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Posner's first theorem and related identities for semiprime rings

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Abstract. We generalize Posner's first theorem and related identities to arbitrary semiprime rings. For instance, Posner's first theorem for semiprime rings is proved as follows: Let R be a semiprime ring with extended centroid C, and let $\delta, D\colon R \to R$ be derivations. Then δD is also a derivation if and only if there exist orthogonal idempotents $e_1, e_2, e_3 \in C$, $e_1{+}e_2{+}e_3 = 1$, and $\lambda \in C$ such that $e_1 D = 0$, $e_2\delta = 0$ and $e_3\big(\delta{-}\lambda D\big) = 0$, where e_2R is 2-torsion free and $2e_3R = 0$.

1. Results

Throughout, unless specially stated, R denotes a semiprime ring; that is, for $a \in R$, $aRa = 0$ implies $a = 0$. We let Q denote the Martindale symmetric ring of quotients of R. The center, denoted by C, of Q is called the extended centroid of R. The center C is a commutative regular self-injective ring. Moreover, C is a field if and only if R is a prime ring (that is, for $a, b \in R$, $aRb = 0$ implies that either $a = 0$ or $b = 0$). We refer the reader to the book [2] for details.

An additive map $D\colon R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$. A classical theorem, which is often called Posner's first theorem, states that if δ, D are derivations of a prime ring R of characteristic not 2 such that δD is also a derivation, then one of δ and D is zero (see [16, Theorem 1]). Let $Z(R)$ denote the center of the ring R. A map $f\colon R \to R$ is called centralizing (resp. commuting) map if $[f(x), x] \in Z(R)$ (resp. $[f(x), x] = 0$)

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for all $x \in R$. POSNER's second theorem states that a prime ring R must be commutative if it admits a nonzero centralizing derivation (see [16, Theorem 2]). We refer the reader to [6, Theorem 1] and [11, Theorem 3] for the one-sided version of Posner's second theorem for semiprime rings. During a few decades, a lot of results concerning additive (centralizing) commuting maps on prime or semiprime rings have been obtained in the literature (see [11], [4], [5], and references therein). In the paper, applying the theory of orthogonal completion of semiprime rings, we give a uniform approach to deal with Posner's first theorem and related identities for semiprime rings.

It is known that a derivation of a semiprime ring R can be uniquely extended to a derivation of its Martindale symmetric ring of quotients Q, and that the idempotents of C are constants of derivations. Moreover, R and Q satisfy the same differential identities (see [13, Theorem 3]). Thus, given a differential identity on R, we try to find a suitable decomposition of Q in terms of finitely many orthogonal central idempotents $e_1, \ldots, e_k$ in C with sum 1, i.e., $Q = e_1Q \oplus \cdots \oplus e_kQ$, and then get the structure of the considered differential identity on e_iQ for $i = 1, \ldots, k$. The second section then gives all preliminary lemmas to find some suitable decompositions of Q.

The first goal of the paper is to generalize Posner's first theorem to arbitrary semiprime rings.

Theorem 1. *Let R be a semiprime ring, and let $\delta, D \colon R \to R$ be nonzero derivations. Then the following are equivalent:*

(1) *δD is a derivation.*

(2) *There exist orthogonal idempotents $e_1, e_2, e_3 \in C$, $e_1+e_2+e_3 = 1$, and $\lambda \in C$ such that $e_1D = 0$, $e_2\delta = 0$ and $e_3\big(\delta - \lambda D\big) = 0$, where e_2R is 2-torsion free and $2e_3R = 0$.*

(3) *There exist orthogonal idempotents $e_1, e_2, e_3 \in C$, $e_1+e_2+e_3 = 1$, and $\lambda \in C$ such that $e_1\delta = 0$, $e_2D = 0$ and $e_3\big(D - \lambda\delta\big) = 0$, where e_2R is 2-torsion free and $2e_3R = 0$.*

As an immediate consequence of Theorem 1, we have the following (see also [11, Theorem 2]).

Corollary 2. *Let R be a 2-torsion free semiprime ring, and let $\delta, D \colon R \to R$ be nonzero derivations. Then δD is also a derivation if and only if there exist orthogonal idempotents $e_1, e_2 \in C$, $e_1 + e_2 = 1$, such that $e_1\delta = 0$ and $e_2D = 0$.*

We remark that the corollary above is also a direct consequence of [11, Theorem 2] and Lemma 8 below.

By an involution $*$ on a ring R we mean that $*$ is an anti-automorphism of R with period 2. An ideal I of R is called a $*$-ideal if $I^* = I$. The ring R is called a $*$-prime ring if any product of two nonzero $*$-ideals of R is nonzero. This is equivalent to saying that, for $a, b \in R$, $aRb = 0$ and $aRb^* = 0$ imply that either $a = 0$ or $b = 0$. Clearly, the characteristic of any $*$-prime ring is well-defined. In a recent paper, Ashraf and Siddeeque proved that if R is a $*$-prime ring, $\mathrm{char}\,(R) \neq 2$, with δD a derivation, where $\delta, D\colon R \to R$ are derivations, then at least one of δ and D is zero if one of δ and D commutes with $*$ (see [1, Corollary 3.1]).

We remark that every $*$-prime ring is a semiprime ring. Let δ be a derivation of a semiprime ring R with involution $*$. We note that if δ commutes with $*$, then $\delta(R)^* = \delta(R)$, but the reverse is not in general true. The following is a generalization of [1, Corollary 3.1].

Corollary 3. *Let R be a $*$-prime ring, where $*$ is an involution on R, and let $\delta, D\colon R \to R$ be derivations such that δD is also a derivation.*

(1) *Suppose that $\delta(R)^* = \delta(R) \neq 0$. Then there exist orthogonal idempotents $f_1, f_2 \in C$, $f_1 + f_2 = 1$, and $\lambda \in C$ such that $f_1 D = 0$ and $f_2(D - \lambda\delta) = 0$, where $f_1 R$ is 2-torsion free and $2f_2R = 0$.*

(2) *Suppose that $D(R)^* = D(R) \neq 0$. Then there exist orthogonal idempotents $f_1, f_2 \in C$, $f_1 + f_2 = 1$, and $\lambda \in C$ such that $f_1\delta = 0$ and $f_2(\delta - \lambda D) = 0$, where $f_1 R$ is 2-torsion free and $2f_2R = 0$.*

The following is another differential identity related to Posner's first theorem (see [12, Theorem 4] and [8, Theorem 1]).

Theorem 4. *Let R be a prime ring, and let $\delta, D\colon R \to R$ be nonzero derivations. Suppose that $\dim_C RC > 4$. Then $\delta D(x) \in Z(R)$ for all $x \in R$ if and only if $\delta = \lambda D$ for some $\lambda \in C$ and $\mathrm{char}\,(R) = 2$.*

Let $\mathbb{Z}\{\widehat{X}\}$ be the free associative $\mathbb{Z}$-algebra in noncommutative indeterminates $X_1, X_2, \ldots$, where $\mathbb{Z}$ is the ring of integers and $\widehat{X} := \{X_1, X_2, \ldots\}$. Given a polynomial $f(X_1, \ldots, X_t) \in \mathbb{Z}\{\widehat{X}\}$ which has zero constant term, a semiprime ring R is called an f-ring if R satisfies f and is called a faithful f-free ring if every nonzero ideal of R does not satisfy f. As usual, the zero ring is defined to be faithful f-free. We let

$$S_m(X_1, \ldots, X_m) := \sum_{\sigma \in \mathrm{Sym}(m)} (-1)^\sigma X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(m)},$$

the standard polynomial of degree m in noncommutative indeterminates X_i for $i = 1, \ldots, m$, where $\mathrm{Sym}(m)$ denotes the permutation group on the set $\{1, 2, \ldots, m\}$.

It is known that a prime ring R is an S_{2n}-ring if and only if $\dim_C RC \leq n^2$ (see [17, Corollary 1] and [7, Theorem p. 17]). The second goal of the paper is to generalize the theorem above to arbitrary semiprime rings.

Theorem 5. *Let R be a semiprime ring, and let $\delta, D\colon R \to R$ be nonzero derivations. Suppose that $\delta D(x) \in Z(R)$ for all $x \in R$. Then there exist orthogonal idempotents $e_1, e_2, e_3, e_4 \in C$, $e_1 + e_2 + e_3 + e_4 = 1$, and $\lambda \in C$ such that $e_1 D = 0$, $e_2\delta = 0$, $e_3\delta = e_3\lambda D$, and both $2e_4R = 0$ and e_4R is an S_4-ring, where e_2R is 2-torsion free, e_3R is faithful S_4-free.*

Lanski proved the following (see [10, Theorem 4]).

Theorem 6 (Lanski 1992)**.** *Let R be a noncommutative prime ring, and let $\delta, D\colon R \to R$ be nonzero derivations. Suppose that $[\delta(x), D(x)] \in Z(R)$ for all $x \in R$. Then $\delta = \lambda D$ for some $\lambda \in C$ except when $\dim_C RC = 4$ and $\mathrm{char}\,(R) = 2$.*

The final goal of the paper is to generalize the theorem above to semiprime rings.

Theorem 7. *Let R be a semiprime ring, and let $\delta, D\colon R \to R$ be nonzero derivations. Suppose that $\big[\delta(x), D(x)\big] \in Z(R)$ for all $x \in R$. Then there exist orthogonal idempotents $e_1, e_2, e_3, e_4 \in C$, $e_1 + e_2 + e_3 + e_4 = 1$, and $\lambda \in C$ such that e_1R is commutative, $e_2D = 0$, $e_3\delta = e_3\lambda D$, and both $2e_4R = 0$ and e_4R is an S_4-ring.*

2. Preliminary results

Recall that R is always a semiprime ring with extended centroid C. The set $\mathbf{B}$ of all idempotents of C forms a Boolean algebra with respect to the operations $e \dot{+} h := e + h - 2eh$ and $e \cdot h := eh$, for all $e, h \in \mathbf{B}$. It is complete with respect to the partial order $e \leq h$ (defined by $eh = e$) in the sense that any subset S of $\mathbf{B}$ has a supremum $\bigvee S$ and an infimum $\bigwedge S$.

We call $\{e_\nu \in \mathbf{B} \mid \nu \in \Lambda\}$ an orthogonal subset of $\mathbf{B}$ if $e_\nu e_\mu = 0$ for $\nu \neq \mu$ and a dense subset of $\mathbf{B}$ if $\sum_{\nu\in\Lambda} e_\nu C$ is an essential ideal of C. A subset T of Q, where $0 \in T$, is called orthogonally complete in the following sense: Given any dense orthogonal subset $\{e_\nu \in \mathbf{B} \mid \nu \in \Lambda\}$ of $\mathbf{B}$, there exists a one-one correspondence between T and the direct product $\prod_{\nu\in\Lambda} Te_\nu$ via the map

$$x \mapsto \langle xe_\nu\rangle \in \prod_{\nu\in\Lambda} Te_\nu, \quad \text{for } x \in T.$$

Therefore, given any subset $\{a_\nu \in T \mid \nu \in \Lambda\}$, there exists a unique $a \in T$ such that $a \mapsto \langle a_\nu e_\nu \rangle$. The element a is written as $\sum_{\nu\in\Lambda}^{\perp} a_\nu e_\nu$ and is characterized by the property that $ae_\nu = a_\nu e_\nu$ for all $\nu \in \Lambda$. In particular, if T is a subring of Q, then the one-to-one correspondence is an isomorphism.

In view of [2, Proposition 3.1.10], Q is orthogonally complete. Moreover, P is a minimal prime ideal of Q if and only if $P = \mathbf{m}Q$, for some $\mathbf{m} \in \mathrm{Spec}(\mathbf{B})$, the spectrum of $\mathbf{B}$ (i.e., the set of all maximal ideals of $\mathbf{B}$) (see [2, Theorem 3.2.15]). In particular, it follows from the semiprimeness of Q that $\bigcap_{\mathbf{m}\in\mathrm{Spec}(\mathbf{B})} \mathbf{m}Q = 0$. We refer the reader to the book [2] for details.

Given an ideal I of R, for $q \in R$ we have $qI = 0$ if and only if $Iq = 0$. Thus, $\mathrm{Ann}_R(I) := \{q \in R \mid qI = 0\}$ is an ideal of R. An ideal J of R is called an annihilator ideal of R if $J = \mathrm{Ann}_R(I)$ for some ideal I of R. An ideal J of R is called essential if $\mathrm{Ann}_R(J) = 0$. This is equivalent to saying that I is an essential right ideal of R. The following is well-known in the literature (see, for instance, [14, Lemma 2.10]).

Lemma 8. *Every annihilator ideal of Q is generated by one central idempotent.*

We refer the reader to [15, Lemma 2.1] for the following lemma.

Lemma 9. *Let R be $n!$-torsion free, where n is a positive integer. Then* $\mathrm{char}\,(Q/\mathbf{m}Q) = 0$ *or a prime $p > n$ for any* $\mathbf{m} \in \mathrm{Spec}(\mathbf{B})$.

Lemma 10. *There exists an idempotent $e \in C$ such that eR is an S_4-ring and $(1-e)R$ is faithful S_4-free.*

Proof. In view of [9, Theorem 2.2], there exists $e \in \mathbf{B}$ such that eQ is an S_4-ring and $(1-e)Q$ is a faithful f-free ring. Clearly, this implies that eR is an S_4-ring and $(1-e)R$ is a faithful f-free ring. $\square$

We refer the reader to [9, Theorem 2.3] for the following.

Lemma 11. *Let $f(X_1, \ldots, X_t) \in \mathbb{Z}\{\hat{X}\}$, where $f(X_1, \ldots, X_t)$ has zero constant term. Then R is faithful f-free iff $Q/\mathbf{m}Q$ does not satisfy f for any* $\mathbf{m} \in \mathrm{Spec}(\mathbf{B})$.

3. Proofs

Throughout, unless specially stated, R always denotes a semiprime ring with extended centroid C. The following due to Bergen is a generalization of Posner's first theorem for prime rings R without the assumption $\mathrm{char}\,(R) \neq 2$ (see [3, Theorem 4.6]).

Theorem 12 (Bergen 1981)**.** *Let R be a prime ring, and let $\delta, D\colon R \to R$ be derivations with $D \neq 0$. Suppose that δD is also a derivation. Then $\delta = \lambda D$, for some $\lambda \in C$.*

It is known that every derivation $D\colon R \to R$ can be uniquely extended to a derivation, denoted by the same D also, of Q. Clearly, $D(e) = 0$ for all $e \in \mathbf{B}$. Given $\mathbf{m} \in \mathrm{Spec}(\mathbf{B})$, since $D(\mathbf{m}Q) \subseteq \mathbf{m}Q$, D canonically induces a derivation, say $\overline{D}$, of $Q/\mathbf{m}Q$. That is, $\overline{D}(\overline{x}) = \overline{D(x)}$ for all $x \in Q$, where $\overline{x} := x + \mathbf{m}Q \in Q/\mathbf{m}Q$.

Lemma 13. *Let $\delta, D\colon R \to R$ be derivations.*

(1) *Let $\Sigma = \{f \in \mathbf{B} \mid f\big(\delta - \lambda D\big) = 0$ for some $\lambda \in C\}$. Then Σ is an ideal of $\mathbf{B}$.*

(2) *Suppose that, given any $\mathbf{m} \in \mathrm{Spec}(\mathbf{B})$, there exists $\lambda_{\mathbf{m}} \in C$ such that $\delta(x) - \lambda_{\mathbf{m}} D(x) \in \mathbf{m}Q$ for all $x \in Q$. Then $\delta = \lambda D$ for some $\lambda \in C$.*

Proof. (1) Indeed, if $g \leq f$ in $\mathbf{B}$ and $f \in \Sigma$, it is clear that $g \in \Sigma$. Given $f, g \in \Sigma$, we claim that $f + g - fg \in \Sigma$. Since $f + g - fg = f + g(1 - f)$ and $f, g(1 - f) \in \Sigma$, we may assume from the start that $fg = 0$. Take $\lambda, \mu \in C$ such that $f\big(\delta - \lambda D\big) = 0$ and $g\big(\delta - \mu D\big) = 0$. Then $(f + g)\big(\delta - (\lambda f + \mu g)D\big) = 0$, implying that $f + g \in \Sigma$, as asserted.

(2) Let Σ be the set defined in (1). If $1 \in \Sigma$, then we are done. Suppose on the contrary that $1 \notin \Sigma$. By Zorn's lemma, an $\mathbf{m} \in \mathrm{Spec}(\mathbf{B})$ exists such that $\Sigma \subseteq \mathbf{m}$. By assumption, there exists $\lambda_{\mathbf{m}} \in C$ such that $\delta(x) - \lambda_{\mathbf{m}} D(x) \in \mathbf{m}Q$, for all $x \in Q$. Since $\{\delta(x) - \lambda_{\mathbf{m}} D(x) \mid x \in Q\}$ is an orthogonally complete subset of Q, it follows from [2, Proposition 3.1.11] that $f\big(\delta - \lambda_{\mathbf{m}} D\big) = 0$ for some $f \in \mathbf{B} \setminus \mathbf{m}$. This is a contradiction, as $f \in \Sigma$. $\square$

Lemma 14. *Let $\delta, D\colon R \to R$ be derivations. Suppose that $\delta D\colon R \to R$ is a derivation, and that $eD \neq 0$ (resp. $e\delta \neq 0$) for any nonzero $e \in \mathbf{B}$. Then $\delta = \lambda D$ (resp. $D = \lambda\delta$) for some $\lambda \in C$.*

Proof. We only give the proof of the case that $eD \neq 0$ for any $0 \neq e \in \mathbf{B}$, because the another case has an analogous argument. Let

$$\Sigma = \{f \in \mathbf{B} \mid f\big(\delta - \lambda D\big) = 0 \text{ for some } \lambda \in C\}.$$

By Lemma 13, Σ is an ideal of $\mathbf{B}$. If $1 \in \Sigma$, then we are done. Suppose on the contrary that $1 \notin \Sigma$. Then there exists $\mathbf{m} \in \mathrm{Spec}(\mathbf{B})$ such that $\Sigma \subseteq \mathbf{m}$.

We first extend derivations $\delta, D\colon R \to R$ to derivations, denoted by the same δ, D, respectively, of Q. Since R and Q satisfy the same differential identities (see [13, Theorem 3]), δD is also a derivation of Q. Note that $\mathbf{m}Q$ is invariant under any derivation of Q. Denote by $\overline{\delta}$, $\overline{D}$ the derivations of $Q/\mathbf{m}Q$ induced canonically by δ, D, respectively. Then $\overline{\delta}\,\overline{D}$ is also a derivation of $Q/\mathbf{m}Q$. Note that $C + \mathbf{m}Q/Q$ is equal to the extended centroid of $Q/\mathbf{m}Q$. In view of Theorem 12, either $\overline{D} = 0$ or $\overline{\delta} = \overline{\lambda_{\mathbf{m}}}\,\overline{D}$, for some $\lambda_{\mathbf{m}} \in C$.

Suppose first that $\overline{D} = 0$. Then $D(Q) \subseteq \mathbf{m}Q$. Since $D(Q)$ is an orthogonally complete subset of Q, by [2, Proposition 3.1.11] there exists $e \in \mathbf{B} \setminus \mathbf{m}$ such that $eD(Q) = 0$, a contradiction. Thus, the latter case holds. That is, $\delta(x) - \lambda_{\mathbf{m}} D(x) \in \mathbf{m}Q$ for all $x \in Q$. Since the subset $\{\delta(x) - \lambda_{\mathbf{m}} D(x) \mid x \in Q\}$ is orthogonally complete, it follows from [2, Proposition 3.1.11] that there exists $g \in \mathbf{B} \setminus \mathbf{m}$ such that $g\big(\delta(x) - \lambda_{\mathbf{m}} D(x)\big) = 0$ for all $x \in Q$. This implies that $g \in \Sigma$, contradicting the fact that $\Sigma \subseteq \mathbf{m}$. □

Lemma 15. *Let $\delta, D\colon R \to R$ be nonzero derivations such that δD is also a derivation. Then the following hold:*

(1) *There exist orthogonal idempotents $e_1, e_2 \in C$, $e_1 + e_2 = 1$, and $\lambda \in C$ such that $e_1 D = 0$ and $e_2\big(\delta - \lambda D\big) = 0$.*

(2) *There exist orthogonal idempotents $e_1, e_2 \in C$, $e_1 + e_2 = 1$, and $\lambda \in C$ such that $e_1 \delta = 0$ and $e_2\big(D - \lambda\delta\big) = 0$.*

Proof. We only give the proof of (1). As before, δ and D can be uniquely extended to derivations, denoted by the same δ and D, respectively, of Q. Moreover, $\delta D\colon Q \to Q$ is also a derivation. In view of Lemma 8, $\mathrm{Ann}_Q(QD(Q)Q) = e_1 Q$, for some $e_1 = e_1^2 \in C$. Let $e_2 := 1 - e_1$, $D_2 := e_2 D$ and $\delta_2 := e_2\delta$. Then $D_2, \delta_2\colon e_2Q \to e_2Q$ are derivations such that $\delta_2 D_2$ is also a derivation.

Note that e_2C is the extended centroid of e_2Q, and $e_2\mathbf{B}$ is the complete Boolean algebra of all idempotents in e_2C. Let $h \in e_2\mathbf{B}$ be such that $hD_2(e_2Q) = 0$. Then $hD(Q) = 0$, implying that $h \in e_1Q$, and so $h = 0$. By Lemma 14, there exists $\lambda \in C$ such that $e_2\big(\delta - \lambda D\big) = 0$. □

Lemma 16. *Let R be 2-torsion free, $\lambda \in C$, and let $D\colon R \to R$ be a derivation. Suppose that the map $x \mapsto \lambda D^2(x)$ for $x \in Q$ is a derivation. Then $\lambda D = 0$.*

Proof. Let $x, y \in Q$. Then $\lambda D^2(xy) = \lambda\big(D^2(x)y + 2D(x)D(y) + xD^2(y)\big)$. On the other hand, since the map $x \mapsto \lambda D^2(x)$ for $x \in Q$ is a derivation, $\lambda D^2(xy) = \lambda D^2(x)y + x\lambda D^2(y)$. Thus, $2\lambda D(x)D(y) = 0$. Replacing y by yx, and using the 2-torsion freeness of Q, we get $\lambda D(x)yD(x) = 0$. The semiprimeness of Q gets that $\lambda D(x) = 0$, for all $x \in Q$. □

Proof of Theorem 1. We only give the proof of "(1) $\Leftrightarrow$ (2)" because that of (1) $\Leftrightarrow$ (3) is similar.

"(1) $\Longrightarrow$ (2)": In view of Lemma 15, there exist orthogonal idempotents $e_1, f \in C$, $e_1 + f = 1$, and $\lambda \in C$ such that $e_1 D = 0$ and $f\big(\delta - \lambda D\big) = 0$. By Lemma 8, there exists an idempotent $e_2 \in fC$ such that $\mathrm{Ann}_{fQ}(2fQ) = (f - e_2)Q$. Set $e_3 := f - e_2$. Then e_1, e_2, e_3 are orthogonal idempotents in C with $e_1 + e_2 + e_3 = 1$, and $e_3\big(\delta - \lambda D\big) = 0$. Moreover, $2e_3 Q = 0$, and $e_2 Q$ are 2-torsion free.

Since $f\big(\delta - \lambda D\big) = 0$, we have $e_2\delta = e_2 f\delta = \lambda e_2 D$, and so $e_2\delta D = \lambda e_2 D^2$, which is a derivation on $e_2 Q$. Since $e_2 Q$ is a 2-torsion free semiprime ring, it follows from Lemma 16 that $\lambda e_2 D = 0$. This implies $e_2\delta = 0$, as asserted.

"(2) $\Longrightarrow$ (1)": It is known that every derivation of R can be uniquely extended to a derivation of Q. Moreover, R and Q satisfy the same differential identities ([13, Theorem 3]). Let $x, y \in R$. Then $\delta D(e_1 x) = \delta\big(e_1 D(x)\big) = 0$, $\delta D(e_2 x) = \big(e_2\delta\big)D(x) = 0$, and so

$$\delta D(x) = \delta D(e_3 x) = \big(e_3\delta\big)D(x) = e_3\lambda D^2(x) = \lambda D^2(e_3 x).$$

Thus, by the fact that $2e_3 Q = 0$, we get

$$\begin{aligned}
&\delta D(xy)\\
&\quad = \lambda D^2(e_3 xy) = \lambda\big(D^2(e_3 x)e_3 y + 2D(e_3 x)D(e_3 y) + e_3 x D^2(e_3 y)\big)\\
&\quad = \lambda\big(D^2(e_3 x)e_3 y + e_3 x D^2(e_3 y)\big) = \lambda\big(D^2(e_3 x)y + xD^2(e_3 y)\big) = \delta D(x)y + x\delta D(y).
\end{aligned}$$

This proves that δD is a derivation of R. $\square$

Proof of Corollary 3. We only give the proof of (1). In view of (3) of Theorem 1, there exist orthogonal idempotents $e_1, e_2, e_3 \in C$, $e_1 + e_2 + e_3 = 1$, and $\lambda \in C$ such that $e_1\delta = 0$, $e_2 D = 0$ and $e_3\big(D - \lambda\delta\big) = 0$, where $e_2 R$ is 2-torsion free and $2e_3 R = 0$.

Note that R is a $*$-prime ring, so is Q. Since $\delta(R)^* = \delta(R) \neq 0$, $Q\delta(R)Q$ is a $*$-ideal of the $*$-prime ring Q. Since $e_1 Q\delta(R)Q = 0$, this implies $e_1 = 0$. Set $f_1 := e_2$ and $f_2 := e_3$. Then $f_1 + f_2 = 1$, $f_1 D = 0$, $f_2\big(D - \lambda\delta\big) = 0$, $f_1 R$ is 2-torsion free and $2f_2 R = 0$, as asserted. $\square$

We now turn to prove Theorem 5.

Lemma 17. *Let R be a faithful S_4-free ring, and let $\delta, D\colon R \to R$ be derivations. Suppose that $\delta D(x) \in Z(R)$ for all $x \in R$, and that $eD \neq 0$ (resp. $e\delta \neq 0$) for any nonzero $e \in \mathbf{B}$. Then $\delta = \lambda D$ (resp. $D = \lambda\delta$) for some $\lambda \in C$.*

Proof. We only give the proof of the case that $eD \neq 0$ for any $0 \neq e \in \mathbf{B}$, because the another case has an analogous argument. Let

$$\Sigma = \{f \in \mathbf{B} \mid f\big(\delta - \lambda D\big) = 0 \text{ for some } \lambda \in C\}.$$

By Lemma 13, Σ is an ideal of $\mathbf{B}$. If $1 \in \Sigma$, then we are done. Suppose on the contrary that $1 \notin \Sigma$. Then there exists $\mathbf{m} \in \mathrm{Spec}(\mathbf{B})$ such that $\Sigma \subseteq \mathbf{m}$.

We first extend derivations $\delta, D\colon R \to R$ to derivations of Q. Since R and Q satisfy the same differential identities (see [13, Theorem 3]), $\delta D(x) \in C$ for all $x \in Q$. Thus, $\overline{\delta}\,\overline{D}(\overline{x}) \in \overline{C}$ $(:= C + \mathbf{m}Q/\mathbf{m}Q)$ for all $\overline{x} \in Q/\mathbf{m}Q$, where $\overline{\delta}, \overline{D}$ are the derivations of $Q/\mathbf{m}Q$ induced by δ, D, respectively. Since Q is faithful S_4-free, $Q/\mathbf{m}Q$ does not satisfy S_4 (see Lemma 11). That is, $\dim_{\overline{C}} Q/\mathbf{m}Q > 4$. In view of Theorem 6, either $\overline{D} = 0$ or $\overline{\delta} = \overline{\lambda}\,\overline{D}$ for some $\lambda \in C$.

Suppose first that $\overline{D} = 0$. Then $D(Q) \subseteq \mathbf{m}Q$. Since $D(Q)$ is an orthogonally complete subset of Q, by [2, Proposition 3.1.11], there exists $e \in \mathbf{B} \setminus \mathbf{m}$ such that $eD(Q) = 0$, a contradiction. Thus, the latter case holds. Then $\delta(x) - \lambda D(x) \in \mathbf{m}Q$ for all $x \in Q$. Since the subset $\{\delta(x) - \lambda D(x) \mid x \in Q\}$ is orthogonally complete, it follows from [2, Proposition 3.1.11] that there exists $g \in \mathbf{B} \setminus \mathbf{m}$ such that $g\big(\delta(x) - \lambda D(x)\big) = 0$ for all $x \in Q$. This implies that $g \in \Sigma$, contradicting the fact that $\Sigma \subseteq \mathbf{m}$. $\square$

Lemma 18. *Let $\delta, D\colon R \to R$ be derivations. Suppose that $\delta D(x) \in Z(R)$ for all $x \in R$. Then there exist orthogonal idempotents $e_1, e_2, e_3 \in C$, $e_1 + e_2 + e_3 = 1$ such that $e_1 D = 0$, $e_2\delta = 0$ and $2e_3R = 0$, where e_2R is 2-torsion free.*

Proof. In view of Lemma 8, there exists an idempotent $e_3 \in C$ such that $\mathrm{Ann}_Q(2Q) = e_3Q$. Then $2e_3R = 0$ and fQ is 2-torsion free, where $f := 1 - e_3$. Applying the same lemma, there exists an idempotent $e_1 \in fC$ such that $\mathrm{Ann}_{fQ}\big(fQD(Q)Q\big) = e_1Q$. Then $e_1D = 0$. Set $e_2 := f - e_1$. Thus, for any $h \in \mathbf{B}$ with $h \leq e_2$, if $hD = 0$, then $h = 0$.

Let $R_2 := R \cap e_2Q$, $Q_2 := e_2Q$, $\mathbf{B}_2 := e_2\mathbf{B}$, $C_2 := e_2C$, and let $\mathrm{Spec}(\mathbf{B}_2)$ be the spectrum of the complete Boolean algebra of $\mathbf{B}_2$. Note that R_2 is a 2-torsion free semiprime ring, and Q_2 is the Martindale symmetric ring of quotients of R_2 (see [2, Proposition 2.3.14]).

Let $\mathbf{m} \in \mathrm{Spec}(\mathbf{B}_2)$. In view of Lemma 9, $Q_2/\mathbf{m}Q_2$ is a 2-torsion free prime ring. Let $\overline{\delta}$, $\overline{D}$ be the derivations of $Q_2/\mathbf{m}Q_2$ induced canonically by δ, D, respectively. Then $\overline{\delta}\,\overline{D}(\overline{x}) \in \overline{C_2}$, for all $\overline{x} \in Q_2/\mathbf{m}Q_2$, where $\overline{C_2}(:= C_2 + \mathbf{m}Q_2/\mathbf{m}Q_2)$ is equal to the extended centroid of R_2. In view of [12, Theorem 4], either $\overline{D} = 0$ or $\overline{\delta} = 0$. Suppose that $\overline{D} = 0$. Then $D(e_2Q) \subseteq \mathbf{m}Q_2$, implying $hD(Q) = 0$ for some $h \leq e_2$ and $h \notin \mathbf{m}$. Hence, $h = 0$, a contradiction. This implies that $\overline{\delta} = 0$. That is, $\delta(Q_2) \subseteq \mathbf{m}Q_2$. Note that $\bigcap_{\mathbf{m} \in \mathrm{Spec}(\mathbf{B}_2)} \mathbf{m}Q_2 = 0$. We get $e_2\delta = 0$. $\square$

Proof of Theorem 5. In view of Lemma 8, there exists an idempotent $e_1 \in C$ such that $\mathrm{Ann}_Q(QD(Q)Q) = e_1Q$. Then $e_1D = 0$. Set $f := 1 - e_1$. By Lemma 10, there exist orthogonal idempotents $e_3, f_2 \in fC$, $e_3 + f_2 = f$, such that e_3R is faithful S_4-free and f_2R is an S_4-ring.

We note that if $e_3D \neq 0$, then $ge_3D \neq 0$, for any $0 \neq g \in e_3C$. Since e_3R is faithful S_4-free, it follows from Lemma 17 that $e_3\delta = \lambda e_3D$ for some $\lambda \in e_3C$. By Lemma 18, there exist orthogonal idempotents $h_1, e_2, e_4 \in f_2C$, $h_1 + e_2 + e_4 = f_2$, such that $h_1D = 0$, $e_2\delta = 0$ and $2e_4R = 0$. Clearly, e_4R is an S_4-ring. Note that $h_1 \leq f$, implying $h_1 = 0$. Thus, orthogonal idempotents e_1, e_2, e_3, e_4 are orthogonal idempotents in C , and $e_1 + e_2 + e_3 + e_4 = 1$. $\square$

Finally, we turn to the proof of Theorem 7.

Lemma 19. *Let R be faithful S_4-free, and let $\delta, D\colon R \to R$ be derivations. Suppose that $\big[\delta(x), D(x)\big] \in Z(R)$ for all $x \in R$. If $eD \neq 0$ for any nonzero $e \in \mathbf{B}$, then $\delta = \lambda D$ for some $\lambda \in C$.*

Proof. Since R and Q satisfy the same differential identities (see [13, Theorem 3]), $\big[\delta(x), D(x)\big] \in C$ for all $x \in Q$. Let $\mathbf{m} \in \mathrm{Spec}(\mathbf{B})$. In view of Lemma 11, $Q/\mathbf{m}Q$ does not satisfy S_4. Let $\overline{\delta}, \overline{D}\colon Q/\mathbf{m}Q \to Q/\mathbf{m}Q$ be the derivations induced canonically by δ, D, respectively. Moreover, $\big[\overline{\delta}(\overline{x}), \overline{D}(\overline{x})\big] \in \overline{C}$, for all $\overline{x} \in Q/\mathbf{m}Q$. Since $\dim_{\overline{C}} Q/\mathbf{m}Q > 4$, it follows from Theorem 6 that either $\overline{D} = 0$ or $\overline{\delta} = \overline{\lambda_{\mathbf{m}}}\,\overline{D}$ for some $\lambda_{\mathbf{m}} \in C$. Note that $\overline{D} \neq 0$, as $eD \neq 0$ for any $0 \neq e \in \mathbf{B}$. Thus $\overline{\delta} = \overline{\lambda_{\mathbf{m}}}\,\overline{D}$; that is, $\delta(x) - \lambda_{\mathbf{m}}D(x) \in \mathbf{m}Q$ for all $x \in Q$. In view of (2) of Lemma 13, $\delta = \lambda D$ for some$\lambda \in C$. $\square$

Lemma 20. *Let $\delta, D\colon R \to R$ be nonzero derivations such that $\big[\delta(x), D(x)\big] \in Z(R)$, for all $x \in R$. Then there exist orthogonal idempotents $e_1, e_2, e_3 \in C$, $e_1 + e_2 + e_3 = 1$, $\lambda \in e_3C$ such that $2e_1R = 0$, $e_2D = 0$ and $e_3\big(\delta - \lambda D\big) = 0$.*

Proof. By Lemma 8, there exists $e_1 \in \mathbf{B}$ such that $\mathrm{Ann}_Q(2Q) = e_1Q$. Then $2e_1R = 0$ and $(1 - e_1)R$ is 2-torsion free. In view of Lemma 8, we have

$$\mathrm{Ann}_{(1-e_1)Q}\Big((1 - e_1)QD(Q)Q\Big) = e_2Q,$$

for some $e_2 \leq 1 - e_1$. Then $e_2D = 0$. Set $e_3 := 1 - e_1 - e_2 \in \mathbf{B}$.

Set $Q_3 := e_3Q$, $\mathbf{B}_3 = e_3\mathbf{B}$ and $C_3 := e_3C$. Let $\mathbf{m} \in \mathrm{Spec}(\mathbf{B}_3)$. Since e_3Q is 2-torsion free, it follows from Lemma 9 that $\mathrm{char}\,(Q_3/\mathbf{m}Q_3) \neq 2$. Let $\overline{\delta}, \overline{D}\colon Q_3/\mathbf{m}Q_3 \to Q_3/\mathbf{m}Q_3$ be the derivations induced canonically by δ, D, respectively. Then, by assumption, we have $[\overline{\delta}(\overline{x}), \overline{D}(\overline{x})] \in \overline{C_3}$, for all $\overline{x} \in Q_3/\mathbf{m}Q_3$. We claim that $\overline{D} \neq 0$. Otherwise, $e_3D(Q_3) \subseteq \mathbf{m}Q_3$. Thus, $fD(Q) = 0$, for some

$f \in e_3\mathbf{B} \setminus \mathbf{m}$. Then $f \le e_2$, and so $f = 0$, a contradiction. In view of Theorem 6, $\overline{\delta} = \overline{\mu_{\mathbf{m}}}\overline{D}$, for some $\mu_{\mathbf{m}} \in e_3C$. That is, $\delta(x) - \mu_{\mathbf{m}}D(x) \in \mathbf{m}Q_3$, for all $x \in e_3Q$. In view of (2) of Lemma 13, $e_3\delta = \lambda e_3 D$, for some $\lambda \in e_3C$. $\square$

Proof of Theorem 7. In view of Lemma 8, $\mathrm{Ann}_Q(Q[R,R]Q) = e_1Q$, for some $e_1 \in \mathbf{B}$. Thus, $e_1[R,R] = 0$ (i.e., e_1R is commutative), and $(1-e_1)R$ is faithful S_2-free. By Lemma 8 again, $\mathrm{Ann}_{(1-e_1)Q}\big((1-e_1)QD(Q)Q\big) = e_2Q$, for some $e_2 \in (1-e_1)\mathbf{B}$. Set $f := 1 - e_1 - e_2 \in \mathbf{B}$.

In view of Lemma 10, there exist orthogonal idempotents $g, h \in fC$, $f = g+h$, such that gR is a faithful S_4-free semiprime ring and hR is a semiprime S_4-ring. Note that $eD \ne 0$, for any $0 \ne e \le g$. In view of Lemma 16, $g(\delta - \mu D) = 0$, for some $\mu \in gC$. By Lemma 20 there exist orthogonal idempotents $e_4, h_2, h_3 \in C$, $e_4 + h_2 + h_3 = h$, $\eta \in h_3C$ such that $2e_4R = 0$, $h_2D = 0$ and $h_3\big(\delta - \eta D\big) = 0$.

Since $h_2D = 0$, we have $h_2 \in e_2C$, and so $h_2 = 0$. Set $e_3 := g + h_3$ and $\lambda := g\mu + h_3\eta \in e_3C$. Then

$$e_3\big(\delta - \lambda D\big) = (g+h_3)\big(\delta - (g\mu + h_3\eta)D\big) = 0.$$

Then e_1, e_2, e_3, e_4 are orthogonal idempotents in C with $\sum_{i=1}^4 e_i = 1$ such that e_1R is commutative, $e_2D = 0$, $e_3\delta = e_3\lambda D$, and both $2e_4R = 0$ and e_4R is an S_4-ring. $\square$

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