

Approximation of functions by nonlinear singular integral operators depending on two parameters

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The preparation of this paper became overshadowed by Professor Wachnicki's death in November 2015. We had intended to write jointly. I have done my best to complete the main ideas, most of which were worked out together. In sorrow, I dedicate this work to his memory.

Grażyna Krech

Abstract. The aim of this paper is to study the behavior of nonlinear singular integral operators of the form

$$T_w(f)(s) = \int_G K_w(s-t, f(t))dt.$$

Here we estimate the rate of convergence at a point s_0 in which a function f is continuous. This is an extension of the paper by ŚWIDERSKI and WACHNICKI [21].

1. Introduction

The approximation theory with nonlinear integral operators of convolution type was introduced by J. MUSIELAK in [20]. For further reading, we refer the reader to [4]–[6], [12], [15], [22], as well as the monographs [8] and [10], where some kinds of convergence results of nonlinear singular integral operators in many

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different spaces have been considered. For further references on approximation results for convolution and non-convolution type operators and their applications, see, e.g., [2]–[3], [7], [11], [13].

Let G be a locally compact abelian group with the Haar measure. The Haar integral of a real-valued function f on $A \subset G$ is denoted by $\int_A f(t)dt$. We denote by $\mathcal{B}(\theta)$ the family of all neighbourhoods of the neutral element θ of G . Let W be a nonempty set of indices with any topology, and w_0 be an accumulation point of W in this topology. We take the family $K = (K_w)_{w \in W}$ of functions $K_w : G \times \mathbb{R} \rightarrow \mathbb{R}$, where $K_w(t, 0) = 0$ for $t \in G$, such that for every $w \in W$, K_w are integrable functions with respect to the first variable for all values of the second variable. The family K will be called a kernel.

We consider a nonlinear integral operator T_w , $w \in W$ of the form

$$T_w(f)(s) = \int_G K_w(s-t, f(t))dt = \int_G K_w(t, f(s-t))dt. \quad (1)$$

In papers [9], [17]–[19], the convergence of integral (1) in some normed or modular spaces has been studied. In [21], the pointwise convergence of the operators T_w in $L_p(G)$ at the point s_0 , where s_0 is a point of continuity of f and $(w, s) \rightarrow (w, s_0)$ in the sense of the product convergence, was investigated. KARSLI [14] and KARSLI–GUPTA [16] investigated the pointwise convergence and the rate of convergence of the operators

$$\int_a^b K_w(t-x, f(t))dt, \quad x \in [a, b] \subset \mathbb{R},$$

on a μ -generalized Lebesgue point x_0 of f in $L_1(a, b)$ as $(x, w) \rightarrow (x_0, w_0)$, or on a discontinuity point x_0 of the first kind.

In the present paper, we investigate the problem of the rate of convergence of operators (1) in the case $G = \mathbb{R}$ or $G = [-\pi, \pi)$ with the addition modulo 2π . We also study the Voronovskaya-type theorem for operators (1), and give examples for which the presented theorems apply.

First, we shall find the conditions which provide the existence and convergence to f of the integral $T_w(f)$.

We assume that the following conditions hold:

- (a) There exists an integrable function $L_w : G \rightarrow \mathbb{R}$ such that

$$|K_w(t, u) - K_w(t, v)| \leq L_w(t)|u - v| \quad (2)$$

for any $w \in W$, $t \in G$, $u, v \in \mathbb{R}$.

(b) There exists a number $M > 0$ such that

$$\int_G L_w(t) dt \leq M \quad \text{for all } w \in W.$$

(c) $\lim_{w \rightarrow w_0} \frac{1}{u} \int_G K_w(t, u) dt = 1$ for every $u \in \mathbb{R} \setminus \{0\}$.

(d) For every $U \in \mathcal{B}(\theta)$,

$$\lim_{w \rightarrow w_0} \int_{G \setminus U} L_w(t) dt = 0.$$

Let $f \in L^p(G)$, $1 \leq p \leq +\infty$. Condition (a) provides that $T_w(f) \in L^p(G)$. Further, denote

$$R_q^U(w) = \begin{cases} \left(\int_{G \setminus U} (L_w(t))^q \right)^{1/q}, & \text{as } 1 \leq q < +\infty, \\ \sup_{G \setminus U} \text{ess } L_w(t), & \text{as } q = +\infty, \end{cases}$$

and assume that

(e) $\lim_{w \rightarrow w_0} R_q^U(w) = 0$ for every $U \in \mathcal{B}(\theta)$ and $\frac{1}{p} + \frac{1}{q} = 1$.

We have the following theorem.

Theorem 1 ([21]). *Let $f \in L^p(G)$, $1 \leq p \leq +\infty$. Assume that the kernel K satisfies (a)–(e). Then*

$$\lim_{(w,s) \rightarrow (w_0,s_0)} T_w(f)(s) = f(s_0),$$

where s_0 is an accumulation point at which f is continuous.

2. The rate of convergence

In this section, we assume that $G = \mathbb{R}$ or $G = [-\pi, \pi)$ with the addition modulo 2π . We prove the following result.

Theorem 2. *Let $f \in L^\infty(G)$. Assume that the kernel K satisfies (a)–(e). Then*

$$|T_w(f)(s) - f(s_0)| \leq \left(2 + 3 \int_G L_w(t) dt \right) \omega(f, \delta) + \left| \int_G K_w(t, f(s_0)) dt - f(s_0) \right|,$$

for every $(w, s) \in W \times G$ such that $|s - s_0| < \delta = \left(\int_G t^2 L_w(t) dt \right)^{1/2}$, where $\omega(f, \delta)$ is a modulus of continuity of f , i.e.,

$$\omega(f, \delta) = \sup_{\substack{s \in G \\ |h| \leq \delta}} |f(s+h) - f(s)|.$$

PROOF. We observe that

$$\begin{aligned} |T_w(f)(s) - f(s_0)| &\leq \left| \int_G K_w(s-t, f(t)) dt - f(s_0) \right| \\ &\leq \left| \int_G K_w(s-t, f(t)) dt - \int_G K_w(s-t, f(s_0)) dt \right| \\ &\quad + \left| \int_G K_w(s-t, f(s_0)) dt - f(s_0) \right| = A_1 + A_2. \end{aligned}$$

By (1) we get

$$A_2 = \left| \int_G K_w(s-t, f(s_0)) dt - f(s_0) \right| = \left| \int_G K_w(t, f(s_0)) dt - f(s_0) \right|.$$

We remark that

$$|f(t) - f(s_0)| \leq \omega(f, |t - s_0|) \leq \left(1 + \frac{(t - s_0)^2}{\delta^2}\right) \omega(f, \delta)$$

for every $\delta > 0$. Hence, by (2) we get

$$\begin{aligned} A_1 &\leq \int_G |f(t) - f(s_0)| L_w(s-t) dt \leq \omega(f, \delta) \int_G \left(1 + \frac{(t - s_0)^2}{\delta^2}\right) L_w(s-t) dt \\ &= \omega(f, \delta) \left[\int_G L_w(s-t) dt + \frac{1}{\delta^2} \int_G (t - s_0)^2 L_w(s-t) dt \right] \\ &= \omega(f, \delta) \left[\int_G L_w(t) dt + \frac{1}{\delta^2} \int_G (s - t - s_0)^2 L_w(t) dt \right] \\ &\leq \omega(f, \delta) \left[\int_G L_w(t) dt + \frac{2}{\delta^2} \int_G t^2 L_w(t) dt + \frac{2(s - s_0)^2}{\delta^2} \int_G L_w(t) dt \right]. \end{aligned}$$

Putting $\delta = \left(\int_G t^2 L_w(t) dt\right)^{1/2}$, we obtain

$$A_1 \leq \omega(f, \delta) \left[3 \int_G L_w(t) dt + 2 \right]$$

for $|s - s_0| < \left(\int_G t^2 L_w(t) dt\right)^{1/2}$. Hence

$$|T_w(f)(s) - f(s_0)| \leq \omega(f, \delta) \left[3 \int_G L_w(t) dt + 2 \right] + \left| \int_G K_w(t, f(s_0)) dt - f(s_0) \right|$$

for $|s - s_0| < \delta = \left(\int_G t^2 L_w(t) dt\right)^{1/2}$. □

Corollary 1. *If $\int_G L_w(t)dt = 1$, then*

$$|T_w(f)(s) - f(s_0)| \leq 5\omega(f, \delta) + \left| \int_G K_w(t, f(s_0))dt - f(s_0) \right|$$

for $|s - s_0| < \delta = \left(\int_G t^2 L_w(t)dt \right)^{1/2}$ and $w \in W$.

Now we give specific examples of operators for which the main theorem applies.

Example 1. In $G = \mathbb{R}$ we consider the natural topology and the Lebesgue integral. Let $u \in \mathbb{R}$. We define

$$K_n(t, u) = K_n(t)u = \sqrt{\frac{n}{\pi}} u e^{-nt^2}, \quad t \in \mathbb{R}$$

for $n \in \mathbb{N}$. The set of indices W is equal to $\mathbb{N}$ and $w_0 = +\infty$. In this case

$$L_n(t) = \sqrt{\frac{n}{\pi}} e^{-nt^2}, \quad t \in \mathbb{R}.$$

Using

$$\int_0^\infty e^{-nt^2} dt = \frac{1}{2} \sqrt{\frac{\pi}{n}}, \quad \int_0^\infty t^2 e^{-nt^2} dt = \frac{1}{4n} \sqrt{\frac{\pi}{n}},$$

we obtain

$$\int_{\mathbb{R}} L_n(t)dt = 1, \quad \int_{\mathbb{R}} t^2 L_n(t)dt = \frac{1}{2n}, \quad \int_{\mathbb{R}} K_n(t, u)dt = u.$$

Therefore,

$$|T_n(f)(s) - f(s_0)| \leq 5\omega(f, \delta) + \left| \int_{\mathbb{R}} K_n(t, f(s_0))dt - f(s_0) \right| = 5\omega(f, \delta)$$

for $|s - s_0| < \delta = \frac{1}{\sqrt{2n}}$.

This is the example of a linear kernel with respect to the second variable, i.e., $K_w(t, u) = L_w(t)u$. This case is widely used in approximation theory, see [10].

Example 2. Take $G = [-\pi, \pi)$ with the operation of addition modulo 2π . In G we consider the natural topology and the Lebesgue integral. Let $u \in \mathbb{R}$. We define

$$K_n(t, u) = \begin{cases} \frac{nu}{2} + \sin \frac{nu}{2}, & \text{if } t \in [-\frac{1}{n}, \frac{1}{n}], \\ 0, & \text{if } t \in [-\pi, \pi) \setminus [-\frac{1}{n}, \frac{1}{n}], \end{cases}$$

for $n \in \mathbb{N}$. The set of indices W is equal to $\mathbb{N}$ and $w_0 = +\infty$. In this case, we find that

$$L_n(t) = \begin{cases} n, & \text{if } t \in [-\frac{1}{n}, \frac{1}{n}], \\ 0, & \text{if } t \in [-\pi, \pi) \setminus [-\frac{1}{n}, \frac{1}{n}]. \end{cases}$$

Moreover,

$$\int_G L_n(t) dt = 2, \quad \int_G t^2 L_n(t) dt = \frac{2}{3n^2}, \quad \int_G K_n(t, u) dt = \frac{2}{n} \sin \frac{nu}{2} + u.$$

Hence

$$|T_n(f)(s) - f(s_0)| \leq 8\omega(f, \delta) + \frac{2}{n} \left| \sin \frac{nf(s_0)}{2} \right| \leq 8\omega(f, \delta) + \frac{2}{n},$$

for $|s - s_0| \leq \delta = \frac{\sqrt{2}}{n\sqrt{3}}$ and $n \in \mathbb{N}$.

Example 3. Analogously, we can consider

$$K_n(t, u) = \begin{cases} n^2 \sin \frac{u}{2n}, & \text{if } t \in [-\frac{1}{n}, \frac{1}{n}], \\ 0, & \text{if } t \in [-\pi, \pi) \setminus [-\frac{1}{n}, \frac{1}{n}]. \end{cases}$$

We take $G = [-\pi, \pi)$ with the operation of addition modulo 2π . The set of indices is equal to $\mathbb{N}$, $w_0 = +\infty$. In this case, we get

$$L_n(t) = \begin{cases} \frac{n}{2}, & \text{for } t \in [-\frac{1}{n}, \frac{1}{n}], \\ 0, & \text{for } t \in [-\pi, \pi) \setminus [-\frac{1}{n}, \frac{1}{n}], \end{cases}$$

and

$$\int_G L_n(t) dt = 1, \quad \int_G t^2 L_n(t) dt = \frac{1}{3n^2}, \quad \int_G K_n(t, u) dt = 2n \sin \frac{u}{2n}.$$

Hence

$$|T_n(f)(s) - f(s_0)| \leq 5\omega(f, \delta) + \left| 2n \sin \frac{f(s_0)}{2n} - f(s_0) \right|,$$

for $|s - s_0| < \delta = \frac{1}{n\sqrt{3}}$, $n \in \mathbb{N}$.

3. Some results for r -times differentiable functions

Let C^r , $r \in \mathbb{N}$, be the class of all r -times differentiable functions $f \in L^\infty(G)$ with derivatives $f^{(k)} \in L^\infty(G)$ for $0 \leq k \leq r$. Observe, that for $f \in C^r$, $r \in \mathbb{N}$, $t \in \mathbb{R}$, we can write

$$f(s) = \sum_{j=0}^r \frac{f^{(j)}(t)}{j!} (s-t)^j + I_r(t, s), \quad s \in \mathbb{R},$$

where

$$I_r(t, s) = \frac{(s-t)^r}{(r-1)!} \int_0^1 (1-u)^{r-1} \left[f^{(r)}(t + u(s-t)) - f^{(r)}(t) \right] du.$$

Let

$$F_r(t, s) = \sum_{j=0}^r \frac{f^{(j)}(t)}{j!} (s-t)^j, \quad s, t \in \mathbb{R},$$

therefore,

$$F_r(t, s) = f(s) - I_r(t, s), \quad s, t \in \mathbb{R}.$$

Assume that $G = \mathbb{R}$ or $G = [-\pi, \pi)$ with the addition modulo 2π and $W = \mathbb{N}$. For $f \in C^r$, we consider the operator

$$T_{n;r}(f)(s) := \int_G K_n(s-t, F_r(t, s)) dt = \int_G K_n(t, F_r(s-t, s)) dt.$$

Theorem 3. *Let $f \in C^r$, and let the kernel K verify conditions (a)–(e). If there exists $n \in \mathbb{N}$ and $M_1(r) > 0$, such that*

$$n^r \int_G t^{2r} L_n(t) dt < M_1(r),$$

then

$$|T_{n;r}(f)(s) - f(s_0)| \leq M_2(r) \frac{1}{r!} n^{-r/2} \omega(f^{(r)}; \delta) + \left| \int_G K_n(t, F_r(s_0, s)) dt - f(s_0) \right|,$$

for $(n, s) \in \mathbb{N} \times G$ such that $|s - s_0| < \delta = (\int_G t^2 L_n(t) dt)^{1/2}$, where $M_2(r)$ is a positive constant.

PROOF. Observe that

$$\begin{aligned} |T_{n,r}(f)(s) - f(s_0)| &\leq \left| \int_G K_n(s-t, F_r(t, s)) dt - \int_G K_n(s-t, F_r(s_0, s)) dt \right| \\ &\quad + \left| \int_G K_n(s-t, F_r(s_0, s)) dt - f(s_0) \right| = A_1 + A_2. \end{aligned}$$

We have

$$A_2 = \left| \int_G K_n(s-t, F_r(s_0, s)) dt - f(s_0) \right| = \left| \int_G K_n(t, F_r(s_0, s)) dt - f(s_0) \right|$$

and

$$A_1 \leq \int_G L_n(s-t) |F_r(t, s) - F_r(s_0, s)| dt.$$

Now, we estimate

$$\begin{aligned} |F_r(t, s) - F_r(s_0, s) - f(s) + f(s_0)| &\leq |F_r(t, s) - f(s)| + |F_r(s_0, s) - f(s_0)| \\ &= |I_r(t, s)| + |I_r(s_0, s)|. \end{aligned}$$

Using the properties of the modulus of smoothness, we get

$$\begin{aligned} |I_r(t, s)| &\leq \frac{|t-s|^r}{(r-1)!} \int_0^1 (1-u)^{r-1} \omega(f^{(r)}; u|t-s|) du \\ &\leq \frac{1}{r!} |t-s|^r \omega(f^{(r)}; |t-s|) \\ &\leq \frac{1}{r!} \omega(f^{(r)}; \delta) \left(|t-s|^r + \frac{1}{\delta} |t-s|^{r+1} \right), \quad \delta > 0. \end{aligned}$$

Therefore,

$$\begin{aligned} A_1 &\leq \int_G L_n(s-t) \left\{ \frac{1}{r!} \omega(f^{(r)}; \delta) \left(|t-s|^r + \frac{1}{\delta} |t-s|^{r+1} \right) \right. \\ &\quad \left. + \frac{1}{r!} \omega(f^{(r)}; \delta) \left(|s_0-s|^r + \frac{1}{\delta} |s_0-s|^{r+1} \right) \right\} dt \\ &= \frac{1}{r!} \omega(f^{(r)}; \delta) \left\{ \int_G L_n(s-t) \left(|t-s|^r + \frac{1}{\delta} |t-s|^{r+1} \right) dt \right. \\ &\quad \left. + \left(|s_0-s|^r + \frac{1}{\delta} |s_0-s|^{r+1} \right) \int_G L_n(s-t) dt \right\} \\ &= \frac{1}{r!} \omega(f^{(r)}; \delta) \left\{ \int_G |t|^r L_n(t) dt + \frac{1}{\delta} \int_G |t|^{r+1} L_n(t) dt \right. \\ &\quad \left. + \left(|s_0-s|^r + \frac{1}{\delta} |s_0-s|^{r+1} \right) \int_G L_n(t) dt \right\}. \end{aligned}$$

From

$$\int_G |t|^r L_n(t) dt \leq \left(\int_G L_n(t) dt \right)^{1/2} \left(\int_G t^{2r} L_n(t) dt \right)^{1/2}$$

and

$$\int_G |t|^{r+1} L_n(t) dt \leq \left(\int_G t^2 L_n(t) dt \right)^{1/2} \left(\int_G t^{2r} L_n(t) dt \right)^{1/2},$$

we obtain for $(n, s) \in \mathbb{N} \times G$ such that $|s - s_0| < \delta = \left(\int_G t^2 L_n(t) dt \right)^{1/2}$, the following inequality:

$$\begin{aligned} A_1 \leq \frac{1}{r!} \omega(f^{(r)}; \delta) & \left\{ \left(\int_G L_n(t) dt \right)^{1/2} \left(\int_G t^{2r} L_n(t) dt \right)^{1/2} \right. \\ & \left. + \frac{1}{\delta} \left(\int_G t^2 L_n(t) dt \right)^{1/2} \left(\int_G t^{2r} L_n(t) dt \right)^{1/2} + 2\delta^r \int_G L_n(t) dt \right\}. \end{aligned}$$

Let

$$n^r \int_G t^{2r} L_n(t) dt < M_1(r),$$

where $M_1(r)$ is some positive constant. Therefore

$$\int_G t^{2r} L_n(t) dt < \frac{M_1(r)}{n^r}.$$

Using this and the above estimate, we get

$$A_1 \leq \frac{1}{r!} \omega(f^{(r)}; \delta) \left\{ \frac{M_1^2(r)}{n^{r/2}} + \frac{M_1(r)}{n^{r/2}} + \frac{2M_1^{r+1}(r)}{n^{r/2}} \right\} \leq M_2(r) \frac{1}{r!} n^{-r/2} \omega(f^{(r)}; \delta)$$

for some $M_2(r) > 0$. Finally, we conclude

$$|T_{n;r}(f)(s) - f(s_0)| \leq M_2(r) \frac{1}{r!} n^{-r/2} \omega(f^{(r)}; \delta) + \left| \int_G K_n(t, F_r(s_0, s)) dt - f(s_0) \right|$$

for $f \in C^r$, $(n, s) \in \mathbb{N} \times G$ such that $|s - s_0| < \delta$, and some $M_2(r) > 0$. $\square$

Now, consider the operator from Example 2. In this case, we derive

$$n^r \int_G t^{2r} L_n(t) dt = n^r \int_{-1/n}^{1/n} t^{2r} n dt = n^{-r} \frac{2}{2r+1} < M_1(r),$$

where $M_1(r)$ is a positive constant. The assumptions of Theorem 3 are fulfilled, and we obtain the following estimation

$$|T_{n;r}(f)(s) - f(s_0)| \leq M_2(r) \frac{1}{r!} n^{-r/2} \omega(f^{(r)}; \delta) + \frac{2}{n},$$

for $(n, s) \in \mathbb{N} \times G$ such that $|s - s_0| < \delta = \frac{\sqrt{2}}{n\sqrt{3}}$, where $M_2(r)$ is a positive constant.

4. A Voronovskaya-type theorem

In this section, we suppose that $G = \mathbb{R}$ or $G = [-\pi, \pi)$ with the addition modulo 2π , $W = \mathbb{N}$ and $w_0 = +\infty$. We prove the following result.

Theorem 4. *Let $f \in L^\infty(G)$, where f is twice differentiable at the point $s_0 \in G$. Let the kernel K verify conditions (a)–(e). Moreover, we suppose that*

$$\begin{aligned} \limsup_{n \rightarrow +\infty} n \int_G |t| L_n(t) dt &= \alpha, & \limsup_{n \rightarrow +\infty} n \int_G t^2 L_n(t) dt &= \beta, \\ \limsup_{n \rightarrow +\infty} n \left| \int_G K_n(t, f(s_0)) dt - f(s_0) \right| &= \gamma, & n^2 \int_G t^4 L_n(t) dt &< M_1^2, \end{aligned}$$

where M_1 is a positive constant. Then

$$\limsup_{\substack{(s,n) \rightarrow (s_0, +\infty) \\ (s,n) \in Z_a^C}} n |T_n(f)(s) - f(s_0)| \leq \gamma + \alpha |f'(s_0)| + \frac{1}{2} \beta |f''(s_0)|, \quad (3)$$

where $Z_a^C = \{(s, n) \in G \times \mathbb{N} : n^a |s - s_0| < C\}$, and $a > 1$, $C > 0$ are fixed numbers.

PROOF. We get

$$\begin{aligned} n |T_n(f)(s) - f(s_0)| &= n \left| \int_G K_n(s-t, f(t)) dt - f(s_0) \right| \\ &\leq n \left| \int_G [K_n(s-t, f(t)) - K_n(s-t, f(s_0))] dt \right| \\ &\quad + n \left| \int_G K_n(s-t, f(s_0)) dt - f(s_0) \right| = I_1 + I_2. \end{aligned}$$

By (1) and the assumption, we get

$$I_2 = n \left| \int_G K_n(t, f(s_0)) dt - f(s_0) \right| \quad \text{and} \quad \limsup_{n \rightarrow +\infty} I_2 = \gamma.$$

By (a), we obtain

$$I_1 \leq n \int_G |f(t) - f(s_0)| L_n(s-t) dt.$$

Applying the Taylor formula,

$$f(t) = f(s_0) + f'(s_0)(t - s_0) + \frac{1}{2}(t - s_0)^2 f''(s_0) + \epsilon(t, s_0)(t - s_0)^2,$$

where $\lim_{t \rightarrow s_0} \epsilon(t, s_0) = 0$ and the function $t \rightarrow \epsilon(t, s_0)$ is of the class $L^\infty(G)$, we see that

$$\begin{aligned} I_1 &\leq n|f'(s_0)| \int_G |t - s_0| L_n(s - t) dt + \frac{n}{2} |f''(s_0)| \int_G (t - s_0)^2 L_n(s - t) dt \\ &\quad + n \int_G |\epsilon(t, s_0)| (s_0 - t)^2 L_n(s - t) dt = |f'(s_0)| I_{11} + \frac{1}{2} |f''(s_0)| I_{12} + I_{13}. \end{aligned}$$

By the assumption, we get

$$\begin{aligned} I_{11} &\leq n \int_G |s - t| L_n(s - t) dt + n \int_G |s - s_0| L_n(s - t) dt \\ &= n \int_G |t| L_n(t) dt + n|s - s_0| \int_G L_n(t) dt \leq n \int_G |t| L_n(t) dt + n^{1-a} C \cdot M, \end{aligned}$$

and

$$\begin{aligned} I_{12} &\leq 2n \int_G (s - t)^2 L_n(s - t) dt + 2n \int_G (s - s_0)^2 L_n(s - t) dt \\ &= 2n \int_G t^2 L_n(t) dt + 2n(s - s_0)^2 \int_G L_n(t) dt \\ &\leq 2n \int_G t^2 L_n(t) dt + 2n^{1-a} |s - s_0| C \cdot M, \end{aligned}$$

for some $M > 0$ and $(n, s) \in Z_a^C$. Hence

$$\lim_{\substack{(s,n) \rightarrow (s_0, +\infty) \\ (s,n) \in Z_a^C}} \left(|f'(s_0)| I_{11} + \frac{1}{2} |f''(s_0)| I_{12} \right) = \alpha |f'(s_0)| + \frac{1}{2} \beta |f''(s_0)|.$$

In order to prove (3), it is sufficient to obtain

$$\lim_{\substack{(s,n) \rightarrow (s_0, +\infty) \\ (s,n) \in Z_a^C}} I_{13} = 0.$$

We get

$$\begin{aligned} I_{13} &\leq 2n \int_G |\epsilon(t, s_0)| (s - t)^2 L_n(s - t) dt + 2n \int_G |\epsilon(t, s_0)| (s - s_0)^2 L_n(s - t) dt \\ &= J_1 + J_2. \end{aligned}$$

By assumption,

$$J_2 \leq 2n^{1-a}C|s - s_0| \int_G |\epsilon(t, s_0)| L_n(s - t) dt \quad \text{for } (s, n) \in Z_a^C.$$

Recalling the Cauchy–Schwarz inequality, we can infer

$$\begin{aligned} J_1 &\leq 2n \left(\int_G \epsilon^2(t, s_0) L_n(s - t) dt \right)^{1/2} \cdot \left(\int_G (s - t)^4 L_n(s - t) dt \right)^{1/2} \\ &\leq 2M_1 \left(\int_G \epsilon^2(t, s_0) L_n(s - t) dt \right)^{1/2}. \end{aligned}$$

By the approximative properties of the convolution (see [10]) and (b), (c), (d), we get

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_G \epsilon^2(t, s_0) L_n(s - t) dt &= \epsilon^2(s_0, s_0) = 0, \\ \lim_{n \rightarrow +\infty} \int_G |\epsilon(t, s_0)| L_n(s - t) dt &= |\epsilon^2(s_0, s_0)| = 0. \end{aligned}$$

Hence

$$\lim_{\substack{(s,n) \rightarrow (s_0, +\infty) \\ (s,n) \in Z_a^C}} I_{13} = 0.$$

Thus, the proof of Theorem 4 is complete. $\square$

We apply Theorem 4 to Example 3. In this case, the assumptions of Theorem 4 are satisfied. Indeed

$$\begin{aligned} n \int_G |t| L_n(t) dt &= \frac{n^2}{2} \int_{-1/n}^{1/n} |t| dt = \frac{1}{2}, \\ n \int_G t^2 L_n(t) dt &= \frac{n^2}{2} \int_{-1/n}^{1/n} t^2 dt = \frac{1}{3n}, \\ n^2 \int_G t^4 L_n(t) dt &= \frac{n^3}{2} \int_{-1/n}^{1/n} t^4 dt = \frac{1}{5n^2}, \end{aligned}$$

$$n \left| \int_G K_n(t, f(s_0)) dt - f(s_0) \right| = \left| 2n^2 \sin \frac{f(s_0)}{2n} - nf(s_0) \right| \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Hence

$$\limsup_{\substack{(s,n) \rightarrow (s_0, +\infty) \\ (s,n) \in Z_a^C}} n |T_n(f)(s) - f(s_0)| \leq \frac{1}{2} |f'(s_0)|.$$

Analogously, we can prove the following result.

Theorem 5. Let $f \in L^\infty(G)$, where f is twice differentiable at the point $s_0 \in G$. Let the kernel K verify conditions (a)–(e). Moreover, we suppose that there exist positive constants M_i , $i = 1, 2, 3, 4$ such that

$$\begin{aligned} n \int_G |t| L_n(t) dt &\leq M_1, & n \int_G t^2 L_n(t) dt &\leq M_2, \\ n \left| \int_G K_n(t, f(s_0)) dt - f(s_0) \right| &\leq M_3, & n^2 \int_G t^4 L_n(t) dt &< M_4. \end{aligned}$$

Then, there exists a constant $M_5 > 0$ such that

$$n |T_n(f)(s) - f(s_0)| \leq M_5$$

in the set $Z_a^C = \{(s, n) \in G \times \mathbb{N} : n^a |s - s_0| < C\}$, where $a > 1$ and C is a positive constant.

Next, we apply Theorem 5 to Example 2. In this case, we get

$$\begin{aligned} n \int_G |t| L_n(t) dt &= 1, & n \int_G t^2 L_n(t) dt &= \frac{2}{3n} \leq \frac{2}{3}, \\ n \left| \int_G K_n(t, u) dt - u \right| &= 2 \left| \sin \frac{nu}{2} \right| < 2, & n^2 \int_G t^4 L_n(t) dt &= \frac{2}{5n^3} < \frac{2}{5}. \end{aligned}$$

Hence

$$n |T_n(f)(s) - f(s_0)| \leq M_5,$$

for $(s, n) \in Z_a^C$ and some $M_5 > 0$.

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