

## Isometric isomorphism of homogeneous space algebras

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**Abstract.** In this paper, we show that two homogeneous spaces with isometrically isomorphic algebras are topologically homeomorphic by themselves. This generalizes the well-known results of J. G. Wendel for group algebras and those of B. E. Johnson for measure algebras.

### 1. Introduction

Harmonic analysis on homogeneous spaces is a very powerful tool to study theoretical and applied physics, as well as engineering. Riemannian symmetric spaces, a very active area of research, are examples of homogeneous spaces. Various subjects related to the topic have been studied by many authors (see [2], [4], [7], [8]). Although homogeneous spaces essentially do not possess a group structure, they are locally compact Hausdorff spaces. Recall that the term “ $X$  is a  $G$ -space” means that  $X$  is a locally compact Hausdorff space on which the topological group  $G$  acts by an action map transitively and continuously. Let  $K$  and  $H$  be two compact subgroups of a locally compact group  $G$ . In 2013, A. GHAANI FARASHAHI defined a convolution on  $L^1(G/K)$  which makes it into a Banach algebra (see [9]). Now, consider two homogeneous spaces  $G/K$  and  $G/H$ . We know that  $G/K$  and  $G/H$  are isomorphic as homogeneous spaces if and only if  $K$  and  $H$  are conjugate, strictly speaking,  $H = g_0 K g_0^{-1}$  for some  $g_0 \in G$  (see [5, Proposition 3.7]). In this paper, we aim to obtain the necessary

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and sufficient conditions on  $G/K$  and  $G/H$  for an isometrically isomorphism to exist between two Banach algebras  $L^1(G/K)$  and  $L^1(G/H)$ .

In [28], J. G. WENDEL showed that two locally compact topological groups with isometrically isomorphism group algebras are themselves isomorphic (algebraic and homeomorphic), and in [22], B. E. JOHNSON showed that the same is true for measure algebras. In this regard, we achieve the same and more general results for  $G$ -spaces, which have many applications in different areas, including computerized tomography, magnetic resonance imaging, radio astronomy, crystallographic analysis, etc. (see [1], [27]). The outline of the rest of this paper is as follows: in Section 2, we present some preliminaries and an overview on homogeneous spaces. Section 3 is allocated to prove some lemmas and propositions which are necessary to prove the main results. Finally, this section is ended by presenting the main theorem of this paper.

## 2. Preliminaries

In the sequel,  $H$  is a closed subgroup of a locally compact group  $G$ , and  $dx, dh$  are the left Haar measures on  $G$  and  $H$ , respectively. We recall that the modular function  $\Delta_G$  is a continuous homomorphism from  $G$  into the multiplicative group  $\mathbb{R}^+$ . Furthermore,

$$\int_G f(y)dy = \Delta_G(x) \int_G f(yx)dy,$$

where  $f \in C_c(G)$ , the space of continuous functions on  $G$  with compact support, and  $x \in G$ . A locally compact group  $G$  is called unimodular if  $\Delta_G(x) = 1$ , for all  $x \in G$ . A compact group  $G$  is always unimodular. Suppose that  $\mu$  is a Radon measure on  $G/H$ . For  $x \in G$ , we denote by  $\mu_x$  the translation of  $\mu$  by  $x$ . Then  $\mu$  is said to be  $G$ -invariant if  $\mu_x = \mu$ , for all  $x \in G$ , and is said to be strongly quasi-invariant, if there is a continuous function  $\lambda : G \times G/H \rightarrow (0, +\infty)$  which satisfies

$$d\mu_x(yH) = \lambda(x, yH)d\mu(yH).$$

If the function  $\lambda(x, \cdot)$  reduces to a constant for each  $x \in G$ , then  $\mu$  is called relatively invariant under  $G$ . We consider a rho-function for the pair  $(G, H)$  as a continuous function  $\rho : G \rightarrow (0, +\infty)$  for which  $\rho(xh) = \Delta_H(h)\Delta_G(h)^{-1}\rho(x)$ , for each  $x \in G$  and  $h \in H$ . It is well-known that  $(G, H)$  admits a rho-function, and for every rho-function  $\rho$ , there is a strongly quasi-invariant measure  $\mu$  on

$G/H$  such that

$$\int_G f(x) \rho(x) dx = \int_{G/H} \int_H f(xh) dh d\mu(xH), \quad (f \in C_c(G)). \quad (2.1)$$

This equation is called the quotient integral formula. The measure  $\mu$  also satisfies

$$\frac{d\mu_x}{d\mu}(yH) = \frac{\rho(xy)}{\rho(y)}, \quad (x, y \in G).$$

Every strongly quasi-invariant measure on  $G/H$  arises from a rho-function in this manner. All of these measures are strongly equivalent (see Proposition 2.54 and Theorem 2.56 of [6]). Therefore, if  $\mu$  is a strongly quasi-invariant measure on  $G/H$ , then the measures  $\mu_x$ ,  $x \in G$ , are all mutually absolutely continuous. It should be remarked that if  $\mu$  is a strongly quasi invariant measure on  $G/H$  which is associated with the rho-function  $\rho$ , then  $\mu$  is relatively invariant if and only if  $\rho(xy) = \frac{\rho(x)\rho(y)}{\rho(e)}$ ,  $x, y \in G$ . Also,  $G/H$  has a  $G$ -invariant Radon measure if and only if the constant function  $\rho(x) = 1$ ,  $x \in G$ , is a rho-function for the pair  $(G, H)$ .

Fix a strongly quasi-invariant measure  $\lambda$  on  $G/H$  which arises from the rho-function  $\rho$ . Put

$$C_c^\rho(G : H) = \{\varphi_{\pi_H}^\rho := \varphi \circ \pi_H \cdot \rho^{1/p} : \varphi \in C_c(G/H)\}. \quad (2.2)$$

Also take  $L^p(G : H) = \overline{C_c^\rho(G : H)}^{\|\cdot\|_p}$ , for all  $1 \leq p < \infty$ . By a similar calculation in [3] and [9], one can see that  $C_c^\rho(G : H)$  is a left ideal of the algebra  $C_c(G)$ . Consider the surjective bounded linear operator  $T_H^p : C_c(G) \rightarrow C_c^\rho(G/H)$  is defined by  $T_H^p(f)(xH) = \int_H \frac{f(xh)}{\rho(xh)^{1/p}} dh$ , and the norm is defined by  $\|\varphi\|_p = \inf \{\|f\|_p : T_H^p(f) = \varphi\}$ . Then we have the extension  $T_H^p : L^p(G) \rightarrow (L^p(G/H), \lambda)$  that is a surjective and bounded linear operator with  $\|T_H^p\| \leq 1$ . For all  $1 \leq p < \infty$ ,  $T_H^p : L^p(G : H) \rightarrow (L^p(G/H), \lambda)$  is an isometric isomorphism between two Banach algebras. Since  $T_H^p$  is surjective, for all  $\varphi$  and  $\psi$  in  $L^p(G/H)$ , there are  $\varphi_{\pi_H}^\rho$  and  $\psi_{\pi_H}^\rho$  in  $L^p(G : H)$  such that  $T_H^p(\varphi_{\pi_H}^\rho) = \varphi$  and  $T_H^p(\psi_{\pi_H}^\rho) = \psi$ . Further,  $\varphi_{\pi_H}^\rho * \psi_{\pi_H}^\rho$  belongs to  $L^p(G : H)$ . So one can define

$$\begin{aligned} \varphi * \psi(xH) &= T_H^p(\varphi_{\pi_H}^\rho * \psi_{\pi_H}^\rho)(xH) \\ &= \int_{G/H} \int_H \varphi(yH) \psi(hy^{-1}xH) \left( \frac{\rho(hy^{-1}x)}{\rho(x)} \right)^{1/p} dh d\lambda(yH), \end{aligned} \quad (2.3)$$

for all  $\varphi \in (L^p(G/H), \lambda)$  and  $\psi \in (L^p(G/H), \lambda)$ . Let  $\sigma \in M(G/H)$  and  $\varphi \in L^p(G/H)$ , then there exist  $\sigma_{P_H} \in M(G : H)$  and  $\varphi_{\pi_H}^\rho \in L^p(G : H)$  such that

$T_H^p(\varphi_{\pi_H}^\rho)$  and  $R_H(\sigma_{P_H}) = \sigma$ . We now define the function  $\sigma * \varphi$  in a natural way as follows:

$$\begin{aligned}\sigma * \varphi(xH) &= T_H^p(\sigma_{P_H} * \varphi_{\pi_H}^\rho)(xH) \\ &= \int_{G/H} \int_H \varphi(hy^{-1}xH) \left( \frac{\rho(hy^{-1}x)}{\rho(x)} \right)^{1/p} dh d\sigma(yH).\end{aligned}\quad (2.4)$$

Also we have

$$\varphi * \sigma(xH) = \int_{G/H} \Delta(y^{-1}) \int_H \varphi(xhy^{-1}H) \left( \frac{\rho(xhy^{-1})}{\rho(x)} \right)^{1/p} dh d\sigma(yH).\quad (2.5)$$

Suppose  $M(G/H)$  denotes the space of all complex bounded Radon measures on  $G/H$ . Then  $(M(G/H), *)$  is a Banach algebra, where  $*$  :  $M(G/H) \times M(G/H) \rightarrow M(G/H)$  is defined by  $\sigma_1 * \sigma_2(\varphi) = R_H(\sigma_{1_{P_H}} * \sigma_{2_{P_H}})(\varphi)$ ,  $\sigma_{i_{P_H}} = \sigma \circ P_H$ , and  $R_H : M(G) \rightarrow M(G/H)$  is defined by  $R_H\mu(\varphi) = \mu(\varphi_\pi^\rho)$ . Moreover, one can show that for all  $1 \leq p < \infty$ ,  $(L^p(G/H), \lambda)$  is a left Banach  $M(G/H)$ -module via the module action  $(\sigma, \varphi) \rightarrow \sigma * \varphi$  (see [3], [9], [26]).

Recall that for a Banach algebra  $\mathcal{A}$ , the bounded linear operator  $\varpi$  of  $\mathcal{A}$  into itself is called a centralizer if  $a(\varpi b) = (\varpi a)b$ , for all  $a, b \in \mathcal{A}$ . One can show that the space of all centralizers of  $\mathcal{A}$  is a commutative sub-algebra of the algebra of all bounded linear operators on  $\mathcal{A}$  (see [25]). Also the bounded linear operator  $\varpi$  of  $\mathcal{A}$  into itself is called a left centralizer if  $\varpi(ab) = (\varpi a)b$ , for all  $a, b \in \mathcal{A}$ . As an example, any left translation on a group algebra is a right centralizer, since  $L_x(f * g) = (L_x f) * g$ , for all  $f, g \in L^1(G)$ .

### 3. Main results

Throughout this section,  $H$  and  $K$  will denote two compact subgroups of a locally compact group  $G$ . Also let  $dh$ ,  $dk$  and  $dx$  be the left Haar measures on them, respectively. From now on, all  $G$ -spaces provided with a relatively invariant measure  $\lambda$  correspond to the rho-function  $\rho$ . The starting point is the following definition.

*Definition 3.1.* A bounded linear operator  $\varpi : (L^1(G/H), \lambda) \rightarrow (L^1(G/H), \lambda)$  is called the right centralizer of  $L^1(G/H)$  if  $\varpi(\varphi * \psi) = \varpi(\varphi) * \psi$ , for all  $\varphi$  and  $\psi$  in  $L^1(G/H)$ .

If  $x \in G$ , define the left and right translations of  $\varphi \in L^1(G/H)$  by  $\ell_x \varphi := T_H(L_x(\varphi_\pi^\rho))$  and  $\mathfrak{R}_x \varphi := T_H(R_x(\varphi_\pi^\rho))$ , where  $L_x$  and  $R_x$  are the known left and right translations of functions in the group algebra  $L^1(G)$ . To avoid confusion, left translations of functions in  $L^1(G/H)$  and  $L^1(G/K)$  through  $x$  will be denoted by  $\ell_{xH}$  and  $\ell_{xK}$ , respectively.

The following lemma gives an example of a right centralizer in  $L^1(G/H)$ .

**Lemma 3.2.** *Let  $x \in G$  and  $\varphi \in L^1(G/H)$ . Then  $\ell_{xH}(\varphi) \in L^1(G/H)$  and the following equalities are satisfied:*

- (i)  $\ell_{xH}(\varphi * \psi) = \ell_{xH}(\varphi) * \psi$  and  $xH \in G/H, \varphi, \psi \in L^1(G/H)$ ;
- (ii)  $\|\ell_{xH}(\varphi)\|_1 = \|\varphi\|_1$  and  $xH \in G/H, \varphi \in L^1(G/H)$ .

PROOF.

(i) Let  $\varphi$  and  $\psi$  be in  $L^1(G/H)$ , then by the definitions of left translation and convolution on  $L^1(G/H)$ , we have

$$\begin{aligned} \ell_{xH}(\varphi * \psi) &= T_H(L_x(\varphi * \psi)_\pi^\rho) = T_H\left(L_x(T_H(\varphi_\pi^\rho * \psi_\pi^\rho))_\pi^\rho\right) \\ &= T_H(L_x(\varphi_\pi^\rho * \psi_\pi^\rho)) = T_H((L_x \varphi_\pi^\rho) * \psi_\pi^\rho) \\ &= T_H(L_x \varphi_\pi^\rho) * T_H(\psi_\pi^\rho) = \ell_{xH}(\varphi) * \psi. \end{aligned}$$

Note that  $L_x$  is a right centralizer of  $L^1(G)$  (for more details, see [6, p. 51]).

(ii) Let  $\varphi$  be in  $L^1(G/H)$  and  $xH \in G/H$ . Then by the definition of left translation  $\ell_{xH}$  and by the relatively invariant property of the measure  $\lambda$ , we have

$$\begin{aligned} \|\ell_{xH}(\varphi)\|_1 &= \int_{G/H} |\ell_{xH} \varphi(yH)| d\lambda(yH) \\ &= \int_{G/H} |T_H(L_x \varphi_\pi^\rho)(yH)| d\lambda(yH) \\ &= \int_{G/H} \left| \int_H \frac{L_x \varphi_\pi^\rho(yh)}{\rho(yh)} dh \right| d\lambda(yH) \\ &= \int_{G/H} \left| \int_H \frac{\varphi_\pi^\rho(x^{-1}yh) \rho(x^{-1}yh)}{\rho(yh)} dh \right| d\lambda(yH) \\ &= \int_{G/H} |\varphi(x^{-1}yH)| \int_H \frac{\rho(x^{-1}yh)}{\rho(yh)} dh d\lambda(yH) \\ &= \int_{G/H} |\varphi(yH)| \int_H \frac{\rho(yh)}{\rho(xyh)} dh d\lambda(xyH) \\ &= \int_{G/H} |\varphi(yH)| d\lambda(yH) = \|\varphi\|_1. \end{aligned}$$

□

**Lemma 3.3.** *Consider the isometric isomorphism  $T_H^p$  between two Banach algebras  $L^p(G : H)$  and  $(L^p(G/H), \lambda)$ . Then  $\varpi$  is a right centralizer of  $L^p(G : H)$  if and only if  $T_H^p \varpi (T_H^p)^{-1}$  is a right centralizer of  $(L^p(G/H), \lambda)$ .*

PROOF. Let  $\varpi$  be a right centralizer of  $L^p(G : H)$ , and put  $\varpi_{G/H} := T_H^p \varpi (T_H^p)^{-1}$ . Then the linearity and boundedness of  $\varpi_{G/H}$  will be deduced from that of  $\varpi$ ,  $T_H^p$  and  $(T_H^p)^{-1}$ . Furthermore, for all  $\varphi, \psi \in L^p(G/H)$ , we have

$$\begin{aligned} \varpi_{G/H}(\varphi * \psi) &= T_H^p \varpi (T_H^p)^{-1}(\varphi * \psi) = T_H^p \varpi (T_H^p)^{-1}(T_H^p(\varphi_{\pi_H}^\rho * \psi_{\pi_H}^\rho)) \\ &= T_H^p \varpi(\varphi_{\pi_H}^\rho * \psi_{\pi_H}^\rho) = T_H^p[(\varpi \varphi_{\pi_H}^\rho) * \psi_{\pi_H}^\rho] \\ &= T_H^p[\varpi \varphi_{\pi_H}^\rho] * T_H^p[\psi_{\pi_H}^\rho] = T_H^p[\varpi (T_H^p)^{-1} \varphi] * \psi = (\varpi_{G/H} \varphi) * \psi, \end{aligned}$$

and note that  $T_H^p[\psi_{\pi_H}^\rho] = \psi$ . The reverse implication can be obtained in a similar way.  $\square$

Now, we prepare some preliminary lemmas which we need in the sequel.

**Lemma 3.4.** *Suppose  $\{e_\alpha\}$  is a left approximate identity of  $L^1(G)$ , and put  $\zeta_\alpha := T_H(e_\alpha)$ . Then  $\{\zeta_\alpha\}$  is a left approximate identity of  $(L^1(G/H), \lambda)$ .*

PROOF. See [9, Proposition 3.3].  $\square$

In the next lemma, we need the fact that  $(L^1(G/H), \lambda)$  can be embedded into  $M(G/H)$  via the embedding  $\psi \mapsto \lambda_\psi$ , where  $d\lambda_\psi(xH) = \psi(xH)d\lambda(xH)$ ,  $xH \in G/H$ . Further,  $\|\psi\|_1 = \|\lambda_\psi\|$  (see [9]).

**Lemma 3.5.** *Let  $\{\psi_\alpha\}$  be a net in  $(L^1(G/H), \lambda)$ . Then there exists a unique complex regular measure of bounded variation  $\sigma$  such that*

- (i)  $\sigma(\gamma) = \lim_{\alpha} \lambda_{\psi_\alpha}(\gamma)$ ,  $\gamma \in C_0(G/H)$ ;
- (ii)  $\|\sigma\| = \lim_{\alpha} \|\psi_\alpha\|$ .

PROOF. Since  $\|\psi_\alpha\|_1 \leq 1$  and  $\|\psi_\alpha\|_1 = \|\lambda_{\psi_\alpha}\|$ , we conclude that  $\{\lambda_{\psi_\alpha}\}$  is a net of bounded positive linear functionals on  $C_0(G/H)$ . So  $\{\lambda_{\psi_\alpha}\}$  is a subset of the closed unit ball in  $C_0(G/H)^*$ , which is weak-\* compact by the Banach-Alaoglu Theorem. Then the set  $\{\lambda_{\psi_\alpha}\}$  has a weak-\* limit point  $F$ . According to the Riesz-Markov Theorem, there exists a unique complex regular measure of bounded variation  $\sigma$  corresponding to the positive linear functional  $F$  such that  $F(\varphi) = \int_{G/H} \varphi(xH) d\sigma(xH)$ ,  $\varphi \in C_0(G/H)$ . Furthermore,

$$\|\sigma\| = \|F\| = \lim_{\alpha} \|\lambda_{\psi_\alpha}\| = \lim_{\alpha} \|\psi_\alpha\|.$$

Since  $F = W^* - \lim_{\alpha} \lambda_{\psi_{\alpha}}$ , we obtain

$$\lim_{\alpha} \lambda_{\psi_{\alpha}}(\gamma) = F(\gamma) = \int_{G/H} \gamma(xH) d\sigma(xH) = \sigma(\gamma),$$

for all  $\gamma \in C_0(G/H)$ .  $\square$

The next theorem asserts that any right centralizer of  $L^1(G/H)$  can be represented as a convolution with a complex regular measure of bounded variation on  $C_c(G/H)$ .

**Theorem 3.6.** *Let  $\varpi$  be a right centralizer of  $L^1(G/H)$ , then there exists a unique complex regular measure of bounded variation  $\sigma$  on  $C_c(G/H)$  such that  $\varpi(\varphi) = \sigma * \varphi$ ,  $\varphi \in L^1(G/H)$  and  $\|\varpi\| = \|\sigma\|$ .*

PROOF. If  $\{e_{\alpha}\}$  is a left approximate identity of  $L^1(G)$ , then by Lemma 3.4,  $\{\zeta_{\alpha} := T_H(e_{\alpha})\}$  is a left approximate identity of  $L^1(G/H)$ . Without loss of generality, we may assume that  $\|\zeta_{\alpha}\| = 1$ . Let  $\varphi \in L^1(G/H)$ , then

$$\varpi\varphi = \varpi(\lim_{\alpha} \zeta_{\alpha} * \varphi) = \lim_{\alpha} \varpi(\zeta_{\alpha} * \varphi) = \lim_{\alpha} (\varpi(\zeta_{\alpha}) * \varphi) = \lim_{\alpha} (\psi_{\alpha} * \varphi), \quad (3.1)$$

where  $\psi_{\alpha} = \varpi(\zeta_{\alpha})$ . According to Lemma 3.5, there exists a unique complex regular measure of bounded variation  $\sigma$  on  $C_0(G/H)$  such that  $\sigma(\gamma) = \lim_{\alpha} \lambda_{\psi_{\alpha}}(\gamma)$ ,  $\gamma \in C_0(G/H)$  and  $\|\sigma\| = \lim_{\alpha} \|\psi_{\alpha}\|$ . Since  $\|\psi_{\alpha}\| = \|\varpi(\zeta_{\alpha})\| \leq \|\varpi\| \|\zeta_{\alpha}\| = \|\varpi\|$ ,

$$\|\sigma\| \leq \|\varpi\|. \quad (3.2)$$

Now, if  $\Upsilon, \Psi \in C_c(G/H)$ , then so is  $\gamma$  defined by

$$\gamma(yH) = \int_{G/H} \int_H \Upsilon(xH) \Psi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh d\lambda(xH).$$

Let  $\varepsilon > 0$ . Since  $\sigma(\gamma) = \lim_{\alpha} \lambda_{\psi_{\alpha}}(\gamma)$ , for all  $\gamma$  in  $C_0(G/H)$ , it follows that for any  $\alpha_0$ , there exists an  $\alpha_1 > \alpha_0$  such that  $|\lambda_{\psi_{\alpha_1}}(\gamma) - \sigma(\gamma)| < \varepsilon$ , for all  $\alpha_1 > \alpha_0$ . Hence

$$\left| \int_{G/H} \gamma(yH) \psi_{\alpha_1}(yH) d\sigma(xH) - \int_{G/H} \gamma(yH) d\sigma(xH) \right| < \varepsilon.$$

Then by substituting,

$$\begin{aligned} & \left| \int_{G/H} \int_{G/H} \int_H \Upsilon(xH) \Psi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh d\lambda(xH) \psi_{\alpha_1}(yH) d\lambda(yH) \right. \\ & \quad \left. - \int_{G/H} \int_{G/H} \int_H \Upsilon(xH) \Psi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh d\lambda(xH) \gamma(yH) d\sigma(xH) \right| < \varepsilon. \end{aligned}$$

By Fubini's Theorem,

$$\left| \int_{G/H} \Upsilon(xH) \int_{G/H} \int_H \Psi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh \psi_{\alpha_1}(yH) d\lambda(yH) d\lambda(xH) \right. \\ \left. - \int_{G/H} \Upsilon(xH) \int_{G/H} \int_H \Psi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh \gamma(yH) d\sigma(xH) d\lambda(xH) \right| < \varepsilon.$$

Using (2.3) and (2.4), we obtain

$$\left| \int_{G/H} \Upsilon(xH) (\psi_{\alpha_1} * \Psi)(xH) d\lambda(xH) - \int_{G/H} \Upsilon(xH) (\sigma * \Psi)(xH) d\lambda(xH) \right| < \varepsilon.$$

By 3.1,  $\psi_{\alpha} * \varphi$  tends to  $\varpi\varphi$  in  $L^1(G/H)$ , and therefore these converge as linear functionals. This means that by passing to the limit through a suitable cofinal subset  $\{\alpha_1\}$  of  $\{\alpha\}$ , for any  $\varphi \in L^1(G/H)$ , we obtain

$$\left| \int_{G/H} \Upsilon(xH) \varpi\varphi(xH) d\lambda(xH) - \int_{G/H} \Upsilon(xH) \sigma * \varphi(xH) d\lambda(xH) \right| < \varepsilon.$$

Thus  $\varpi\varphi$  and  $\sigma * \varphi$  are equal as linear functionals on  $C_c(G/H)$ , so they are equal easily on  $C_0(G/H)$  by  $\overline{C_c(G/H)} = C_0(G/H)$ . Then by an approximation treatment, these are equal on  $L^1(G/H)$ , so  $\|\varpi\varphi\|_1 = \|\sigma * \varphi\|_1$ . Hence by [9, Proposition 2.18],  $\|\varpi\| \leq \|\sigma\|$ . Considering 3.2, we finally conclude that  $\|\sigma\| = \|\varpi\|$ . To prove the uniqueness, let  $\sigma_1$  and  $\sigma_2$  satisfy the desired condition of this theorem. Then  $\sigma_1 * \varphi = \varpi\varphi = \sigma_2 * \varphi$ , for all  $\varphi \in C_c(G/H)$ , and this implies that  $\sigma_1 = \sigma_2$ .  $\square$

The next two lemmas are both needed to prove Theorem 3.9.

**Lemma 3.7.** *Let  $\sigma$  be a complex regular measure of bounded variation on  $C_c(G/H)$ , and*

$$\left| \int_{G/H} \varphi(yH) d\sigma(yH) \right| = \int_{G/H} |\varphi(yH)| d|\sigma|(yH), \quad \varphi \in C_c(G/H).$$

Then

$$\left| \int_{G/H} \varphi(yH) d\sigma(yH) \right| = \int_{G/H} |\varphi(yH)| d\sigma(yH), \quad \varphi \in C_c(G/H).$$

PROOF. Let

$$\Theta(\varphi) := \left| \int_{G/H} \varphi(yH) d\sigma(yH) \right| = \int_{G/H} |\varphi(yH)| d|\sigma|(yH).$$

Consider a non-negative function  $\varphi$  in  $C_c(G/H)$  such that  $\Theta(\varphi) \neq 0$ . Using the assumption, there exists a unique  $c_\varphi \in \mathbb{C}$  with absolute value 1 such that

$$\int_{G/H} \varphi(yH) d\sigma(yH) = c_\varphi \int_{G/H} \varphi(yH) d|\sigma|(yH).$$

Similarly, for a non-negative function  $\psi$  in  $C_c(G/H)$  with  $\Theta(\psi) \neq 0$ , there exists a unique  $c_\psi \in \mathbb{C}$  with absolute value 1 such that

$$\int_{G/H} \psi(yH) d\sigma(yH) = c_\psi \int_{G/H} \psi(yH) d|\sigma|(yH).$$

Also for  $\varphi + \psi$ , there exists a unique  $c_{\varphi+\psi} \in \mathbb{C}$  with absolute value 1 such that

$$\int_{G/H} (\varphi + \psi)(yH) d\sigma(yH) = c_{\varphi+\psi} \int_{G/H} (\varphi + \psi)(yH) d|\sigma|(yH).$$

Hence we have

$$\begin{aligned} c_\varphi \int_{G/H} \varphi(yH) d|\sigma|(yH) + c_\psi \int_{G/H} \psi(yH) d|\sigma|(yH) \\ = c_{\varphi+\psi} \int_{G/H} \varphi(yH) d|\sigma|(yH) + c_{\varphi+\psi} \int_{G/H} \psi(yH) d|\sigma|(yH). \end{aligned}$$

Given the uniqueness of constants, we deduce that  $c_\varphi = c_\psi = c_{\varphi+\psi}$  so that the coefficient is independent of the choice of  $\varphi \in C_c(G/H)$ . Then for any  $\varphi$  in  $C_c(G/H)$  with  $\Theta(\varphi) \neq 0$ , there exists a unique  $c \in \mathbb{C}$  with absolute value 1 such that

$$\int_{G/H} \varphi(yH) d\sigma(yH) = c \int_{G/H} \varphi(yH) d|\sigma|(yH).$$

This means that  $d\sigma(yH) = d|\sigma|(yH)$ , and finally

$$\left| \int_{G/H} \varphi(yH) d\sigma(yH) \right| = \int_{G/H} |\varphi(yH)| d\sigma(yH); \varphi \in C_c(G/H).$$

In fact, this result holds for all real continuous functions which have a limit at infinity.  $\square$

**Lemma 3.8.** Fix  $\sigma$  as a normalized measure in  $M(G/H)$  with  $|\sigma(\varphi)| = \sigma|\varphi|$ ,  $\varphi \in C_0(G/H)$ . If  $F : C_0(G/H) \rightarrow \mathbb{C}$  is the mapping defined by  $F(\varphi) := \sigma(\varphi)$ , then  $F$  is a point functional.

PROOF. It is clear that  $F$  is a positive linear functional. Using the Riesz Representation Theorem,  $\|F\| = \|\sigma\| = 1$ . Further, if  $\min(\varphi, \psi) = 0$  for all non-negative  $\varphi, \psi \in C_0(G/H)$ . Then  $|\varphi + \psi| = |\varphi - \psi|$ , so

$$\begin{aligned} \int_{G/H} (\varphi(yH) + \psi(yH)) d\sigma(yH) &= \int_{G/H} |\varphi(yH) - \psi(yH)| d\sigma(yH) \\ &= \left| \int_{G/H} (\varphi(yH) - \psi(yH)) d\sigma(yH) \right|. \end{aligned}$$

Using the hypothesis  $|\sigma(\varphi)| = \sigma|\varphi|$ , we get  $\left| \int_{G/H} \varphi(yH) d\sigma(yH) + \int_{G/H} \psi(yH) d\sigma(yH) \right| = \left| \int_{G/H} \varphi(yH) d\sigma(yH) - \int_{G/H} \psi(yH) d\sigma(yH) \right|$ . Thus  $\min(F(\varphi), F(\psi)) = 0$ . Finally, by the Kakutani Theorem,  $F$  is a point functional (see [23]), that is, there exists  $x_\sigma \in G$  such that  $F(\varphi) = \sigma(\varphi) = \varphi(x_\sigma)$ ,  $\varphi \in C_0(G/H)$ .  $\square$

In the next theorem, we present necessary and sufficient conditions on a bounded linear operator  $A$  to be a right centralizer in  $L^1(G/H)$ .

**Theorem 3.9.** Let  $\varpi : L^1(G/H) \rightarrow L^1(G/H)$  be a bounded linear operator, then there exists  $x_0 \in G$  such that  $\varpi = c\ell_{x_0H}$  for some  $c \in \mathbb{C}(|c| = 1)$  if and only if the following statements are satisfied:

- (i)  $\varpi$  is a right centralizer on  $L^1(G/H)$ ;
- (ii)  $\varpi$  preserves the norm.

PROOF. Let  $\varpi = c\ell_{x_0H}$  for some  $x_0 \in G$  and  $c \in \mathbb{C}(|c| = 1)$ . Then it is obvious to see that  $\varpi$  is a right centralizer on  $L^1(G/H)$  and it preserves the norm by Lemma 3.2. Conversely, suppose that  $\varpi$  is a right centralizer and it preserves the norm. Therefore, by Theorem 3.8, there exists a unique complex regular measure of bounded variation  $\sigma$  on  $C_c(G/H)$  such that  $\varpi\varphi = \sigma * \varphi$  and  $\|\varpi\| = \|\sigma\|$ ,  $\varphi \in C_c(G/H)$ . So for all  $\varphi \in C_c(G/H)$ , we have  $\|\varphi\|_1 = \|\varpi\varphi\|_1 = \|\sigma * \varphi\|_1 \leq \|\sigma\| \|\varphi\|_1$ . The last inequality follows by [9, Proposition 2.18]. So

$$\begin{aligned} &\int_{G/H} |\sigma * \varphi(xH)| \lambda(xH) \\ &= \int_{G/H} \left| \int_{G/H} \int_H \varphi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh \sigma(yH) \right| d\lambda(xH) = \|\varphi\|_1. \end{aligned} \quad (3.3)$$

Since  $|\sigma|$  is a regular measure,

$$\begin{aligned} & \left| \int_{G/H} \int_H \varphi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh d\sigma(yH) \right| \\ & \leq \int_{G/H} \left| \int_H \varphi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh \right| d|\sigma|(yH). \end{aligned} \quad (3.4)$$

If strict inequality holds in 3.4 on a set of positive measure, then by 3.3 we get

$$\begin{aligned} \|\varphi\|_1 & < \int_{G/H} \int_{G/H} \left| \int_H \varphi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh \right| d|\sigma|(yH) d\lambda(xH) \\ & = \int_{G/H} \int_{G/H} \left| \int_H \varphi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh \right| d|\sigma|(yH) d\lambda(xH). \end{aligned}$$

By replacing  $x$  by  $yh^{-1}x$ , we get

$$\begin{aligned} \|\varphi\|_1 & < \int_{G/H} \int_{G/H} \left| \int_H \varphi(xH) dh \right| d\lambda(xH) d|\sigma|(yH) \\ & = \int_{G/H} \int_{G/H} |\varphi(xH)| d\lambda(xH) d|\sigma|(yH) = \|\varphi\|_1 d\|\sigma\|. \end{aligned}$$

Using the equality  $\|\varpi\| = \|\sigma\|$  and knowing that  $\|\varpi\| = 1$ , we conclude that  $\|\varphi\|_1 < \|\varphi\|_1 \|\varpi\| = \|\varphi\|_1$ . But this is a contradiction. Thus inequality 3.3 is an equality for a.e.  $xH \in G/H$ . But since both sides of the this equality are continuous functions of  $xH$ , the equality holds everywhere:

$$\begin{aligned} & \left| \int_{G/H} \int_H \varphi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh d\sigma(yH) \right| \\ & = \int_{G/H} \left| \int_H \varphi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh \right| d|\sigma|(yH). \end{aligned}$$

If  $x \in H$ , and if we replace  $\varphi(yH)$  by  $\varphi(y^{-1}H)$ ,

$$\left| \int_{G/H} \int_H \varphi(hyH) \frac{\rho(hyx)}{\rho(x)} dh d\sigma(yH) \right| = \int_{G/H} \left| \int_H \varphi(hyH) \frac{\rho(hyx)}{\rho(x)} dh \right| d|\sigma|(yH).$$

Thus

$$\left| \int_{G/H} \varphi(yH) d\sigma(yH) \right| = \int_{G/H} |\varphi(yH)| d|\sigma|(yH).$$

By Lemma 3.7, we obtain

$$\left| \int_{G/H} \varphi(yH) d\sigma(yH) \right| = \int_{G/H} |\varphi(yH)| d\sigma(yH).$$

Now, let us define  $F : C_0(G/H) \rightarrow \mathbb{C}$  by  $F(\varphi) := \sigma(\varphi)$ ,  $\varphi \in C_0(G/H)$ . Then, since  $\sigma(|\varphi|) = |\sigma(\varphi)|$ , Lemma 3.8 assures us that  $F$  is a point functional. Then there exists  $x_0 = x_\sigma \in G$  such that  $F(\varphi) = \varphi(x_0H)$ ,  $\varphi \in C_0(G/H)$ , so  $\sigma(\varphi) = F(\varphi) = \varphi(x_0H)$ . But  $\varpi(\varphi)(xH) = \sigma \star \varphi(xH) = \int_{G/H} \int_H \varphi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} \times dh d\sigma(yH) = \int_H \varphi(hx_0^{-1}xH) \frac{\rho(hx_0^{-1}x)}{\rho(x)} dh = \ell_{x_0H} \varphi(xH)$ .  $\square$

The following lemma has a significant role in the proof of the main Theorem 3.13.

**Lemma 3.10.** *Let  $T : L^1(G/K) \rightarrow L^1(G/H)$  be an algebraic isomorphism, both algebras are real or complex, which does not increase norms, and also let  $\ell_{xK}$  be a left translation on  $L^1(G/K)$ . Put  $\gamma_{xK} := T\ell_{xK}T^{-1}$ . Then there exist  $c_{xK}$  in  $\mathbb{C}$  ( $|c_{xK}| = 1$ ) and a unique  $yH = \tau(xK)$  in  $G/H$  such that  $\gamma_{xK} = c_{xK}\ell_{yH}$ .*

PROOF. Considering Theorem 3.9 and the boundedness of the linear operator  $\gamma_{xK}$ , it suffices to show that  $\gamma_{xK}$  is a norm-preserving right centralizer in  $L^1(G/H)$ . To this end, let  $\varphi_H$  and  $\psi_H$  be in  $L^1(G/H)$ , then

$$\begin{aligned} \gamma_{xK}(\varphi_H \star \psi_H) &= T\ell_{xK}T^{-1}(\varphi_H \star \psi_H) = T\ell_{xK}(T^{-1}\varphi_H \star T^{-1}\psi_H) \\ &= T(\ell_{xK}(T^{-1}\varphi_H) \star T^{-1}\psi_H) = T(\ell_{xK}(T^{-1}\varphi_H) \star T^{-1}\psi_H) \\ &= T\ell_{xK}T^{-1}(\varphi_H) \star \psi_H = (\gamma_{xK}\varphi_H) \star \psi_H. \end{aligned}$$

Then  $\gamma_{xK}$  is a right centralizer. Now, we want to show that  $\gamma_{xK}$  preserves the norm in  $L^1(G/H)$ . First, assume that  $\{e_\alpha\}$  is a left approximate identity of  $L^1(G/K)$ , and put  $\zeta_\alpha := T_H(e_\alpha)$ . By surjectivity of  $T$ , it is concluded that to any  $\varphi_H$  in  $L^1(G/H)$  there exists  $\varphi_K \in L^1(G/K)$  such that  $\varphi_H = T\varphi_K$ . Thus

$$\begin{aligned} \lim_\alpha \|\varphi_K \star \zeta_\alpha - \varphi_K\| &= \lim_\alpha \|T\varphi_K \star Te_\alpha - T\varphi_K\| \\ &= \lim_\alpha \|T(\varphi_K \star e_\alpha - \varphi_K)\| \leq \lim_\alpha \|\varphi_K \star e_\alpha - \varphi_K\|. \end{aligned}$$

The last term of the previous inequality converges to 0, so  $\{\zeta_\alpha\}$  is a left approximate identity of  $L^1(G/H)$ . Now, we have

$$\begin{aligned} \gamma_{xK}\varphi_H &= \gamma_{xK}\left(\lim_\alpha \zeta_\alpha \star \varphi_H\right) = \lim_\alpha (\gamma_{xK}\zeta_\alpha) \star \varphi_H \\ &= \lim_\alpha (T\ell_{xK}T^{-1}\zeta_\alpha) \star \varphi_H = \lim_\alpha (T\ell_{xK}e_\alpha) \star \varphi_H. \end{aligned}$$

Then we have

$$\begin{aligned} \|\lim_{\alpha} \gamma_{xK} \varphi_H\| &= \|\lim_{\alpha} (T\ell_{xK} e_{\alpha}) \star \varphi_H\| \leq \lim_{\alpha} \|(T\ell_{xK} e_{\alpha})\| \|\varphi_H\| \\ &\leq \lim_{\alpha} \|(T\ell_{xK} e_{\alpha})\| \|\varphi_H\| \leq \lim_{\alpha} \|(\ell_{xK} e_{\alpha})\| \|\varphi_H\| \\ &\leq \lim_{\alpha} \|(e_{\alpha})\| \|\varphi_H\| = \|\varphi_H\|. \end{aligned}$$

Then  $\gamma_{xK}$  is a contraction in  $L^1(G/H)$ , and since  $(\gamma_{xK})^{-1} = \gamma_{x^{-1}K}$ , so it is a contraction in  $L^1(G/H)$  in a similar way. So it preserves the norm. Thus by Theorem 3.9, there exists  $c_{xK}$  in  $\mathbb{C}$  ( $|c_{xK}| = 1$ ), and there exists  $yH \in G/H$  such that  $\gamma_{xK} = c_{xK} \ell_{yH}$ . It is worth noting that  $yH$  corresponds to  $xK$ .  $\square$

**Lemma 3.11.** *The mappings  $xK \mapsto c_{xK}$  and  $xK \mapsto \tau(xK)$  defined in Lemma 3.10 are of continuous homomorphism-type of  $G/K$  to, respectively, the circle group  $\mathbb{T}$  and the  $G$ -space  $G/H$ ;  $\tau$  is  $1 - 1$ .*

PROOF. Let  $x_1, x_2 \in G$  and  $\ell_{x_1H}, \ell_{x_2H}$  denote the left translations in  $L^1(G/H)$ . Then the two mappings are of homomorphism-type by Lemma 3.10, and we have the following equations:

$$\begin{aligned} c_{x_1x_2K} \ell_{y_1y_2H} &= T\ell_{x_1x_2K} T^{-1} = T\ell_{x_1K} \ell_{x_2K} T^{-1} = T\ell_{x_1K} T^{-1} T\ell_{x_2K} T^{-1} \\ &= c_{x_1K} \ell_{y_1H} c_{x_2K} \ell_{y_2H} = c_{x_1K} c_{x_2K} \ell_{y_1H} \ell_{y_2H}, \end{aligned}$$

Now,  $\tau$  is the product of  $\tau_1 : xK \mapsto \ell_{xK}$ ,  $\tau_2 : \ell_{xK} \mapsto T\ell_{xK} T^{-1} = c_{xK} \ell_{yH}$  and  $\tau_3 : c_{xK} \ell_{yH} \mapsto yH$ . Due to [6, Proposition 2.41], the mapping  $x \mapsto L_x$  is continuous in strong operator topology. So  $\|L_x f - L_{x_0} f\|_1 \rightarrow 0$ , as  $x \rightarrow x_0$ , for all  $f \in L^1(G)$ . Then  $\|\ell_x \varphi - \ell_{x_0} \varphi\|_1 = \|T_H(L_x(\varphi_{\pi_H}^{\rho})) - T_H(L_{x_0}(\varphi_{\pi_H}^{\rho}))\|_1 = \|L_x(\varphi_{\pi_H}^{\rho}) - L_{x_0}(\varphi_{\pi_H}^{\rho})\|_1 \rightarrow 0$ , as  $x \rightarrow x_0$ , for all  $\varphi \in L^1(G/H)$ . Hence the first mapping is continuous in strong operator topology. The second one is continuous by boundedness of  $T$  and  $T^{-1}$ . So it suffices to show that the third mapping is continuous. In the first step, we prove  $L_y \mapsto y$  is a continuous homomorphism of the group of all left translations to  $G$ ; let  $V$  be a neighborhood of  $1 \in G$ , then there exists a symmetric neighborhood  $W$  of  $1$  with finite measure  $\omega$  such that  $WW^{-1} \subseteq V$ . Put  $N_I := \{L_y : \|L_y \chi_W - \chi_W\|_1 < \omega\}$ . If  $L_y \in N_I$ , then  $y \in V$ . To verify this, suppose  $y \notin V$ , then  $y \notin WW^{-1}$ , so  $yW$  and  $W$  are disjoint sets. Hence

$$\begin{aligned} \|L_y \chi_W - \chi_W\|_1 &= \int_G |\chi_W(y^{-1}x) - \chi_W(x)| dx \\ &= \int_G |\chi_W(y^{-1}x)| dx + \int_G |\chi_W(x)| dx = \|L_y \chi_W\|_1 + \|\chi_W\|_1 = 2\omega. \end{aligned}$$

This contradiction completes the proof of continuity of  $L_y \mapsto y$ . For the rest of the proof of the lemma, let  $V'$  be an arbitrary neighborhood of  $\pi_H(y_0) = y_0H \in G/H$ , and put  $U := \pi_H^{-1}(V')$  that includes  $y_0 \in G$ . Then by the continuity of  $L_y \mapsto y$ , there exists  $N_{L_{y_0}}$  such that  $y \in U$  for all  $L_y \in N_{L_{y_0}}$ . Now, it suffices to take

$$\mathcal{N}_{\ell_{y_0}} := \{\ell_y : L_y \in N_{L_{y_0}}\}.$$

Since  $N_{L_{y_0}}$  is open, for any  $L_y \in N_{L_{y_0}}$ , there exists  $\varepsilon > 0$  such that  $B(L_y, \varepsilon) \subseteq N_{L_{y_0}}$ . Suppose  $\ell_{y_1} \in \mathcal{N}_{\ell_{y_0}}$ , then  $\mathcal{B} := \{\ell_y : L_y \in B(L_{y_1}, \varepsilon)\} \subseteq \mathcal{N}_{\ell_{y_0}}$ . So  $\mathcal{N}_{\ell_{y_0}}$  is open. Finally, if  $yH \notin V'$ , then  $y \notin U$ . So  $L_y \notin N_{L_{y_0}}$ , hence  $\ell_y \notin \mathcal{N}_{\ell_{y_0}}$ , and this implies that  $\ell_y \mapsto y$  is continuous. Finally,  $xK \mapsto c_{xK}$  is continuous, since  $c_{xK}I$  is the product of the uniformly bounded and continuous functions  $T\ell_{xK}T^{-1}$  and  $\ell_{x^{-1}H}$ . To show that  $\tau$  is 1-1, let  $x_1, x_2 \in G$  and  $\tau(x_1K) = \tau(x_2K)$ . Then by Lemma 3.10, we have  $c_{x_1K}T\ell_{x_1K}T^{-1} = c_{x_2K}T\ell_{x_2K}T^{-1}$  so that  $c_{x_2^{-1}K}\ell_{x_1K} = c_{x_1^{-1}K}\ell_{x_2K}$ , then  $c_{x_1K}c_{x_2^{-1}K}\ell_{x_2^{-1}x_1K} = I_K$ , and this implies  $x_1K = x_2K$ .

Then we are done, the proof is completed.  $\square$

The following lemma states that the right translations span the space of all right centralizers.

**Lemma 3.12.** *Let  $\varpi$  be a right centralizer of  $L^1(G/H)$ , then  $\varpi$  is a strong limit point of the set of finite linear combinations of left translations.*

PROOF. Considering the Hahn-Banach Theorem, it is enough to show that if  $\Lambda$  is an arbitrary strongly continuous linear functional on the operators on  $L^1(G/H)$ , which vanishes on the left translations, then  $\Lambda(\varpi) = 0$ . It is well known that any strongly continuous linear functional  $\Lambda$  on the space of all operators  $\{T\}$  on a Banach space  $L^1(G/H)$  can be written as  $\Lambda(T) = \sum_{j=1}^n \varphi_j^*(T(\varphi_j))$ , where  $\varphi_j \in L^1(G/H)$  and  $\varphi_j^* \in (L^1(G/H))^*$ . Since  $\varphi_j^* \in (L^1(G/H))^*$ , and  $(L^1(G/H))^*$  is the Banach space of all locally measurable functions which a.e. are bounded. We have

$$\Lambda(T) = \sum_{j=1}^n \int_{G/H} \psi_j(xH)T(\varphi_j)(xH)d\lambda(xH),$$

where  $\psi_j$  are locally measurable functions which a.e. are bounded, and  $\lambda$  is a relatively invariant measure on  $G/H$  which arises from the rho-function  $\rho$ . If  $\Lambda \equiv 0$  on the space of all left translations, then

$$\sum_{j=1}^n \int_{G/H} \psi_j(xH)\ell_{yH}\varphi_j(xH)d\lambda(xH) = 0, \quad (yH \in G/H),$$

or, equivalently,

$$\sum_{j=1}^n \int_{G/H} \psi_j(xH) \int_H \varphi_j(y^{-1}xH) \frac{\rho(y^{-1}xh)}{\rho(xh)} dh d\lambda(xH) = 0, \quad (yH \in G/H). \quad (3.5)$$

Now, by Theorem 3.6, there exists a unique complex regular measure of bounded variation  $\sigma$  on  $C_c(G/H)$  such that  $\varpi(\varphi) = \sigma * \varphi$ ,  $\varphi \in L^1(G/H)$  and  $\|\varpi\| = \|\sigma\|$ . Then

$$\begin{aligned} \Lambda(\varpi) &= \sum_{j=1}^n \int_{G/H} \psi_j(xH) \varpi(\varphi_j)(xH) d\lambda(xH) = \sum_{j=1}^n \int_{G/H} \psi_j(xH) \sigma * \varphi_j(xH) d\lambda(xH) \\ &= \sum_{j=1}^n \int_{G/H} \psi_j(xH) \int_{G/H} \int_H \varphi_j(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh d\sigma(yH) d\lambda(xH) \\ &= \sum_{j=1}^n \int_{G/H} \psi_j(xH) \int_{G/H} \int_H \varphi_j(y^{-1}xH) \frac{\rho(y^{-1}xh)}{\rho(xh)} dh d\sigma(yH) d\lambda(xH) \\ &= \sum_{j=1}^n \int_{G/H} \int_{G/H} \psi_j(xH) \int_H \varphi_j(hy^{-1}xH) \frac{\rho(y^{-1}xh)}{\rho(xh)} dh d\lambda(xH) d\sigma(yH) = 0. \end{aligned}$$

Hence the proof is complete.  $\square$

**Theorem 3.13.** *Let  $T : L^1(G/K) \rightarrow L^1(G/H)$  be an algebraic isomorphism, both algebras real or complex, which does not increase norms. The mapping  $\tau : G/K \rightarrow G/H$  defined in Lemma 3.10 is a homeomorphism.*

PROOF. By Lemma 3.10,  $\tau$  is a continuous injection, then it is enough to show that  $\tau$  is surjective and  $\tau^{-1}$  is continuous. The mapping  $\tau^{-1}$  is the product of  $\tau_1^{-1} : \ell_{xK} \mapsto xK$ ,  $\tau_2^{-1} : T\ell_{xK}T^{-1} = c_{xK}\ell_{yH} \mapsto \ell_{xK}$  and  $\tau_3^{-1} : yH \mapsto c_{xK}\ell_{yH}$ . Considering Lemma 3.10, the proofs of the continuity of  $\tau_1^{-1}$ ,  $\tau_2^{-1}$  and  $\tau_3^{-1}$  are the same as those of  $\tau_3$ ,  $\tau_2$  and  $\tau_1$ , respectively.

To see that  $\tau$  is surjective, as our first step, we shall show that  $\tau(G/K)$  is closed in  $G/H$ . Let  $\{\tau(x_\alpha K)\} \subseteq \tau(G/K)$  be a directed sequence in  $G/H$  that converges to  $yH$  in  $G/H$ . Since  $yH \mapsto \ell_{yH}$  is continuous,  $\ell_{\tau(x_\alpha K)}$  tends to  $\ell_{yH}$ . So  $T^{-1}\ell_{\tau(x_\alpha K)}T = c_{x_\alpha K}^{-1}\ell_{x_\alpha K}$ ,  $c_{x_\alpha K} \in \mathbb{C}$ ,  $|c_{x_\alpha K}| = 1$  (see Lemma 3.10). Considering Theorem 3.9,  $A = \text{strong-}\lim_{\alpha} c_{x_\alpha K}^{-1}\ell_{x_\alpha K}$  is an isometric right centralizer, so it has the form  $c_{xK}\ell_{xK}$ , for some  $xK \in G/K$  and  $c_{xK} \in \mathbb{C}$ ,  $|c_{xK}| = 1$ . Therefore,  $T^{-1}\ell_{yH}T = c_{xK}\ell_{xK}$ , so  $\ell_{yH} = c_{xK}T\ell_{xK}T^{-1}$ , which, using Lemma 3.10, implies that  $yH = \tau(xK)$ .

As the final step, we show that  $\tau(G/K) = G/H$ . Suppose that there is no preimage in  $G/K$  for some  $yH$  in  $G/H$ . Then  $\varpi := T^{-1}\ell_{yH}T$  is a right centralizer of  $L^1(G/K)$ . Thus by Theorem 3.12, it holds that  $\varpi = \text{strong-}\lim_{\alpha} \ell_{\alpha}$ , where  $\ell_{\alpha} = \sum_{i=1}^{n_{\alpha}} c_{\alpha_i} \ell_{x_{\alpha_i}K}$ ,  $c_{\alpha_i} \in \mathbb{C}$ ,  $|c_{\alpha_i}| = 1$ . Then  $T\varpi T^{-1} = \ell_{yH} = \text{strong-}\lim_{\alpha} \sum_{i=1}^{n_{\alpha}} c_{\alpha_i} T\ell_{x_{\alpha_i}K}T^{-1} = \text{strong-}\lim_{\alpha} \sum_{i=1}^{n_{\alpha}} c'_{\alpha_i} \ell_{y_{\alpha_i}H}$ , for some  $y_{\alpha_i}H \in G/H$  and  $c'_{\alpha_i} \in \mathbb{C}$ ,  $|c'_{\alpha_i}| = 1$ . Now, since  $\tau(G/K)$  is closed in  $G/H$ , there exists a neighborhood  $\omega$  of  $H$  in  $G/H$  such that  $\lambda(\omega) < \infty$  and  $y\omega\omega^{-1} \cap \tau(G/K)$  is empty, where  $\lambda$  is a relatively invariant measure on  $G/H$ . Put  $\varphi := \chi_{\omega}$  and  $\varphi_{\alpha} := \ell_{\alpha}(\varphi)$ . Let  $zH \in \omega$ . If  $y_{\alpha_i}H \in \tau(\omega)$ , or equivalently,  $y^{-1}y_{\alpha_i}H \notin \omega\omega^{-1}$ , then  $y_{\alpha_i}^{-1}zH \in \omega(z^{-1}y_{\alpha_i}H \in \omega^{-1})$  implies that  $y^{-1}y_{\alpha_i}H = y^{-1}zz^{-1}y_{\alpha_i}H \in \omega\omega^{-1}$ , which is a contradiction. Therefore,  $\varphi(y^{-1}zH) = 1$  concludes that  $\varphi_{\alpha_i}(zH) = 0$ . Hence we have

$$\begin{aligned} \|\ell_{\alpha}(\varphi) - \ell_{yH}(\varphi)\|_1 &\geq \|\varphi_{\alpha} - \ell_{yH}(\varphi)\|_1 \\ &= \int_{y\omega} |\varphi_{\alpha}(zH) - \varphi(y^{-1}zH)| d\lambda(zH) = \int_{y\omega} 1 d\lambda(zH) = \lambda(y\omega), \end{aligned}$$

which contradicts that  $\{\ell_{\alpha}\}$  tends to  $\ell_{yH}$  in strong operator topology. With this, the proof is completed.  $\square$

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