Publ. Math. Debrecen
46 / 1-2 (1995), 187–193

On some properties of values of a class of arithmetical functions

By JÓZSEF BUKOR (Nitra) and JÁNOS TÓTH (Nitra)

Euler's totient function φ has the following properties:
For an arbitrary natural number m there exists a natural number n such that

$$\frac{\varphi(n-1)}{\varphi(n)} > m \quad \text{and} \quad \frac{\varphi(n+1)}{\varphi(n)} > m \tag{1}$$

and a natural number n' such that

$$\frac{\varphi(n')}{\varphi(n'-1)} > m \quad \text{and} \quad \frac{\varphi(n')}{\varphi(n'+1)} > m \tag{2}$$

(cf. [2], also [1] pp. 208–209).

The next properties of φ are proved in [3] and are explicit in [4]:
For any two natural numbers m and k there exists a natural number n such that

$$\frac{\varphi(n+i)}{\varphi(n+i-1)} > m \qquad \text{for } i = 1, 2, \ldots, k \tag{3}$$

and a natural number n' such that

$$\frac{\varphi(n'+i-1)}{\varphi(n'+i)} > m \qquad \text{for } i = 1, 2, \ldots, k \tag{4}$$

The same properties (1)–(4) are valid for the function σ, where $\sigma(n)$ is the sum of the natural divisors of a natural number n (cf. [1] p. 246).

It is natural to ask, whether for $\frac{\sigma}{\varphi}$ properties (1)–(4) are still valid. The answer is positive, moreover there exists a class of positive multiplicative arithmetic functions satisfying properties (1)–(4).

Let p_n denote the n–th prime number. We have

Theorem 1. *Let $f(n)$ be a multiplicative arithmetic function satisfying the following conditions:*

(i) $\liminf_{n\to\infty} f(p_n)^n < +\infty$.

(ii) For an arbitrary natural number n, $f(n) \geq 1$ *and for any two primes* p *and* q, $p > q$ *and an arbitrary natural number* α, $f(p) \leq f(p^\alpha) \leq f(q)$.

(iii) The series $\sum_{n=1}^{\infty} (f(p_n) - 1)$ *diverges.*

Then for any positive K *and natural number* l *there is a natural number* k *and a positive constant* c *for which*

$$f(k+2i) < c \quad \text{and} \quad f(k+2i+1) > K \cdot c \qquad \text{for} \quad i = 0, 1, \ldots, l. \tag{5}$$

Proof. For a fixed l there is a natural number m such that $p_m > 2l$ and $l \mid m$.
Let $c_1 = f(2)^{m-1}$. It is known that the condition *(iii)* is equivalent with the statement $\prod_{n=1}^{\infty} f(p_n) = +\infty$. From the conditions *(i)–(iii)* follows that there exists a positive number c_2 such that for infinitely many n's $f(p_n)^n < c_2$. Therefore there is a natural number n such that

$$f(p_n)^n < c_2 \qquad \text{and} \qquad \prod_{\substack{j \equiv 0 \pmod{l} \\ m<j<n}} f(p_j) > K \cdot c_1 \cdot c_2 \tag{6}$$

Let us consider the system of congruences

$$x \equiv 2l - 2i \left(\mathrm{mod} \prod_{\substack{j \equiv i \pmod{l} \\ m \leq j \leq n}} p_j \right), \qquad i = 0, 1, \ldots, l-1.$$

By the Chinese Remainder Theorem there exists a number x_0 satisfying these congruences such that

$$0 < x_0 < \prod_{j=m}^{n} p_j. \tag{7}$$

Hence we get a sequence of natural numbers $a_0, a_1, \ldots, a_{l-1}$ for which

$$x_0 = 2l - 2i + a_i \cdot \prod_{\substack{j \equiv i \pmod{l} \\ m \leq j \leq n}} p_j, \qquad i = 0, 1, \ldots, l-1. \tag{8}$$

Put $k = x_0 - (2l+1)$. It is easy to see that $k > 0$. Let us estimate the values $f(k+2i)$ and $f(k+2i+1)$ for $i \in \{0, 1, \ldots, l\}$.

Clearly $k+2i \le k+2l$. From (7) and (8) we have $k+2i < \prod_{j=m}^{n} p_j$ and $k+2i$ is not divisible by the prime numbers p_j $(j = m, m+1, \dots, n)$. Let $\prod_{j=1}^{m-1} p_j^{\alpha_j} \prod_{j=1}^{r} q_j^{\beta_j}$ be a factorization of $k+2i$ into prime numbers, where $\alpha_j \ge 0$ $(j = 1, 2, \dots, m-1)$ and $\beta_j \ge 1$ $(j = 1, 2, \dots, r)$. Then $q_j > p_m$ $(j = 1, 2, \dots, r)$ and clearly $r < n$. Thus using *(ii)* we have

$$f(k+2i) = \prod_{j=1}^{m-1} f(p_j^{\alpha_j}) \prod_{j=1}^{r} f(q_j^{\beta_j}) < f(2)^{m-1} \cdot f(p_n)^n.$$

Therefore

$$f(k+2i) < c_1 \cdot c_2$$

and *(ii)*, (8), (6) together yield

$$f(k+2i+1) = f\left(a_i \prod_{\substack{j \equiv i \pmod{l} \\ m \le j \le n}} p_j\right) \ge$$

$$\ge \prod_{\substack{j \equiv i \pmod{l} \\ m \le j \le n}} f(p_j) \ge \prod_{\substack{j \equiv 0 \pmod{l} \\ m < j < n}} f(p_j) > K \cdot c_1 \cdot c_2. \quad \square$$

In view of Theorem 1 we immediately have

Corollary 1. *Let $f(n)$ be a multiplicative function satisfying conditions (i)–(iii) of Theorem 1. Then for any positive number K and natural number l there exists a natural number k such that*

$$\frac{f(k+2i+1)}{f(k+2i)} > K \quad \text{and} \quad \frac{f(k+2i+1)}{f(k+2i+2)} > K \quad \text{for} \quad i = 0, 1, \dots, l-1.$$

Remark 1. Note that the functions $\frac{\sigma(n)}{\varphi(n)}$, $\frac{\sigma(n)}{n}$, $\frac{n}{\varphi(n)}$ and the multiplicative function $h(n) = \prod_{k : p_k \mid n} (1 + \frac{1}{k})$ satisfy conditions *(i)–(iii)*. Properties (1) and (2) can be visualized on the diagram of φ as a "valley" and a "peak", respectively. Thus by Corollary 1 we can say that the diagrams of the functions $\frac{\sigma(n)}{\varphi(n)}$, $\frac{\sigma(n)}{n}$, $\frac{n}{\varphi(n)}$ and $h(n)$ contain arbitrary number of arbitrary big "peaks" and "valleys", moreover alternately.

Remark 2. It is easy to verify that if the multiplicative functions f, g satisfy the conditions *(i)–(iii)* then $c \cdot f^{\alpha} \cdot g^{\beta}$ $(c \ge 1, \alpha, \beta \ge 0$, not both zero) satisfies them too. Consequently, there are infinitely many multiplicative functions satisfying conditions *(i)–(iii)*.

Theorem 2. *Let $f(n)$ be a multiplicative function satisfying conditions (i)–(iii) of Theorem 1. Then for any given positive K and natural number l there exists a natural number k, such that*

$$\frac{f(k+i+1)}{f(k+i)} > K \qquad \text{for } i = 1, 2, \dots, l-1 \tag{9}$$

and a natural number k, such that

$$\frac{f(k+i)}{f(k+i+1)} > K \qquad \text{for } i = 1, 2, \dots, l-1. \tag{10}$$

Proof. We prove (9), the proof of (10) is analogous.
Set $L = \binom{l+1}{2}$, $c_1 = f(2)^{2L}$. In virtue of conditions *(i)–(iii)* we get that for some natural number n

$$\prod_{\substack{j \equiv n \pmod L \\ 2L \le j \le n}} f(p_j) > c_1 \cdot c_2 \cdot K \quad \text{and} \quad f(p_n)^n < c_2. \tag{11}$$

Let m be the smallest natural number with the properties $p_m > l$ and $m \equiv n \pmod L$. Clearly $m < 2L$. Put $n_1 = \frac{n-m}{L}$. Let us consider the system of congruences

$$\begin{aligned}
x + 1 &\equiv 0 \left(\mathrm{mod} \prod_{j=1}^{n_1} p_{m+jL} \right) \\
x + 2 &\equiv 0 \left(\mathrm{mod} \prod_{j=1}^{n_1} p_{m+jL-1} \cdot p_{m+jL-2} \right) \\
x + 3 &\equiv 0 \left(\mathrm{mod} \prod_{j=1}^{n_1} p_{m+jL-3} \cdot p_{m+jL-4} \cdot p_{m+jL-5} \right) \\
&\dots \\
x + l &\equiv 0 \left(\mathrm{mod} \prod_{j=1}^{n_1} p_{m+jL-\binom{l}{2}} \cdot p_{m+jL-\binom{l}{2}-1} \cdots p_{m+jL-\binom{l}{2}-(l-1)} \right)
\end{aligned} \tag{12}$$

From the Chinese Remainder Theorem follows the existence of a natural number k, $0 < k < \prod_{j=m+1}^{n} p_j$ satisfying the system of congruences, i.e. there exist natural numbers $a_1, a_2, \dots, a_l$ for which

$$k + i = a_i \prod_{j=1}^{n_1} \prod_{t=0}^{i-1} p_{m+jL-\binom{i}{2}-t} \qquad i = 1, 2, \dots, l. \tag{13}$$

For simplicity we use the notation $h(j,i,t)$ instead of $m+Lj-\binom{i}{2}-t$. Let a factorization of a_i be of the form

$$a_i = \prod_{j=1}^{m} p_j^{\alpha_j} \cdot \prod_{j=1}^{n_1} \prod_{t=0}^{i-1} p_{h(j,i,t)}^{\alpha_{h(j,i,t)}} \cdot \prod_{j=1}^{r} q_j^{\beta_j},$$

where $\alpha_j \geq 0$ for $j = 1,2,\ldots,n$ and $\beta_j \geq 1$ for the primes q_j $(j = 1,2,\ldots,r)$ which are greater than p_n. Clearly $r < n$. According to *(ii)* we have

$$\begin{aligned} f(k+i) &= f\left(\prod_{j=1}^{m} p_j^{\alpha_j} \cdot \prod_{j=1}^{n_1} \prod_{t=0}^{i-1} p_{h(j,i,t)}^{\alpha_{h(j,i,t)}+1} \cdot \prod_{j=1}^{r} q_j^{\beta_j}\right) \geq \\ &\geq \prod_{j=1}^{n_1} \prod_{t=0}^{i-1} f\left(p_{h(j,i,t)}^{\alpha_{h(j,i,t)}+1}\right). \end{aligned} \tag{14}$$

The inequalities

$$f\left(\prod_{j=1}^{m} p_j^{\alpha_j}\right) < f(2)^m < f(2)^{2L} = c_1 \quad \text{and} \quad f\left(\prod_{j=1}^{r} q_j^{\beta_j}\right) < f(p_n)^n < c_2$$

imply

$$\begin{aligned} f(k+i) &= f\left(\prod_{j=1}^{m} p_j^{\alpha_j}\right) \cdot \prod_{j=1}^{n_1} \prod_{t=0}^{i-1} f\left(p_{h(j,i,t)}^{\alpha_{h(j,i,t)}+1}\right) \cdot f\left(\prod_{j=1}^{r} q_j^{\beta_j}\right) < \\ &< c_1 \cdot c_2 \cdot \prod_{j=1}^{n_1} \prod_{t=0}^{i-1} f\left(p_{h(j,i,t)}^{\alpha_{h(j,i,t)}+1}\right). \end{aligned} \tag{15}$$

Using inequalities (14), (15) and (11) we have

$$\frac{f(k+i+1)}{f(k+i)} > \frac{1}{c_1 c_2} \prod_{j=1}^{n_1} \left(\prod_{t=0}^{i-1} \frac{f\left(p_{h(j,i+1,t)}^{\alpha_{h(j,i+1,t)}+1}\right)}{f\left(p_{h(j,i,t)}^{\alpha_{h(j,i,t)}+1}\right)} \cdot f\left(p_{h(j,i+1,i)}^{\alpha_{h(j,i+1,i)}+1}\right) \right) >$$

$$> \frac{1}{c_1 c_2} \prod_{j=1}^{n_1} f(p_{m+jL}) > K$$

for $i = 1,2,\ldots,l-1$ which proves the theorem. □

With a slight modification of the proofs of Theorem 1 and Theorem 2 we can prove the following

Corollary 2. *Let $f(n)$ be a multiplicative function satisfying the conditions (i)–(iii) of Theorem 1. Then for any positive number K and natural number l there exists a natural number k for which*

$$(-1)^{i+1}(f(k+i)-f(k+i-1))>K,$$
$$(-1)^{i+1}(f(k+i)-f(k+i+1))>K,\quad (i=0,1,\dots,l-1)$$

and there exists a natural number k' such that

$$f(k'+i+1)-f(k'+i)>K,\quad (i=0,1,\dots,l-1)\,.$$

Corollary 3. *Let $g(n)$ be a multiplicative arithmetic function satisfying the following conditions:*

(i) $\limsup_{n\to\infty} g(p_n)^n>0$.

(ii) For an arbitrary natural number n, $g(n)\le 1$ and for any two primes p and q, $p>q$ and an arbitrary natural number α, $g(p)\ge g(p^\alpha)\ge g(q)$.

(iii) The series $\sum_{n=1}^{\infty}\frac{1-g(p_n)}{g(p_n)}$ *diverges.*

Then, for any positive K and natural number l there are natural numbers k and k' for which

$$\frac{g(k+2i+1)}{g(k+2i)}>K,\quad \frac{g(k+2i+1)}{g(k+2i+2)}>K\quad \textit{and}$$
$$\frac{g(k'+i+1)}{g(k'+i)}>K,\quad (i=1,2,\dots,l-1).$$

Proof. The proof follows immediately from the fact that the function $f=\frac{1}{g}$ satisfies conditions *(i)–(iii)* of Theorem 1. □

Remark 3. From [5] follows that if $f(p_n)>1$, $\lim_{n\to\infty} f(p_n)=1$ and $\sum_{n=1}^{\infty}(f(p_n)-1)$ diverges for a multiplicative function f, then the set $\{f(n);n\in\mathbb{N}\}$ is dense in $(1,+\infty)$. It is an easy exercise to show that conditions *(i)–(iii)* of Theorem 1 imply the above conditions. Therefore the set of values of an arithmetic function satisfying conditions *(i)–(iii)* from Theorem 1 is a dense set in $(1,+\infty)$.

References

[1] W. Sierpiński, Teoria liczb II, *PWN*, *Warszawa*, 1959.
[2] A. Schinzel, W. Sierpiński, Sur quelques propriétés des fonctions $\varphi(n)$ et $\sigma(n)$, *Bull. Acad. Polon. Sci.* **III 2** (1954), 463–466.
[3] A. Schinzel, Qualques théorèmes sur les fonctions $\varphi(n)$ et $\sigma(n)$, *Bull. Acad. Polon. Sci.* **III 2** (1954), 467–469.
[4] P. Erdős, A. Schinzel, Distributions of the values of some arithmetical functions, *Acta Arith.* **6** (1961), 473–485.
[5] Š. Porubský, Über die Dichtigkeit der Werte multiplikativer Funktionen, *Math. Slovaca* **29** (1979), 69–72.

JÓZSEF BUKOR
COLLEGE OF EDUCATION
DEPARTMENT OF MATHEMATICS
FARSKÁ 19.
949 01 NITRA
SLOVAKIA
E-MAIL: BUKOR@UNITRA.SK

JÁNOS TÓTH
COLLEGE OF EDUCATION
DEPARTMENT OF MATHEMATICS
FARSKÁ 19.
949 01 NITRA
SLOVAKIA
E-MAIL: TOTH@UNITRA.SK