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On the simultaneous equations $\sigma(2^a) = p^{f_1}q^{g_1}$, $\sigma(3^b) = p^{f_2}q^{g_2}$, $\sigma(5^c) = p^{f_3}q^{g_3}$

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Abstract. Let $\sigma(N)$ denote the sum of divisors of N. We shall solve the simultaneous equations $\sigma(2^a) = p^{f_1}q^{g_1}$, $\sigma(3^b) = p^{f_2}q^{g_2}$, $\sigma(5^c) = p^{f_3}q^{g_3}$ with p, q distinct primes.

1. Introduction

As usual, let $\sigma(N)$ denote the sum of divisors of N, and $\omega(N)$ the number of distinct prime factors of N. In [18], the author has shown that there are only finitely many odd superperfect numbers (i.e., the number satisfying $\sigma(\sigma(N)) = 2N$) with bounded numbers of distinct prime factors, by proving that the simultaneous equations $\sigma(p_i^{e_i}) = q_1^{f_{1i}} \cdots q_k^{f_{ki}}$ for $k+1$ prime powers $p_i^{e_i} (i = 1, 2, \ldots, k+1)$ cannot have solutions with $p_1, \ldots, p_{k+1}$ all small. In this paper, we use the method developed in [18] to solve the simultaneous equations $\sigma(2^a) = p^{f_1}q^{g_1}$, $\sigma(3^b) = p^{f_2}q^{g_2}$, $\sigma(5^c) = p^{f_3}q^{g_3}$ with p, q distinct primes.

Wakulicz [16] has shown that all solutions of the purely exponential diophantine equation $2^n - 5^m = 3$ are $(n, m) = (2, 0), (3, 1)$ and $(7, 3)$, from which Makowski and Schinzel [9] derived that the equation $\sigma(2^a) = \sigma(5^c)$ has only the solution $(a, c) = (4, 2)$. We note that it is easy to show that $\sigma(2^a) = \sigma(3^b)$ has no nontrivial solution, and $\sigma(3^b) = \sigma(5^c)$ also has no nontrivial solution. Bugeaud and Mignotte [3] have shown that neither of $\sigma(2^a)$, $\sigma(3^b)$, $\sigma(5^c)$ can be a perfect power except $\sigma(3) = 2^2$ and $\sigma(3^4) = 11^2$. Moreover, they have shown

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that the only perfect powers $\frac{x^n-1}{x-1}$, with $x = z^t$, $z \le 10$, are $(3^5-1)/2 = 11^2$ and $(7^4-1)/6 = 20^2$.

Now we shall state our result.

Theorem 1.1. *The simultaneous equations*

$$\sigma(2^a) = p^{f_1}q^{g_1}, \quad \sigma(3^b) = p^{f_2}q^{g_2}, \quad \sigma(5^c) = p^{f_3}q^{g_3}, \tag{1}$$

with $a, b, c > 0$, $f_1, f_2, f_3, g_1, g_2, g_3 \ge 0$ and p, q distinct primes, have only the following solutions:

(i) $(a, b, c) = (1, 1, 1)$,

(ii) $(a, b, c) = (4, 1, 2)$,

(iii) $(a, b, c) = (4, 4, 2)$ *and*

(iv) $(a, c) = (4, 2)$ *and $\sigma(3^b)$ is prime.*

In other words, if $\omega(\sigma(2^a3^b5^c)) \le 2$, then (a, b, c) must satisfy one of the above conditions.

Our result is related to the Nagell–Ljunggren equation

$$\frac{x^m-1}{x-1} = y^n, \qquad x \ge 2,\ y \ge 2,\ m \ge 3,\ n \ge 2, \tag{2}$$

which has been conjectured to have only three solutions $(x, y, m, n) = (3, 11, 5, 2)$, $(7, 20, 4, 2)$ and $(18, 7, 3, 3)$. Some of recent remarkable results concerning to the Nagell–Ljunggren equation are [2], [3], [4], [11] and [12]. Our result leads us to conjecture that the diophantine equation

$$\frac{x^\ell-1}{x-1} = y^m z^n \tag{3}$$

has only finitely many solutions in integers $x \ge 2$, $z \ge y \ge 2$ and $\ell, m, n \ge n_0$ for some constant n_0. The *abc*-conjecture, which Mochizuki [13] claims to prove, would allow us to take $n_0 = 3$. More exactly, assuming the *abc*-conjecture, we could prove that any integer solution of (3) with $\ell \ge 3$, $m \ge 1$, $n \ge 2$, $1 \le y < z$, and x^ℓ sufficiently large must satisfy $(\ell, m, n) = (4, 1, 2)$, $(3, 1, 3)$ or $(\ell, n) = (3, 2)$.

Indeed, applying the *abc*-conjecture to the equation $1 + (x-1)y^m z^n = x^\ell$, we see that for any given $\epsilon > 0$, the inequality

$$x^2yz > x(x-1)yz \ge \prod_{p \mid (x-1)x^\ell y^m z^n} p > x^{\ell(1-\epsilon/2)} \tag{4}$$

would hold for a sufficiently large x^ℓ. Hence, the inequality

$$\frac{2}{\ell} + \frac{\ell - 1}{\ell} \times \frac{2}{n + \min\{n, m\}} > 1 - \epsilon \tag{5}$$

would also hold for a sufficiently large x^ℓ. In particular, taking $\epsilon = 1/15$, we see that the left of (5) must be greater than $14/15$ for a sufficiently large x^ℓ. Recalling that $n \geq 2$ and $m \geq 1$, we obtain that $\ell \leq 4$. We must have $n + \min\{n, m\} = 3$ for $\ell = 4$, and $n + \min\{n, m\} \leq 4$ for $\ell = 3$. So, either of $(\ell, m, n) = (4, 1, 2)$, $(3, 1, 3)$ or $(\ell, n) = (3, 2)$ should hold for a sufficiently large x^l.

2. Preliminary lemmas

In this section, we introduce some preliminary lemmas. One is Matveev's lower bound for linear forms of logarithms [10].

Lemma 2.1. *Let $a_1, a_2, \ldots, a_n$ be positive integers such that $\log a_1, \ldots, \log a_n$ are not all zero and $A_j \geq \max\{0.16, \log a_j\}$ for each $j = 1, 2, \ldots, n$. Moreover, we put*

$$B = \max\{1, |b_1| A_1/A_n, |b_2| A_2/A_n, \ldots, |b_n|\},$$
$$\Omega = A_1 A_2 \ldots A_n, \qquad C_0 = 1 + \log 3 - \log 2,$$
$$C_1(n) = \frac{8}{(n-1)!} e^{n+1} (2n+3)(n+2)(4(n+1))^{n+1} \times (4.4n + 5.5 \log n + 7), \tag{6}$$

and

$$\Lambda = b_1 \log a_1 + \ldots + b_n \log a_n. \tag{7}$$

Then we have

$$\log |\Lambda| > -C_1(n)(C_0 + \log B) \max\left\{1, \frac{n}{6}\right\} \Omega. \tag{8}$$

The next lemmas deal with some arithmetical properties of values of cyclotomic polynomials. Lemma 2.2 is a basic and well-known result of this area. Lemma 2.2 has been proved by Zsigmondy [19], and rediscovered by many authors such as Dickson [6] and Kanold [7]. We need only the special case $b = 1$, for which this lemma had already been proved by Bang [1]. See also Theorem 6.4A.1 in [14].

Lemma 2.2. *If $a > b \geq 1$ are coprime integers, then $a^n - b^n$ has a prime factor which does not divide $a^m - b^m$ for any $m < n$, unless $(a, b, n) = (2, 1, 6)$, $a - b = n = 1$, or $n = 2$, and $a + b$ is a power of 2.*

Let $o_p(a)$ denote the residual order of $a \pmod p$. Lemma 2.2 immediately gives the following result.

Lemma 2.3. *If $(a^e-1)/(a-1) = p^{f_1}q^{f_2}$ for some integers a, e, f_1, f_2 and primes $p < q$, then we have $(a, e, p, q, f_1, f_2) = (2, 6, 3, 7, 2, 1)$, $e = r$ or $e = r^2$ for some prime r. Moreover, in the case $e = r$, then we have $p = r$, $o_q(a) = r$ or $o_p(a) = o_q(a) = r$. In the case $e = r^2$, we have $(p^{f_1}, q^{f_2}) = ((a^r-1)/(a-1), (a^{r^2}-1)/(a^r-1))$, $(p^{f_1}, q^{f_2}) = ((a^{r^2}-1)/(a^r-1), (a^r-1)/(a-1))$ or $(a, e, p, f_1) = (2^m-1, 4, 2, m+1)$ for some integer m.*

Proof. If e has at least two distinct prime factors and $(a, e) \neq (2, 6)$, then e must have at least four distinct divisors. By Lemma 2.2, for each divisor $d > 1$ of e, $(a^d-1)/(a-1)$ has a prime factor which does not divide $(a^m-1)/(a-1)$ for any $m < d$, and therefore $(a^e-1)/(a-1)$ must have at least three distinct prime factors, contrary to the assumption. Hence, $(a, e) = (2, 6)$ or e must be a prime power. If $e = r^l$ is power of a prime r, then, for each $k \leq l$, $(a^{r^k}-1)/(a-1)$ has a prime factor which divides none of $(a^m-1)/(a-1)$ with $m < r^k$. Thus we must have $l \leq 2$.

If $e = r$, then $o_p(a) = 1$ or r. If $o_p(a) = 1$, then $a \equiv 1 \pmod p$ and $(a^r-1)/(a-1) \equiv r \pmod p$. Hence, we must have $p = r$. Now we see that $q > p = r$ and $o_q(a) = r$, since $o_q(a) = 1$ should yield $q = r$ as before, which is clearly a contradiction. If $o_p(a) = r$, then $r \equiv 1 \pmod p$, and therefore $q > p > r$. Hence, we must have $o_p(a) = o_q(a) = r$.

If $e = r^2$, then $(a^{r^2}-1)/(a-1)$ must have a prime factor not dividing $(a^r-1)/(a-1)$. Hence, $(a^r-1)/(a-1)$ must be a prime power. If a prime p (or q) divides both $(a^r-1)/(a-1)$ and $(a^{r^2}-1)/(a^r-1)$, then, by Lemma 6.4A.2 in [14], we must have $o_p(a) = 1$ and $p = r$ (or $o_q(a) = 1$ and $q = r$). However, by Lemma 2.2, this occurs only if $r = 2$ and $a+1 = 2^m$ for some integer m. □

The following lemma is proved in [3], as mentioned in the Introduction.

Lemma 2.4. *Let a, e, x, f be positive integers with $a, x, f > 1$ and $e > 2$. The equation $(a^e-1)/(a-1) = x^f$ has no solution but $(a, e, x, f) = (3, 5, 11, 2)$, $(7, 4, 20, 2)$ in integers $2 \leq a \leq 10$, $e > 2$, $x > 1$, $f > 1$.*

Using results mentioned in the Introduction, we can immediately solve some special case of our main theorem.

Lemma 2.5. *Choose $a < b$ from the first three primes $2, 3, 5$. If $(a^e-1)/(a-1) = p^k$ and $(b^f-1)/(b-1) = p^l$ for some integers e, f, k, l and some prime p, then $(a^e, b^f) = (2^5, 5^3)$ and $p = 31$, $k = l = 1$.*

PROOF. In the case $k = l = 1$ and $(a^e - 1)/(a - 1) = (b^f - 1)/(b - 1)$, as observed in the Introduction, we have $(a^e, b^f) = (2^5, 5^3)$.

Lemma 2.4 yields that the perfect power case must arise from $(3^5-1)/2 = 11^2$ or $(3^2 - 1)/2 = 2^2$. In this case, we must have $2^e - 1 = 2$ or 11 or $(5^f - 1)/4 = 2$ or 11, which is clearly impossible. □

3. Bounding the smallest exponent

For convenience, we put $a_1 = 2$, $a_2 = 3$, $a_3 = 5$ and $e_1 = a + 1$, $e_2 = b + 1$, $e_3 = c + 1$. In this section, we would like to give an absolute and explicit upper bound for the smallest one among $a_i^{e_i}$'s, which is the main part of our argument.

Lemma 3.1. *For each $i = 1, 2, 3$, we have*

$$e_i \log a_i < E_i = C_i \log p \log q (\log\log p + C_{i+3}), \tag{9}$$

where $C_1 = 1.422 \times 10^{10}$, $C_2 = 1.226 \times 10^{12}$, $C_3 = 1.795 \times 10^{12}$, $C_4 = 23.3$, $C_5 = 27.8$, $C_6 = 28.1$.

PROOF. Let $\Lambda_i = f_i \log p + g_i \log q + \log(a_i - 1) - e_i \log a_i = \log(1 - a_i^{-e_i})$ for $i = 1, 2, 3$. It immediately follows from Matveev's theorem that

$$-\log|\Lambda_1| < C(3)\left(C_0 + \log\left(\frac{e_1 \log 2}{\log q}\right)\right) \log 2 \log p \log q, \tag{10}$$

and

$$-\log|\Lambda_j| < C(4)\left(C_0 + \log\left(\frac{e_j \log a_j}{\log q}\right)\right) \log 2 \log a_j \log p \log q, \tag{11}$$

for $j = 2, 3$.

Now we shall prove (9) in the case $i = 1$. We may assume that $e_1 > 10^{10} \log q / \log 2$. Since $0 < |\Lambda_1| = -\log(1 - 2^{-e_1}) < 1/(2^{e_1} - 1)$, we have

$$-\log|\Lambda_1| > \log(2^{e_1} - 1) > (1 - 10^{-10}) e_1 \log 2. \tag{12}$$

Combining upper and lower bounds for Λ_1, we obtain

$$\begin{aligned}\frac{e_1 \log 2}{\log q} &< (1 + 10^{-10})\left(C_0 + \log\left(\frac{e_1 \log 2}{\log q}\right)\right) C(3) \log 2 \log p \\ &< 1.244 \times 10^{10} \log p \log\left(\frac{e_1 \log 2}{\log q}\right).\end{aligned} \tag{13}$$

Hence, observing that $1.244 \times 10^{10} \log p \geq 1.244 \times 10^{10} \log 2$, we obtain

$$\begin{aligned} \frac{e_1 \log 2}{\log q} &< 1.143 \times (1.244 \times 10^{10} \log p) \log(1.244 \times 10^{10} \log p) \\ &< 1.422 \times 10^{10} (\log\log p + 23.3), \end{aligned} \tag{14}$$

giving (9) in the case $i = 1$.

Next we shall prove (9) in the case $i = 2$. We may assume that $e_2 > 10^{10} \log q / \log 3$ as in the previous case. From $0 < |\Lambda_2| = -\log(1 - 3^{-e_2}) < 1/(3^{e_2} - 1)$, we see that

$$-\log|\Lambda_2| > \log(3^{e_2} - 1) > (1 - 10^{-10}) e_2 \log 3, \tag{15}$$

and therefore

$$\begin{aligned} \frac{e_2 \log 3}{\log q} &< (1 + 10^{-10}) \left(C_0 + \log\left(\frac{e_2 \log 3}{\log q} \right) \right) C(4) \log 2 \log 3 \log p \\ &< 1.089 \times 10^{12} \log p \log\left(\frac{e_2 \log 3}{\log q} \right). \end{aligned} \tag{16}$$

This gives (9) in the case $i = 2$.

Similarly, (9) in the case $i = 3$ follows from

$$\begin{aligned} \frac{e_3 \log 5}{\log q} &< (1 + 10^{-10}) \left(C_0 + \log\left(\frac{e_3 \log 5}{\log q} \right) \right) C(4) \log 2 \log 5 \log p \\ &< 1.595 \times 10^{12} \log p \log\left(\frac{e_3 \log 5}{\log q} \right). \end{aligned} \tag{17}$$

This completes the proof of the lemma. □

Next, we shall show that we cannot have all of $a_i^{e_i}$'s large.

Lemma 3.2. *Let x be the smallest among $a_i^{e_i}$'s. Let $h_1 = f_2 g_3 - f_3 g_2$, $h_2 = f_3 g_1 - f_1 g_3$ and $h_3 = f_1 g_2 - f_2 g_1$ and $H = \max |h_i|$. Then*

$$\log x \leq \log\left(\frac{7H}{4} \right) + C(3)(C_0 + \log((e_1 + 3)H)) \log 2 \log 3 \log 5. \tag{18}$$

Proof. We begin by observing that

$$(2^{e_1} - 1)^{h_1} \left(\frac{3^{e_2} - 1}{2} \right)^{h_2} \left(\frac{5^{e_3} - 1}{4} \right)^{h_3} = 1. \tag{19}$$

Now we put

$$\begin{aligned}\Lambda &= (e_1h_1 - h_2 - 2h_3)\log 2 + e_2h_2\log 3 + e_3h_3\log 5\\ &= h_1\log\left(\frac{2^{e_1}}{2^{e_1}-1}\right) + h_2\log\left(\frac{3^{e_2}}{3^{e_2}-1}\right) + h_3\left(\log\frac{5^{e_3}}{5^{e_3}-1}\right). \end{aligned} \tag{20}$$

Then we have

$$0 < |\Lambda| \le H\left(\frac{1}{2^{e_1}-1} + \frac{1}{3^{e_2}-1} + \frac{1}{5^{e_3}-1}\right) \le \frac{7H}{4x}, \tag{21}$$

and therefore

$$\log|\Lambda| \le -\log x + \log\left(\frac{7H}{4}\right). \tag{22}$$

It follows from the assumption $e_i > 0$ that $\Lambda \neq 0$. Hence, Matveev's lower bound gives

$$\log|\Lambda| \ge -C(3)(C_0 + \log((e_1+3)H))\log 2\log 3\log 5. \tag{23}$$

Combining (22) and (23), we obtain (18). □

The third step is to obtain upper bounds for each e_i.

Lemma 3.3. *Unless $x = p = 31$, we have $e_1 < 4.44 \times 10^{52}$, $e_2 < 2.54 \times 10^{54}$ and $e_3 < 2.55 \times 10^{54}$, and $H < 2.89 \times 10^{68}$.*

Proof. We may assume without the loss of generality that $p < q$. We begin by considering the case $q \mid x$. In this case, we have

$$\log q < \log x < \log\left(\frac{7H}{4}\right) + C(3)(C_0 + \log((e_1+3)H))\log 2\log 3\log 5. \tag{24}$$

We note that

$$H \le C_2C_3\log p\log q(\log\log p + C_5)(\log\log p + C_6), \tag{25}$$

since it follows from Lemma 3.1 that

$$f_i < e_i\log a_i/\log p_i < C_i\log q(\log\log p + C_{i+3}) \tag{26}$$

and

$$g_i < e_i\log a_i/\log q_i < C_i\log p(\log\log p + C_{i+3}). \tag{27}$$

Hence, we obtain $\log p < \log q < 4.35 \times 10^{12}$.

Now we consider the case $p < q$ and $q \nmid x$. Put i to be the index such that $x = (a_i^{e_i} - 1)/(a_i - 1)$, j, k be the other two and

$$\begin{aligned}\Lambda' &= e_j h_j \log a_j + e_k h_k \log a_k - h_j \log(a_j - 1) - h_k \log(a_k - 1) + h_i \log x \\ &= h_j \log\left(\frac{a_j^{e_j}}{a_j^{e_j} - 1}\right) + h_k \log\left(\frac{a_k^{e_k}}{a_k^{e_k} - 1}\right). \end{aligned} \tag{28}$$

It follows from Lemma 2.5 that if $(a_j^{e_j} - 1)/(a_j - 1) = p^{f_j}$ or $(a_k^{e_k} - 1)/(a_k - 1) = p^{f_k}$, then $a_i^{e_i} = 2^5$ or 5^3 and $x = p = 31$. Hence, we see that both numbers $(a_j^{e_j} - 1)/(a_j - 1), (a_k^{e_k} - 1)/(a_k - 1)$ must be divisible by q unless $x = p = 31$.

Thus we obtain

$$0 < \Lambda' < H\left(\frac{1}{a_j^{e_j} - 1} + \frac{1}{a_k^{e_k} - 1}\right) \le \frac{3H}{2q}. \tag{29}$$

As in the previous case, Matveev's theorem now gives

$$\log|\Lambda'| \ge -C(4)\left(C_0 + \log\left(\frac{E_3 H}{\log x}\right)\right)\log 2 \log 3 \log 5 \log x. \tag{30}$$

Combining (29) and (30), we obtain

$$\log q \le \log\left(\frac{3H}{2}\right) + C(4)\left(C_0 + \log\left(\frac{E_3 H}{\log x}\right)\right)\log 2 \log 3 \log 5 \log x. \tag{31}$$

Since

$$E_3 = C_3 \log p \log q(\log\log p + C_6) \le C_3 \log x \log q(\log\log x + C_6) \tag{32}$$

and

$$H < C_2 C_3 (\log q)^2 (\log\log q + C_5)(\log\log q + C_6), \tag{33}$$

combining (18) and (31), we obtain $\log q < 3.45 \times 10^{27}$. Moreover, we have

$$\begin{aligned}\log p = \log x &< \log\left(\frac{7H}{4}\right) + C(3)(C_0 + \log((e_1 + 3)H))\log 2 \log 3 \log 5 \\ &< 7.22 \times 10^{12}. \end{aligned} \tag{34}$$

So, we conclude that in both cases, we have $\log p < 7.22 \times 10^{12}$ and $\log q < 3.45 \times 10^{27}$. Now Lemma 3.1 immediately gives that $e_1 < 4.44 \times 10^{52}$, $e_2 < 2.54 \times 10^{54}$ and $e_3 < 2.55 \times 10^{54}$. Finally, the upper bound $H < 2.89 \times 10^{68}$ follows from $H < C_2 C_3 (\log p)(\log q)(\log\log p + C_6)(\log\log q + C_5)$. □

Now, using the lattice reduction algorithm, we shall obtain feasible upper bounds.

Lemma 3.4. *We have* $\log x < 354.8$. *Moreover, if* $p < q$ *and* q *divides* x, *then* $\log x < 249.5$.

PROOF. We begin by noting that we can assume $x \neq 31$ without the loss of generality.

In order to reduce our upper bounds, we use the LLL lattice reduction algorithm introduced in [8]. Let M be the matrix defined by $m_{12} = m_{13} = m_{21} = m_{23} = 0$, $m_{11} = m_{22} = \gamma$ and $m_{3i} = \lfloor C\gamma \log a_i \rfloor$ for $i = 1, 2, 3$, where C and γ are constants chosen later. Let L denote the LLL-reduced matrix of M, and $l(L)$ the shortest length of vectors in the lattice generated by the column vectors of L.

From the previous lemma, we know that Λ has coefficients with absolute values at most $H \max\{e_1 + 3, e_2, e_3\} < 7.37 \times 10^{122}$. It is implicit in the proof of Lemma 3.7 of DE WEGER's book [17] that if $l(\Gamma) > X_1\sqrt{16 + 4\gamma}$ and $X_1 \geq 7.37 \times 10^{122}$, then $|\Lambda| > X_1/(C\gamma)$.

Taking $C = 10^{370}$, $\gamma = 2$, we can confirm that $l(\Gamma) > X_1\sqrt{16 + 4\gamma}$, and therefore we obtain that $|\Lambda| > 3.685 \times 10^{-248}$. Hence, we have

$$\log x < \log\left(\frac{7H}{4}\right) - \log|\Lambda| < 727.94. \tag{35}$$

We choose the index i such that $x = (a_i^{e_i} - 1)/(a_i - 1)$, and let j, k be the others. From the above estimate for x, we derive that

$$e_i \leq \left\lfloor \frac{\log 2x}{\log a_i} \right\rfloor \leq 1051. \tag{36}$$

We consider the case $p < q$ and q does not divide x. From (31) we obtain $\log q < 3.337 \times 10^{17}$. Lemma 3.1 gives that

$$|h_i| < C_2C_3 \log x \log q(\log\log x + C_6)(\log\log q + C_5) < 1.264 \times 10^{48}, \tag{37}$$

$$|h_j| = |f_i g_k| < C_3 \log x(\log\log q + C_6) < 8.944 \times 10^{16}, \tag{38}$$

$$|e_j| < C_3 \log x \log q(\log\log q + C_6)/\log 2 < 4.306 \times 10^{34}, \tag{39}$$

and similar upper bounds hold for $|h_k|$ and $|e_k|$, respectively. Hence, Λ has coefficients with absolute values at most 3.852×10^{51}. Using the LLL-reduction again with $C = 10^{157}$ and $\gamma = 2$, we obtain $|\Lambda| > 1.926 \times 10^{-106}$, and therefore $\log x < \log(7H/4) - \log|\Lambda| < 354.8$.

Next, we consider the case $p < q$ and q divides x. In this case, we have $\log p < \log q \le \log x < 727.94$. We choose the index i such that $x = (a_i^{e_i} - 1)/(a_i - 1)$ and let j, k be the other two. Lemma 3.1 gives that

$$|h_i| < C_2 C_3 \log^2 x (\log\log x + C_5)(\log\log x + C_6) < 1.392 \times 10^{33}, \tag{40}$$

$$|h_j| \le \max|f_i g_k, f_k g_i| < C_3 \log x (\log\log x + C_6) < 4.533 \times 10^{16}, \tag{41}$$

$$|e_j| < C_3 \log^2 x (\log\log x + C_6)/\log 2 < 4.761 \times 10^{19}, \tag{42}$$

and similar upper bounds hold for $|h_k|$ and $|e_k|$, respectively. Combining these upper bounds with (36), we see that Λ has coefficients with absolute values at most 2.159×10^{36}. We use the LLL-reduction again with $C = 10^{111}$ and $\gamma = 2$, we obtain $|\Lambda| > 1.079 \times 10^{-75}$, and therefore $\log x < \log(7H/4) - \log|\Lambda| < 249.5$. This proves the lemma. □

4. Checking the small ranges

The final step is checking all possibilities of x. We note that from The Cunningham Project (see [15] or [5]), we know all prime factors of x's below our upper bounds.

For $x = (a_i^{e_i} - 1)/(a_i - 1)$, we should check the residual orders of the other prime a_j modulo x. A summary is given in Tables 1–6, where Pn such as P13 in the row $e_1 = 49$ denotes a prime with n digits, and (n) indicates that the residual order is a multiple of n. For example, putting $x = 2^{347} - 1 = pq$ with $p < q$, $o_q(3)$ is divisible by 6, since $q - 1$ is divisible by $2^3 \times 3^2$ and $3^{(q-1)/8}$, $3^{(q-1)/3} \not\equiv 1 \pmod q$, although $3^{(q-1)/4} \equiv 1 \pmod q$, which yields that $(3^{e_2} - 1)/2 = p^{f_2} q^{g_2}$ with $g_2 > 0$ is impossible.

If $p = x = 2^{e_1} - 1$ is prime, then $e_1 \le 511$, and therefore e_1 must belong to the set

$$\{2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127\}.$$

Among them, there exists no e_1 such that $o_x(3) = 1$, being a prime or the square of a prime, as we can see from Table 1. Hence, by Lemma 2.3, we must have $(3^{e_2} - 1)/2 = q^{g_2}$. By Lemma 2.5, $(5^{e_3} - 1)/4$ must be divisible by $p = x$. Hence, by Lemma 2.3, $o_x(5) = 1$ or $o_x(5)$ must be a prime or the square of a prime, and therefore, from Table 1, $e_1 = e_3 = 2$ or $e_1 = 5$, $e_3 = 3$. If $e_1 = e_3 = 2$, then $(5^{e_3} - 1)/4 = 6 = 2 \times 3$, and therefore $(3^{e_2} - 1)/2$ must be a power of 2, yielding that $e_2 = 2$. If $e_1 = 5$ and $e_3 = 3$, then $p = 31$ and $(3^{e_2} - 1)/2 = q^{g_2}$, yielding that $e_2 = 5$ or $(3^{e_2} - 1)/2 = q$.

If $x = 2^{e_1} - 1$ is not a prime power, then $e_1 \leq 359$, and therefore e_1 must belong to the set

$$\{4, 6, 9, 11, 23, 37, 41, 49, 59, 67, 83, 97, 101, 103, 109, 131,$$
$$137, 139, 149, 167, 197, 199, 227, 241, 269, 271, 281, 293, 347\}.$$

Hence, we can write $x = 2^{e_1} - 1 = pq$ for distinct primes $p < q$. By Lemma 2.5, $(3^{e_2} - 1)/2 = p^{g_2}$ and $(5^{e_3} - 1)/4 = p^{g_3}$ cannot simultaneously hold. In other words, at least one of these two integers must be divisible by q. But, for no e_1 in the above set, $o_q(5)$ is 1, a prime or prime-square, as can be seen from Table 2. Hence, $(3^{e_2} - 1)/2$ must be divisible by q. The only e_1 for which $o_q(3)$ is 1, a prime or prime-square is $e_1 = 4$. Then we must have $x = 2^4 - 1 = 3 \times 5$ and $(p, q) = (3, 5)$. But this implies that e_2 is divisible by 4, and $(3^{e_2} - 1)/2$ must be divisible by 2. Hence, $(3^{e_2} - 1)/2$ cannot be of the form $p^{f_2}q^{g_2}$. Hence, it cannot occur that $x = 2^{e_1} - 1$ is not a prime power.

If $x = (3^{e_2} - 1)/2 = p^{f_2}$ is prime or prime power, then

$$e_2 \in \{2, 3, 5, 7, 13, 71, 103\}.$$

For none of them, $o_p(2) = 1, 6$ or a prime power. Hence, as above, $(5^{e_3} - 1)/4$ must be divisible by p. Since $o_p(5)$ must be 1 or a prime power, we must have $e_2 \in \{2, 3, 5\}$. If $e_2 = 2$, then $p = 2$ and $e_3 = 2$, which yields that $q = 3$ and $e_1 = 2$. If $e_2 = 3$, then $p = 13$ and $e_3 = 4$, which is impossible since $(5^{e_3} - 1)/4 = 156 = 2^2 \times 3 \times 13$ has three distinct prime factors. If $e_2 = 5$, then $p = 11$ and $e_3 = 5$. Hence, $(5^{e_3} - 1)/4 = 781 = 11 \times 71$. This implies that $2^{e_1} - 1 = 11^{f_1}71^{g_1}$, which is impossible since $2^{10} - 1 = 3 \times 11 \times 31$ and $2^{35} - 1 = 31 \times 71 \times 127 \times 122921$.

If $x = (3^{e_2} - 1)/2$ is not a prime power, then

$$e_2 \in \{9, 11, 17, 19, 23, 37, 43, 59, 61, 223\}.$$

Hence, we can write $x = (3^{e_1} - 1)/2 = pq$ for distinct primes $p < q$ with $p, q \neq 31$. However, $o_q(2)$ or $o_q(5)$ can never be 1, 6, or a prime power among the above e_2's. Hence, both $2^{e_1} - 1$ and $(5^{e_3} - 1)/4$ must be a power of p. By Lemma 2.5, we must have $p = 31$, which is impossible as mentioned above.

If $x = (5^{e_3} - 1)/4$ is a prime power, then

$$e_3 \in \{3, 7, 11, 13, 47, 127, 149, 181\}.$$

Among them, no e_3 gives a prime power (or one) residual order 3 (mod x), and only $e_3 = 3$ makes the residual order 2 (mod x) acceptable in view of Lemma 2.3.

Hence, $p = 31$, $e_3 = 3$, $e_1 = 5$ and $(3^{e_2} - 1)/2 = q^{f_2}$, which implies that $e_2 = 5$ or $(3^{e_2} - 1)/2 = q$.

If $x = (5^{e_3} - 1)/4$ is not a prime power, then

$$e_3 \in \{2, 5, 17, 23, 31, 41, 43, 59, 71\}.$$

Hence, we can write $x = (5^{e_3} - 1)/4 = pq$ for distinct primes $p < q$. None of such $e_3 > 2$ gives an acceptable residual order 2 (mod q) or 3 (mod q) in view of Lemma 2.3. Hence, we see that neither $2^{e_1} - 1$ nor $(3^{e_2} - 1)/2$ can be divisible by q, and both must be a power of p, contrary to Lemma 2.5. Hence, we must have $e_3 = 2$, $(p, q) = (2, 3)$. This yields that $e_1 = e_2 = 2$.

This completes the proof of Theorem 1.1. □

Table 1. The residual orders of $3, 5$ modulo p for $p = 2^{e_1} - 1$.

e_1	$o_p(3)$	$o_p(5)$
2	N/A	2
3	6	6
5	30	3
7	126	42
13	910	1365
17	131070	65535
19	524286	74898
31	715827882	195225786
61	(10)	(15)
89	(6)	(84)
107	(6)	(6)
127	(6)	(6)

Table 2. The residual orders of $3, 5$ modulo p, q for $pq = 2^{e_1} - 1$, $p < q$.

e_1	$2^{e_1} - 1 = pq$	$o_p(3)$	$o_q(3)$	$o_p(5)$	$o_q(5)$
9	7×73	6	12	6	72
11	23×89	11	88	22	44
23	47×178481	23	178480	46	44620
37	223×616318177	222	308159088	222	616318176
41	13367×164511353	6683	164511352	13366	164511352
49	$127 \times$ P13	126	(8)	42	(8)
59	$179951 \times$ P13	89975	(8)	89975	(8)
67	$193707721 \times$ P12	96853860	(6)	8071155	(6)
83	$167 \times$ P23	83	(10)	166	(166)

Continued on next page

Table 2 – *Continued from previous page*

e_1	$2^{e_1}-1=pq$	$o_p(3)$	$o_q(3)$	$o_p(5)$	$o_q(5)$
97	11447 × P26	5723	(194)	11446	(194)
101	P13 × P14	(303)	(303)	(303)	(303)
103	2550183799 × P22	(166)	(206)	(249)	(309)
109	745988807 × P24	(11663)	(118)	(214)	(118)
131	263 × P38	131	(74)	262	(74)
137	P20 × P22	(274)	(66290053)	(1202723)	(66290053)
139	P13 × P30	(6)	(6)	(6)	(15)
149	P20 × P25	(745)	(16)	(745)	(8)
167	2349023 × P44	(26)	(22)	(26)	(22)
197	7487 × P56	(3743)	(394)	(38)	(394)
199	P12 × P49	(14)	(1393)	(8)	(1393)
227	P18 × P52	(8)	(35)	(8)	(497)
241	22000409 × P66	(8)	(5114261)	(482)	(5114261)
269	13822297 × P74	(6)	(6)	(6)	(22)
271	15242475217 × P72	(8)	(542)	(8)	(15)
281	80929 × P80	(8)	(278)	(6)	(417)
293	P26 × P63	(6)	(6)	(8)	(6)
347	P23 × P82	(6)	(6)	(21)	(8)

Table 3. The residual orders of $2, 5$ modulo p for $p^{f_2}=(3^{e_2}-1)/2$.

e_2	$o_p(2)$	$o_p(5)$
2	N/A	1
3	12	4
5	10	5
7	1092	364
13	398580	30660
71	(8)	(8)
103	(12)	(14)

Table 4. The residual orders of $2, 5$ modulo p, q for $pq = (3^{e_2} - 1)/2$, $p < q$.

e_2	$(3^{e_2} - 1)/2 = pq$	$o_p(2)$	$o_q(2)$	$o_p(5)$	$o_q(5)$
9	13×757	12	756	4	756
11	23×3851	11	3850	22	1925
17	1871×34511	935	595	935	3451
19	1597×363889	532	181944	532	22743
23	47×1001523179	23	(46)	46	(1073)
37	$13097927 \times$ P12	(9731)	8594564351	(74)	(74)
43	$431 \times$ P18	43	215	(22)	(22)
59	$14425532687 \times$ P18	(3953)	(118)	(106)	(10679)
61	$603901 \times$ P24	201300	(12)	150975	(145)
223	P26 $\times$ P81	(446)	(12)	(6)	(446)

Table 5. The residual orders of $2, 3$ modulo p for $p = (5^{e_3} - 1)/4$.

e_3	$o_p(2)$	$o_p(3)$
3	5	30
7	6510	6510
11	1220703	369910
13	61035156	1211015
47	(94)	(6)
127	(18)	(18)
149	(10)	(6)
181	(12)	(15)

Table 6. The residual orders of $2, 3$ modulo p, q for $pq = (5^{e_3} - 1)/4$, $p < q$.

e_3	$(5^{e_3} - 1)/4 = pq$	$o_p(2)$	$o_q(2)$	$o_p(3)$	$o_q(3)$
2	2×3	N/A	2	1	N/A
5	11×71	10	35	5	35
17	409×466344409	204	3429003	204	116586102
23	$8971 \times$ P12	8970	(8)	8970	2306995565
31	$1861 \times$ P18	1860	(15)	310	(6)
41	$2238236249 \times$ P19	279779531	(8)	(8)	(8)
43	$1644512641 \times$ P21	(8)	(15)	(8)	(10)
59	P17 $\times$ P25	(12)	(9)	(6)	(118)
71	$569 \times$ P47	284	(142)	568	(452610863706241)

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