

On iterative roots of order n of some multifunctions with a unique set-value point

By GRAŻYNA LYDZIŃSKA (Katowice)

Abstract. In [6], W. JARCZYK and W. ZHANG considered the existence of square iterative roots of multifunctions with exactly one set-value point, and next, together with L. LI and J. JARCZYK, continued that study in [9]. In this paper, we improve and generalize the main results from [6] and one of the theorems from [9]. Moreover, our generalization also deals with the iterative roots of order n .

1. Introduction

Throughout this paper, we assume that X is an arbitrary set.

Iterative roots of order n (where n is a positive integer, $n \geq 2$) of a given mapping $f : X \rightarrow X$ are functions $g : X \rightarrow X$ satisfying the condition

$$g^n = \underbrace{g \circ \cdots \circ g}_{n\text{-times}} = f.$$

The existence of iterative roots for single-valued functions is a problem initially formulated and studied in 1815 by C. BABBAGE [1]. Since then, it has been extensively studied by many mathematicians. For details, you can see the books [14] by GY. TARGONSKI, [7] by M. KUCZMA, and [8] by M. KUCZMA, B. CHOCZEWSKI and R. GER. Some results have been presented in the survey papers [2] and [3]. To see both recent and historical results, it is worth reading [4], written by W. JARCZYK. Some natural ideas of using set-valued functions (instead of g) have been examined by T. POWIERZA in [11], [12], [13], and together with W. JARCZYK in [5].

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We can consider replacing single-valued functions by set-valued functions both for f and g (for details, see Preliminaries). Current results of such researches for f with a unique set-value point and $n = 2$, we can find in [6], [9] and [10]. Below we present some theorems from the first two of them.

Let $\#A$ denote the cardinality of a subset $A \subset X$. Fix $c \in X$. In what follows, $\mathcal{F}_c(X)$ stands for the set of all multifunctions $f : X \rightarrow 2^X$ satisfying the following two conditions:

- (i) $\#f(c) > 1$;
- (ii) $\#f(x) = 1$, for every $x \in X \setminus \{c\}$.

Moreover, we define

$$\mathcal{F}_c^{\{c\}}(X) := \{f \in \mathcal{F}_c(X) : \{c\} \text{ is a value of } f\}.$$

In [6], W. JARCZYK and W. ZHANG considered the existence of square iterative roots of multifunctions from the class $\mathcal{F}_c^{\{c\}}(X)$, and obtained the following main results.

Theorem A ([6, Theorem 1]). *Let $f \in \mathcal{F}_c^{\{c\}}(X)$. If there exists a positive integer k such that*

- (i) $\#f(c) > k$; and
- (ii) $\#\{x \in X : f(x) = \{y\}\} \leq k$, for every $y \in X$,

then f has no square iterative roots.

Theorem B ([6, Theorem 2]). *Let $f \in \mathcal{F}_c^{\{c\}}(X)$. If $c \in f(c)$, then f has no square iterative roots.*

In [9], the same authors, together with L. LI and J. JARCZYK, studied mainly the multifunctions for which $\{c\}$ is not a value, i.e., from the class $\mathcal{F}_c(X) \setminus \mathcal{F}_c^{\{c\}}(X)$, and X is an interval (similarly, in [10], only such a case was considered with an arbitrary set X), but in the first part of the paper, they proved the following theorem.

We say that a multifunction $f : X \rightarrow 2^X$ is *one-to-one on a set* $A \subset X$ if $f(x_1) \neq f(x_2)$ for all different $x_1, x_2 \in A$.

Theorem C ([9, Theorem 1]). *Let $f \in \mathcal{F}_c^{\{c\}}(X)$. Then the multifunction f has no square iterative roots, one-to-one on the set $f(c)$. If, in addition, f is one-to-one on $f(c)$, then f has no square iterative roots at all.*

In this paper, we continue the study of the existence of iterative roots of the multifunctions from $\mathcal{F}_c^{\{c\}}(X)$ and improve the necessary conditions which

have been shown in Theorems A, B and C. Moreover, our considerations are not limited to square iterative roots, we obtain results also for the iterative roots of order n .

2. Preliminaries

For $f : X \rightarrow 2^Y$, the image $f(A)$ of a set $A \subset X$ is defined by

$$f(A) := \bigcup_{x \in A} f(x).$$

The composition $g \circ f$ of set-valued functions $f : X \rightarrow 2^Y$ and $g : Y \rightarrow 2^Z$ is given by

$$(g \circ f)(x) := g(f(x)).$$

It is easy to notice that the above operation is associative.

For every $n \in \mathbb{N}$, we can define the n -th iterate of $g : X \rightarrow 2^X$ as the composition of n copies of g :

$$g^n := \underbrace{g \circ \cdots \circ g}_{n\text{-times}}.$$

Let $f : X \rightarrow 2^X$, $n \in \mathbb{N}$ and $n \geq 2$. A multifunction $g : X \rightarrow 2^X$ is called an iterative root of order n of multifunction f if

$$g^n = f.$$

Of course, if $g : X \rightarrow 2^X$ is an iterative root of $f : X \rightarrow 2^X$, then $f \circ g = g \circ f$.

Remark 1. Let $f \in \mathcal{F}_c^{\{c\}}(X)$ and $n \in \mathbb{N}$, $n \geq 2$. If $g : X \rightarrow 2^X$ is an iterative root of order n of multifunction f , then $g(c) \neq \{c\}$.

PROOF. Suppose that $g(c) = \{c\}$. We obtain

$$f(c) = g^{n-1}(g(c)) = g^{n-1}(c) = \cdots = \{c\}.$$

This contradiction completes the proof. $\square$

Lemma 1. Let $f \in \mathcal{F}_c^{\{c\}}(X)$ and $n \in \mathbb{N}$, $n \geq 2$. If $g : X \rightarrow 2^X$ is an iterative root of order n of multifunction f , then

$$\#g^{n-k}(c) = 1$$

for every $k \in \mathbb{N}$ which is a divisor of n , $k \neq n$.

PROOF. Let $k \in \mathbb{N}$, $k \neq n$ be a divisor of n . Therefore, for some $m \in \mathbb{N}$, $m > 1$, we have $n = mk$. Since f has only non empty values, g has too. Consequently, g^{n-k} has only non empty values. Suppose that

$$\#g^{n-k}(c) > 1. \quad (1)$$

Let $x_0 \in X$ satisfy condition $f(x_0) = \{c\}$. Take an $x \in g^{n-k}(x_0)$. We have

$$g^k(x) \subset g^k(g^{n-k}(x_0)) = g^n(x_0) = f(x_0) = \{c\},$$

thus

$$g^k(x) = \{c\}, \quad (2)$$

and consequently,

$$f(x) = g^{n-k}(g^k(x)) = g^{n-k}(c).$$

According to (1), we get $\#f(x) > 1$, and hence $x = c$. Thus, by (2), we obtain that $g^k(c) = \{c\}$, and

$$f(c) = g^{mk}(c) = g^{(m-1)k}(g^k(c)) = g^{(m-1)k}(c) = \dots = g^k(c) = \{c\},$$

which contradicts the assumption on f and completes the proof. $\square$

3. Main result

Theorem 1. *Let $f \in \mathcal{F}_c^{\{c\}}(X)$ and $n \in \mathbb{N}$, $n \geq 2$. If $g : X \rightarrow 2^X$ is an iterative root of order n of f , then there exists a $k \in \mathbb{N}$, $1 \leq k < n$ such that*

$$g^k|_{f(c)} \text{ is constant and single-valued.} \quad (\text{G})$$

More precisely,

- (i) if $g^{n-1}(c) \neq \{c\}$, then condition (G) holds for $k = n - 1$;
- (ii) if $g^{n-1}(c) = \{c\}$, then $n > 2$ and $g^k|_{f(c)} = \{c\}$ for $k = n - 2$.

PROOF. Of course, g has only non-empty values.

At first consider the case $g^{n-1}(c) \neq \{c\}$. By Lemma 1, we have $\#g^{n-1}(c) = 1$, and since

$$g^{n-1}(f(c)) = f(g^{n-1}(c)),$$

we obtain

$$\#g^{n-1}(f(c)) = \#f(g^{n-1}(c)) = 1.$$

Thus $g^{n-1}|_{f(c)}$ is constant and single-valued.

Now assume that $g^{n-1}(c) = \{c\}$. By Remark 1, we obtain $n \geq 3$. We have

$$g^{n-1}(g^{n-1}(c)) = g^{n-1}(c) = \{c\},$$

whence

$$g^{2(n-1)}(c) = \{c\}.$$

For $k := 2(n-1) - n = n-2$, we obtain

$$g^k(f(c)) = g^k(g^n(c)) = g^{k+n}(c) = g^{2(n-1)}(c) = \{c\}.$$

Thus $g^{n-2}|_{f(c)} = \{c\}$. □

Now we present the main result of this paper.

Theorem 2. *Let $f \in \mathcal{F}_c^{\{c\}}(X)$ and $n \in \mathbb{N}$, $n \geq 2$. If f has an iterative root of order n , then $f|_{f(c)}$ is constant and single-valued.*

PROOF. Assume that there exists a multifunction $g : X \rightarrow 2^X$ such that $f = g^n$. By Theorem 1, there exists a $k \in \mathbb{N}$, $1 \leq k < n$ for which the condition (G) is satisfied. Let $g^k|_{f(c)} = \{y_0\}$ for some $y_0 \in X$. Then

$$f(f(c)) = g^n(f(c)) = g^{n-k}(g^k(f(c))) = g^{n-k}(y_0).$$

Let $a \in f(c)$ be such that $a \neq c$. Observe that

$$f(a) = g^{n-k}(g^k(a)) = g^{n-k}(y_0).$$

Therefore,

$$f(f(c)) = f(a),$$

whence, by our assumption on a , we get $\#f(f(c)) = 1$. Thus $f|_{f(c)}$ is constant and single-valued. □

Remark 2. Notice that for a multifunction $f \in \mathcal{F}_c^{\{c\}}(X)$, the following conditions are equivalent:

- (i) $f|_{f(c)}$ is single-valued;
- (ii) $c \notin f(c)$.

In the last part of this paper, we show that the above theorem is stronger than the theorems mentioned in the Introduction. For Theorem C it is obvious (see Theorem 1 and Theorem 2). For Theorem B it is a consequence of Remark 2 and Theorem 2.

To prove that Theorem A follows from Theorem 2, assume that $f \in \mathcal{F}_c^{\{c\}}(X)$ satisfies the assumptions (i) and (ii) of Theorem A with $k \in \mathbb{N}$. We will prove that $f|_{f(c)}$ is not constant, which will complete our proof. Suppose that $f|_{f(c)}$ is constant. Then $f|_{f(c)}$ is single-valued, and

$$f(c) \subset \{x \in X : f(x) = \{y\}\}, \quad \text{for some } y \in X,$$

whence, by assumption (ii) of Theorem A, we get $\#f(c) \leq k$, contrary to (i) of Theorem A.

Moreover, notice that for the function f in Figure 1, none of the above-mentioned theorems from papers [6] and [9] gives us an answer to the question if f has iterative square roots. Due to Theorem 2, we know that such a function does not have iterative roots of any order.

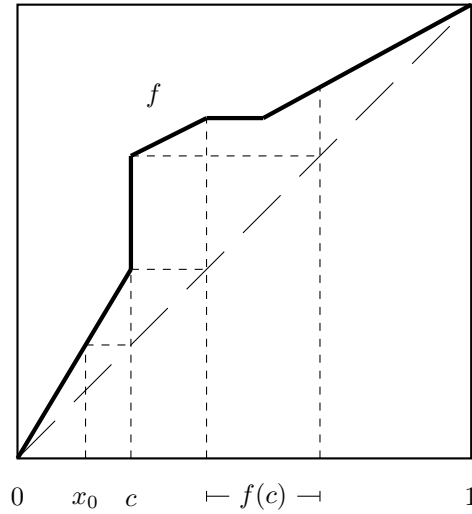


Figure 1: f does not have iterative roots of any order.

Example 1. Let $f : [0, 1] \rightarrow 2^{[0,1]}$ be given by

$$f(x) = \begin{cases} (0, 1], & x = 0, \\ \{0\}, & x \in (0, 1]. \end{cases}$$

Notice that $f \in \mathcal{F}_c^{\{c\}}(X)$ with $c = 0$ and $f|_{f(c)} = f|_{(0,1]}$ is constant and single-valued, but f has no iterative root of order 2. Indeed, suppose that $g : [0, 1] \rightarrow 2^{[0,1]}$ is a square iterative root of f . Due to Lemma 1, we have $\#g(0) = 1$, and by Remark 1, we get $g(0) \neq \{0\}$. Thus $g(0) \subset (0, 1] = f(0)$, and in consequence,

$$f(0) = g(g(0)) \subset g(f(0)).$$

Hence, by Theorem 1, we deduce that $\#f(0) = 1$, which is not true. It shows that the condition in Theorem 2 is necessary, but it is not sufficient.

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GRAŻYNA LYDZIŃSKA
INSTITUTE OF MATHEMATICS
UNIVERSITY OF SILESIA
BANKOWA 14
PL-40-007 KATOWICE
POLAND

E-mail: lydzinska@math.us.edu.pl

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