

The n -dimensional hyperbolic space in $\mathbf{E}^{4n-3}$

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Abstract. In this paper we will construct an isometric immersion of the n -dimensional hyperbolic space into the euclidian space $\mathbf{E}^{4n-3}$ with a modification of DANILO BLANUŠA's [2] immersion into $\mathbf{E}^{6n-5}$ [2].

§1. Introduction

In (1955) BLANUŠA [2] gave a beautiful construction for the isometric embedding of the complete hyperbolic plane $\mathbf{H}^2$ into $\mathbf{E}^6$ and also for the isometric immersion of the n -dimensional hyperbolic space into $\mathbf{E}^{6n-5}$. This immersion is of class $\mathbf{C}^\infty$, but it is not analytic. A construction of an analytical embedding of the hyperbolic plane into $\mathbf{E}^n$ (with sufficiently large n) is unknown even these days.

In 1960 ROZENDORN [1] published a paper noting that every metric $ds^2 = du^2 + f^2(u) dv^2$ can be immersed in $\mathbf{E}^5$ using Blanuša's method.

The immersion we are dealing with is a modification of BLANUŠA's construction, therefore we shortly recall it. BLANUŠA considered the surface $\Phi(u, v)$ in $\mathbf{E}^6$ described in Cartesian coordinates $x_1, x_2, \dots, x_6$ by the functions

$$(1) \quad x_1(u, v) = x_1(u) = \int_0^u \sqrt{1 - f_1'(y)^2 - f_2'(y)^2} dy$$

$$(2) \quad x_2(u, v) = f_1(u) \sin(v\psi_1(u))$$

$$(3) \quad x_3(u, v) = f_1(u) \cos(v\psi_1(u))$$

$$(4) \quad x_4(u, v) = f_2(u) \sin(v\psi_2(u))$$

$$(5) \quad x_5(u, v) = f_2(u) \cos(v\psi_2(u))$$

$$(6) \quad x_6(u, v) = x_6(v) = v \quad -\infty < u, v < \infty$$

where

$$\begin{aligned} f_1(u) &= \frac{\varphi_1(u) \sinh u}{\psi_1(u)}, & f_2(u) &= \frac{\varphi_2(u) \sinh u}{\psi_2(u)} \\ \psi_1(u) &= e^{5+2[\frac{1+|u|}{2}]}, & \psi_2(u) &= e^{6+2[\frac{|u|}{2}]} \end{aligned}$$

($[x]$ denotes the integer part of x), and

$$\begin{aligned} \varphi_1(u) &= \left(\frac{1}{A} \int_0^{1+u} \frac{\sin(\pi x)}{e^{\sin^{-2}(\pi x)}} dx \right)^{1/2}, \\ \varphi_2(u) &= \left(\frac{1}{A} \int_0^u \frac{\sin(\pi x)}{e^{\sin^{-2}(\pi x)}} dx \right)^{1/2} \\ A &= \int_0^1 \frac{\sin(\pi x)}{e^{\sin^{-2}(\pi x)}} dx, \quad A \approx 0.141327. \end{aligned}$$

The functions $\varphi_1(u)$ and $\varphi_2(u)$ have the properties

$$\begin{aligned} 0 \leq \varphi_1(u) \leq 1, \quad 0 \leq \varphi_2(u) \leq 1, \\ \varphi_1(u)^2 + \varphi_2(u)^2 = 1, \quad \varphi_1(u) = \varphi_2(u+1), \quad u \in \mathbb{R}. \end{aligned}$$

ψ_2 has a discontinuity for even integers u , and so has ψ_1 for odd integers u , but at these points φ_1 and φ_2 vanish together with all of their derivatives, and so the functions f_1 resp. f_2 are of class $\mathbf{C}^\infty$. Besides,

$$f_i'(u) = \frac{\varphi_i(u) \cosh u + \varphi_i'(u) \sinh u}{\psi_i(u)}, \quad i = 1, 2$$

while ψ_i is a step function and has zero derivatives. Furthermore we have

$$\begin{aligned} |f_i'(u)| &< \frac{e^{|u|}|\varphi_i(u)| + e^{|u|}|\varphi_i'(u)|}{\psi_i(u)} < \frac{19e^{|u|}}{\psi_i(u)} < 19e^{|u|-(4+|u|)} \\ &= \frac{19}{e^4} < \frac{1}{\sqrt{2}}, \quad i = 1, 2. \end{aligned}$$

Using the above properties of the functions f_i , it is easy to see that

$$\sqrt{1 - f_1'(y)^2 - f_2'(y)^2}$$

is real for any value of u .

BLANUŠA has shown (see [2]) that (1)–(6) give a one-to-one $\mathbf{C}^\infty$ mapping of the (u, v) plane $\mathbb{R}^2$ into $\mathbf{E}^6$, and the metric of $\Phi(u, v)$ induced by $\mathbf{E}^6$ is

$$(7) \quad ds^2 = du^2 + \cosh^2 u dv^2,$$

and hence Φ has a constant negative curvature. (7) can be considered as the metric of $\mathbf{H}^2(\mathbf{u}, \mathbf{v})$.

Theorem. (BLANUŠA'S [2] p. 218). *Functions (1)–(6) ($u, v \in \mathbb{R}$) define an isometric $\mathbf{C}^\infty$ embedding of the hyperbolic plane into $\mathbf{E}^6$.*

§2. The hyperbolic plane in $\mathbf{E}^5$

By omitting x_6 , we get a surface Σ of $\mathbf{E}^5$ with metric $ds^2 = du^2 + \sinh^2 u dv^2$. $\Sigma \subset \mathbf{E}^5$ has singularity only when $u = 0$, which corresponds to the origin $(0, 0, 0, 0, 0)$ of $\mathbf{E}^5$. For integers u the images of the parametric lines are closed. We wish to illustrate the appearance of a surface having one singular point in $\mathbf{E}^5$ using an analog of the projection from $\mathbf{E}^4$ onto $\mathbf{E}^3$ (See Figure 1). At any other point the metric is positive definite and the curvature is -1 . So we obtain

Figure 1.

Parallel projection of a surface with constant negative curvature in $\mathbf{E}^4$ into $\mathbf{E}^3$.

The image is also a surface with constant negative curvature in $\mathbf{E}^3$.

Proposition 1. *The surface Σ given by (1)–(5) ($u, v \in \mathbb{R}$) is a surface with constant negative curvature, having only one singular point.*

In 1955 AMSLER [3] proved that each surface of $\mathbf{E}^3$ with constant negative curvature has an edge (i.e. it contains a curve consisting of singularities), and showed the nonexistence in $\mathbf{E}^3$ of surfaces of constant negative curvature with singularity consisting of one point alone. Proposition 1 shows that AMSLER's theorem fails to be valid in $\mathbf{E}^n$ if $n \geq 5$.

If we change $\sinh u$ to $\cosh u$ in f_1 and f_2 , then (1)–(6) give a surface with the metric $ds^2 = du^2 + (\cosh^2 u + 1) dv^2$, which is a metric of Efimov type and its curvature is

$$K(u, v) = -\frac{1 + 12e^{2u} + 6e^{4u} + 12e^{6u} + e^{8u}}{(1 + 6e^{2u} + e^{4u})^2} < -1/4.$$

If we omit $\mathbf{x}_6$ again, we get a surface $\tilde{\Sigma}$ in $\mathbf{E}^5$ with constant negative curvature.

$\tilde{\Sigma}$ is given by

$$(8) \quad x_1(u, v) = x_1(u) = \int_0^u \sqrt{1 - g_1'(y)^2 - g_2'(y)^2} dy$$

$$(9) \quad x_2(u, v) = g_1(u) \cos(v\psi_1(u)),$$

$$(10) \quad x_3(u, v) = g_1(u) \sin(v\psi_1(u)),$$

$$(11) \quad x_4(u, v) = g_2(u) \cos(v\psi_2(u)),$$

$$(12) \quad x_5(u, v) = g_2(u) \sin(v\psi_2(u)),$$

$$g_1(u) = \frac{\varphi_1(u) \cosh u}{\psi_1(u)}, \quad g_2(u) = \frac{\varphi_2(u) \cosh u}{\psi_2(u)}, \quad u, v \in \mathbb{R}.$$

A calculation shows that the metric of $\tilde{\Sigma}$ in $\mathbf{E}^5$ is also (7) and this can be considered as the metric of $\mathbf{R}^2(\mathbf{u}, \mathbf{v}) \equiv \mathbf{H}^2(\mathbf{u}, \mathbf{v})$.

Theorem. *Functions given by (8)–(12) $u, v \in \mathbb{R}$ define an isometric $\mathbf{C}^\infty$ immersion of the hyperbolic plane into $\mathbf{E}^5$.*

PROOF. BLANUŠA has shown that φ_1/ψ_1 and φ_2/ψ_2 are of class $\mathbf{C}^\infty$. From this follows that $\mathbf{g}_1$ and $\mathbf{g}_2$ also have this property. We show that

$$x_1(u, v) = \int_0^u \sqrt{1 - g_1'(y)^2 - g_2'(y)^2} dy$$

is real for any value of u . The majoring of g_i' is totally analogous to that of f_i'

$$\begin{aligned} g_i'(u) &= \frac{\varphi_i(u) \cosh u + \varphi_i'(u) \sinh u}{\psi_i(u)}, \quad i = 1, 2; \\ |g_i'(u)| &< \frac{e^{|u|}|\varphi_i(u)| + e^{|u|}|\varphi_i'(u)|}{\psi_i(u)} < \frac{19e^{|u|}}{\psi_i(u)} < 19e^{|u|-(4+|u|)} \\ &= \frac{19}{e^4} < \frac{1}{\sqrt{2}}, \quad i = 1, 2. \end{aligned}$$

It follows that $x_1(u, v)$ is real for any value of u ; moreover $\frac{\partial x_i}{\partial u} \nparallel \frac{\partial x_i}{\partial v}$ ($i = 1, \dots, 5$), therefore (8)–(12) is an immersion, indeed. $\square$

We remark that the above calculation is analogous to that of BLANUŠA.

§3. The n -dimensional hyperbolic space in $\mathbf{E}^{4n-3}$

In order to map the n -dimensional hyperbolic space into $\mathbf{E}^{6n-5}$ isometrically, Blanuša constructed two new functions. These are the following

$$F_1(u) = \frac{\varphi_1(\frac{1}{u})}{\psi_1(\frac{1}{u})} \sqrt{\frac{1}{u^2} - e^{-2u}}, \quad F_2(u) = \frac{\varphi_2(\frac{1}{u})}{\psi_2(\frac{1}{u})} \sqrt{\frac{1}{u^2} - e^{-2u}},$$

where $\varphi_1, \varphi_2, \psi_1, \psi_2$ are the same as in §1.

Let $x_0, x_{r1}, x_{r2}, \dots, x_{r6}$ ($r = 1, \dots, n-1$) denote a Cartesian coordinate system in $\mathbf{E}^{6n-5}$, and let u, v_r ($r = 1, \dots, n-1$) ($u > 0$ and $v_r \in \mathbb{R}$) be the parameter domain endowed with the metric

$$(13) \quad ds^2 = dx_0^2 + \sum_{r=1}^{n-1} \sum_{s=1}^6 dx_{rs}^2 = \frac{1}{u^2} \left(du^2 + \sum_{r=1}^{n-1} dv_r^2 \right).$$

So we have got the hyperbolic space $\mathbf{H}^n$.

Theorem (BLANUŠA [2] p. 225). *The functions*

$$\begin{aligned} x_0(u, v_r) &= x_0(u) = \int_1^u \sqrt{\frac{1}{y^2} - F_1'(y)^2 - F_2'(y)^2 - e^{-2y}} dy, \\ x_{r1}(u, v_r) &= \frac{e^{-u}}{\sqrt{n-1}} \cos(\sqrt{n-1}v_r), \\ x_{r2}(u, v_r) &= \frac{e^{-u}}{\sqrt{n-1}} \sin(\sqrt{n-1}v_r) \end{aligned}$$

$$\begin{aligned}
x_{r3}(u, v_r) &= \frac{F_1(u)}{\sqrt{n-1}} \cos(\sqrt{n-1}v_r\psi_1(\frac{1}{u})), \\
x_{r4}(u, v_r) &= \frac{F_1(u)}{\sqrt{n-1}} \sin(\sqrt{n-1}v_r\psi_1(\frac{1}{u})), \\
x_{r5}(u, v_r) &= \frac{F_2(u)}{\sqrt{n-1}} \cos(\sqrt{n-1}v_r\psi_2(\frac{1}{u})), \\
x_{r6}(u, v_r) &= \frac{F_2(u)}{\sqrt{n-1}} \sin(\sqrt{n-1}v_r\psi_2(\frac{1}{u})), \\
&\quad -\infty < v_r < \infty, \quad (r = 1, \dots, n-1), \quad 0 < u.
\end{aligned}$$

define a $\mathbf{C}^\infty$ isometric immersion of the n -dimensional hyperbolic space into $\mathbf{E}^{6n-5}$ with the metric (13).

In order to reduce the number of dimensions from $6n-5$ to $4n-3$, we need to find a metric

$$ds^2 = g_{11}(u) du^2 + \sum_{r=1}^{n-1} g_{r+1,r+1}(u) dv_r^2$$

with constant negative curvature, where $g_{22} = g_{33} = \dots = g_{nn} \equiv f(u)^2$.

The following observation helps us to generalize a two-dimensional metric $g_{11} = g_{11}(u)$, $g_{12} = 0$, $g_{22} = g_{22}(u)$ with constant negative curvature to an n -dimensional metric $g_{ii} = g_{ii}(u)$ and $g_{ij} = 0, i \neq j$ with constant negative curvature.

Proposition. *The metric*

$$(14) \quad ds^2 = x(u)du^2 + f^2(u)dv^2$$

has curvature $K = -1$ if and only if

$$(15) \quad x'f' + 2fx^2 - 2f''x = 0.$$

Furthermore, this differential equation has the following particular solutions

$$(16) \quad x_1 = \frac{f'^2}{f^2 - 1}; \quad x_2 = \frac{f'^2}{f^2}; \quad x_3 = \frac{f'^2}{1 + f^2}.$$

PROOF. Suppose that (14) has curvature $K = -1$. Then, using the elementary formula

$$(17) \quad K(u, v) = \frac{\left(\frac{\partial t_1}{\partial u} + \frac{\partial t_2}{\partial v}\right)}{2d},$$

where $d = \sqrt{g_{11} g_{22} - g_{12}^2}$, $t_1 = \frac{g_{12} \frac{\partial g_{11}}{\partial v}}{d g_{11}} - \frac{\partial g_{22}}{\partial u}$, $t_2 = \frac{2 \frac{\partial g_{11}}{\partial u}}{d} - \frac{\partial g_{11}}{\partial v} - \frac{g_{12} \frac{\partial g_{11}}{\partial u}}{d g_{11}}$ (see e.g. [4]). In our case $d = f\sqrt{x}$, $t_1 = -2\frac{f'}{\sqrt{x}}$, $t_2 = 0$ so we obtain the differential equation (15).

Conversely (14) and (15) yield $K = -1$ by (17). Finally, a simple substitution shows that the functions (16) satisfy the equation (15).

Examples 1. If $f(u) = \cosh u$ then we obtain for $x(u) = x_1(u)$, $x(u) = x_2(u)$, $x(u) = x_3(u)$ the metrics $ds^2 = du^2 + \cosh^2 u dv^2$, $ds^2 = \tanh^2 u du^2 + \cosh^2 u dv^2$ and $ds^2 = \frac{\sinh^2 u}{\cosh^2 u + 1} du^2 + \cosh^2 u dv^2$, respectively.

Examples 2. In case $f(u) = \frac{1}{u}$ we obtain for $x(u) = x_1(u)$, $x(u) = x_2(u)$, $x(u) = x_3(u)$ the metrics $ds^2 = \frac{1}{u^2(1-u^2)} du^2 + \frac{1}{u^2} dv^2$, $ds^2 = \frac{1}{u^2} (du^2 + dv^2)$ and $ds^2 = \frac{1}{u^2(1+u^2)} du^2 + \frac{1}{u^2} dv^2$, respectively.

Let us consider the mapping

$$\rho : \mathbb{R}^n(u, v_r) \longrightarrow \mathbf{E}^{4n-3}$$

given in Cartesian coordinates by

$$(18) \quad x_0(u, v_r) = x_0(u) = \int_0^u \sqrt{1 - f_1'(y)^2 - f_2'(y)^2} dy$$

$$(19) \quad x_{r1}(u, v_r) = f_1(u) \sin(v_r \psi_1(u))$$

$$(20) \quad x_{r2}(u, v_r) = f_1(u) \cos(v_r \psi_1(u))$$

$$(21) \quad x_{r3}(u, v_r) = f_2(u) \sin(v_r \psi_2(u))$$

$$(22) \quad x_{r4}(u, v_r) = f_2(u) \cos(v_r \psi_2(u))$$

$$-\infty < u, v_r < \infty \quad (r = 1, \dots, n-1)$$

where

$$f_1(u) = \frac{\varphi_1(u)e^u}{\psi_1(u)}, \quad f_2(u) = \frac{\varphi_2(u)e^u}{\psi_2(u)}$$

and $\varphi_1, \varphi_2, \psi_1, \psi_2$ are the same as in §1.

Theorem (the main result). *The mapping ρ defines a $\mathbf{C}^\infty$ isometric immersion of the n -dimensional hyperbolic space $\mathbf{H}^n$ into $\mathbf{E}^{4n-3}$.*

To prove the Theorem we need the next

Lemma. *The n -dimensional metric*

$$(23) \quad ds^2 = (f'/f)^2 du^2 + f^2(u) \sum_{i=2}^n dv_i^2$$

has curvature $K = -1$. In other words

$$(24) \quad R_{ijkl} = -(g_{ik}g_{jl} - g_{il}g_{jk}).$$

PROOF. We get by an easy calculation that the Christoffel symbols of the first kind are

$$c_{ijk} = \begin{cases} \frac{f(u)'f(u)''}{f(u)^2} - \frac{f(u)'^3}{f(u)^3}, & i = j = k = 1 \\ f(u)f'(u), & i = 1, j = k > 1 \\ -f(u)f'(u), & i = j > 1, k = 1 \\ 0, & \text{otherwise,} \end{cases}$$

and the Christoffel symbols of the second kind are

$$C_{ij}^k = \begin{cases} \frac{f(u)^2}{f(u)'^2} \left(\frac{f(u)'f(u)''}{f(u)^2} - \frac{f(u)'^3}{f(u)^3} \right), & i = j = k = 1 \\ \frac{f'(u)}{f(u)}, & i = 1, j = k > 1 \\ -\frac{f(u)^3}{f'(u)}, & i = j > 1, k = 1 \\ 0, & \text{otherwise.} \end{cases}$$

From these it follows that

$$(25) \quad R_{ijkl} = \begin{cases} -f'(u)^2, & i = k = 1, j = l > 1 \\ -f(u)^4, & i = k > 1, j = l > 1 \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand,

$$(26) \quad g_{ik}g_{jl} - g_{il}g_{jk} = \begin{cases} f'(u)^2, & i = k = 1, j = l > 1 \\ f(u)^4, & i = k > 1, j = l > 1 \\ 0, & \text{otherwise.} \end{cases}$$

From (25) and (26) we obtain (24). By the Lemma, (23) implies that $R_{ijkl} = -(g_{ik}g_{jl} - g_{il}g_{jk})$ and thus $K = -1$ is true.

PROOF of the Theorem. Functions (18)–(22) are $\mathbf{C}^\infty$, so ρ is also $\mathbf{C}^\infty$.

Straightforward calculations show that the images of the tangents to the parametric lines are linearly independent and hence $\rho : \mathbb{R}^n \longrightarrow \mathbf{E}^{4n-3}$ is an immersion. The induced metric of $\rho(\mathbb{R}^n)$ is

$$ds^2 = du^2 + e^{2u} \sum_{i=2}^n dv_i^2.$$

Thus, according to the Lemma, it has curvature $K = -1$, and therefore ρ determines an isometric immersion of $\mathbf{H}^n = (\rho(\mathbb{R}^n), ds^2)$ into $\mathbf{E}^{4n-3}$.

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(Received March 23, 1993; revised December 5, 1994)