)$ which is impossible. Therefore $E_2(R/C) \cap O(G,p) = \emptyset$. By Lemma 3.4, $E_3(R) = E_3(R/C)$ and by statement 1) of our theorem G/G' is p-divisible for all $p \in E_3(R/C) = E_3(S)$. Therefore the group ring SG satisfies the conditions of Theorem 3.5 and consequently,

$$A^2(SG) = A^3(SG).$$

In (13) $n \geq 2$ and consequently from the above equation it follows that

$$A^n(RG) = A^{n+1}(RG). \tag{15}$$

This completes the proof of the theorem.

Theorem 3.7. *Let R be a commutative ring with unity of characteristic zero,$E_2(R) \cap O(G,p) = M$ and let the augmentation terminal $\tau_R(G)$ of G with respect to R be finite. Then M is a finite set, and $\tau_R(G) \leq 2$ if M is empty, and*

$$\tau_R(G) = \max_{p\in M}\left\{\tau_{K_p}(G)\right\} = \max_{p\in M}\left\{\tau^{\circ}(A(K_pG/G_p))\right\} \geq 2$$

otherwise.

PROOF. By Theorem 3.6, M is a finite set. If $M = \emptyset$, then by Theorem 3.5, $\tau_R(G) \leq 2$.

Suppose that $M \neq \emptyset$. From Theorem 3.4 it follows that

$$\tau_K(G) = \max_{p\in M}\left\{\tau_{K_p}(G)\right\} = \max_{p\in M}\left\{\tau^{\circ}(A(K_pG/G_p))\right\} \geq \max_{p\in M}\left\{\tau_p(G)\right\}.$$

From (14) and (15) we have that $\tau_R(G) \leq \tau_K(G)$. It is easy to se that $\tau_R(G) \geq \tau_K(G)$. Consequently $\tau_R(G) = \tau_K(G)$ which proves the theorem.

Remark. Theorems 3.3, 3.4, and 3.6 were announced in [4].

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