

Erratum and addendum to the paper: “On a class of projective Ricci flat Finsler metrics”

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Abstract. In this paper, we correct an error in the formula (24) in [1] and modify the formula of the projective Ricci curvature for Randers metrics. Based on these, we simplify and optimize the main results in [1].

1. Projective Ricci flat Randers metrics

By definition, the projective Ricci curvature of a Finsler metric is given by

$$\mathbf{PRic} = \mathbf{Ric} + \frac{n-1}{n+1} \mathbf{S}_{|m} y^m + \frac{n-1}{(n+1)^2} \mathbf{S}^2. \quad (1.1)$$

For a Randers metric $F = \alpha + \beta$ on M , let G^i and ${}^\alpha G^i$ denote the geodesic coefficients of F and α , respectively. Then G^i and ${}^\alpha G^i$ are related by

$$G^i = {}^\alpha G^i + \alpha s^i_0 + \frac{1}{2F} \{-2\alpha s_0 + r_{00}\} y^i. \quad (1.2)$$

For the related definitions and notations, see [1]. The Ricci curvature of $F = \alpha + \beta$ is given by

$$\mathbf{Ric} = {}^\alpha \mathbf{Ric} + (2\alpha s^m_{0;m} - 2t_{00} - \alpha^2 t^m_m) + (n-1)\Xi, \quad (1.3)$$

Mathematics Subject Classification: 53B40, 53C60.

Key words and phrases: Randers metric, Ricci curvature, S -curvature, projective Ricci curvature.

This paper is supported by the National Natural Science Foundation of China (11371386), and the Science Foundation of Chongqing Normal University (17XLB022).

where

$$\Xi := \frac{3}{4F^2}(r_{00} - 2\alpha s_0)^2 + \frac{1}{2F}[4\alpha(q_{00} - \alpha t_0) - (r_{00;0} - 2\alpha s_{0;0})]. \quad (1.4)$$

Further, the S -curvature of $F = \alpha + \beta$ is given by

$$\mathbf{S} = (n+1) \left[\frac{e_{00}}{2F} - (s_0 + \rho_0) \right], \quad (1.5)$$

where $e_{00} = r_{00} + 2\beta s_0$. From (1.5), we have

$$\mathbf{S}_{y^m} = (n+1) \left[\frac{r_{m0} + b_m s_0 + \beta s_m}{F} - \frac{(\alpha^{-1} a_{mj} y^j + b_m)(r_{00} + 2\beta s_0)}{2F^2} - s_m - \rho_m \right], \quad (1.6)$$

where we have used $F_{y^m} = \alpha^{-1} a_{jm} y^j + b_m$.

By (1.2), we have

$$\begin{aligned} G_m^i &= \alpha G_m^i + \alpha_{y^m} s_0^i + \alpha s_m^i - \frac{F_{y^m}}{2F^2} (-2\alpha s_0 + r_{00}) y^i \\ &\quad + \frac{1}{F} (-\alpha_{y^m} s_0 - \alpha s_m + r_{m0}) y^i + \frac{1}{2F} (-2\alpha s_0 + r_{00}) \delta_m^i. \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{S}_{|m} y^m &= y^m \frac{\partial \mathbf{S}}{\partial x^m} - G_m^l y^m \frac{\partial \mathbf{S}}{\partial y^l} = \mathbf{S}_{;m} y^m - \left[2\alpha s_0^m + \frac{1}{F} (-2\alpha s_0 + r_{00}) y^m \right] \frac{\partial \mathbf{S}}{\partial y^m} \\ &= \mathbf{S}_{;m} y^m - 2\alpha s_0^m \mathbf{S}_{y^m} - \frac{\mathbf{S}}{F} (-2\alpha s_0 + r_{00}). \end{aligned} \quad (1.7)$$

From (1.5) and (1.6), we obtain

$$\begin{aligned} \mathbf{S}_{;m} y^m &= (n+1) \left\{ \frac{1}{2F} r_{00;0} + \frac{1}{F} r_{00} s_0 + \frac{1}{F} \beta s_{0;0} - \frac{1}{2F^2} e_{00} r_{00} - s_{0;0} - \rho_{0;0} \right\} \\ &= (n+1) \left\{ \frac{1}{2F} r_{00;0} + \frac{1}{2F^2} (2\alpha s_0 - r_{00}) r_{00} - \frac{1}{F} \alpha s_{0;0} - \rho_{0;0} \right\}, \end{aligned} \quad (1.8)$$

$$\begin{aligned} 2\alpha s_0^m \mathbf{S}_{y^m} &= \frac{2(n+1)}{F} \alpha q_{00} + \frac{2(n+1)}{F^2} \alpha^2 s_0^2 - \frac{2(n+1)}{F} \alpha^2 t_0 \\ &\quad - \frac{(n+1)}{F^2} \alpha s_0 r_{00} - 2(n+1) \alpha (\rho_m s_0^m), \end{aligned} \quad (1.9)$$

$$\frac{\mathbf{S}}{F} (-2\alpha s_0 + r_{00}) = \frac{n+1}{F} \left\{ -\frac{2}{F} \alpha r_{00} s_0 + \frac{2}{F} \alpha^2 s_0^2 + \frac{1}{2F} r_{00}^2 + (2\alpha s_0 - r_{00}) \rho_0 \right\}. \quad (1.10)$$

Here, we have corrected (24) in [1] as (1.9). By (1.8), (1.9) and (1.10) and (1.7), we obtain

$$\begin{aligned} \frac{n-1}{n+1} \mathbf{S}_{|m} y^m = (n-1) \left\{ \frac{1}{2F} r_{00;0} + \frac{4}{F^2} \alpha r_{00} s_0 - \frac{1}{F^2} r_{00}^2 - \frac{1}{F} \alpha s_{0;0} - \rho_{0;0} - \frac{2}{F} \alpha q_{00} \right. \\ \left. - \frac{4}{F^2} \alpha^2 s_0^2 + \frac{2}{F} \alpha^2 t_0 + 2\alpha(\rho_m s_0^m) - \frac{2}{F} \alpha s_0 \rho_0 + \frac{1}{F} \rho_0 r_{00} \right\}. \end{aligned} \quad (1.11)$$

Here, we have corrected (26) in [1] and modified it into (1.11). Further, we have

$$\begin{aligned} \frac{n-1}{(n+1)^2} \mathbf{S}^2 \\ = (n-1) \left\{ \frac{1}{4F^2} r_{00}^2 + \frac{1}{F^2} \alpha^2 s_0^2 - \frac{1}{F^2} \alpha r_{00} s_0 + \rho_0^2 - \frac{1}{F} \rho_0 r_{00} + \frac{2}{F} \alpha \rho_0 s_0 \right\}. \end{aligned} \quad (1.12)$$

Substituting (1.3), (1.11) and (1.12) into (1.1), we obtain the following formula for the projective Ricci curvature of $F = \alpha + \beta$:

$$\mathbf{PRic} = {}^\alpha \mathbf{Ric} + 2\alpha s_{0;m}^m - 2t_{00} - \alpha^2 t_m^m + (n-1) \left\{ 2\alpha(\rho_m s_0^m) - \rho_{0;0} + \rho_0^2 \right\}. \quad (1.13)$$

Here, (1.13) is the revised version of (28) in [1].

Now Theorem 1.1 in [1] can be simplified and optimized as follows.

Theorem 1.1. *Let $F = \alpha + \beta$ be a Randers metric on a manifold M of dimension n . Then F is a projective Ricci flat metric if and only if α and β satisfy the following equations:*

$${}^\alpha \mathbf{Ric} = t_m^m \alpha^2 + 2t_{00} + (n-1)[\rho_{0;0} - \rho_0^2], \quad (1.14)$$

$$s_{0;m}^m = -(n-1)(\rho_m s_0^m). \quad (1.15)$$

PROOF. The proof of the sufficiency of the condition in Theorem 1.1 is immediate. To prove the necessity, let us assume that $\mathbf{PRic} = 0$. Equation (1.13) is equivalent to

$$\Xi_2 \alpha^2 + \Xi_1 \alpha + \Xi_0 = 0, \quad (1.16)$$

where

$$\Xi_2 = -t_m^m, \quad (1.17)$$

$$\Xi_1 = 2s_{0;m}^m + 2(n-1)(\rho_m s_0^m), \quad (1.18)$$

$$\Xi_0 = {}^\alpha \mathbf{Ric} - 2t_{00} + (n-1)[-\rho_{0;0} + \rho_0^2]. \quad (1.19)$$

Replacing y with $-y$ in (1.16), and then adding the obtained equation to (1.16) yields

$$0 = \Xi_2 \alpha^2 + \Xi_0, \quad (1.20)$$

$$0 = \Xi_1 \alpha. \quad (1.21)$$

By (1.20), we obtain

$${}^\alpha \mathbf{Ric} = t_m^m \alpha^2 + 2t_{00} + (n-1)[\rho_{0;0} - \rho_0^2].$$

From (1.21), we obtain

$$s_{0;m}^m = -(n-1)(\rho_m s_0^m).$$

□

Further, we can restate Theorem 1.1 as follows.

Theorem 1.2. *Let $F = \alpha + \beta$ be a Randers metric on a manifold M of dimension n . Then F is a projective Ricci flat metric if and only if α and β satisfy the following equations:*

$${}^\alpha \mathbf{Ric} = t_m^m \alpha^2 + 2t_{00} - (n-1) \left[\frac{r_{0;0} + s_{0;0}}{1-b^2} + \frac{3(r_0 + s_0)^2}{(1-b^2)^2} \right], \quad (1.22)$$

$$s_{0;m}^m = \frac{n-1}{1-b^2}(q_0 + t_0). \quad (1.23)$$

We must point out that Theorem 1.1 and Theorem 1.2 here are just the revised versions of Theorem 1.1 and Theorem 1.2 in [1], respectively. We have cancelled condition (iii) in Theorem 1.1 and Theorem 1.2 in [1].

2. Application: projective Ricci flat Randers metrics with isotropic S -curvature

Let F be a Finsler metric on an n -dimensional manifold M . Assume that F is of isotropic S -curvature, i.e., $\mathbf{S} = (n+1)cF$. Then

$$\mathbf{S}_{|m} = (n+1)c_m F,$$

$$\mathbf{PRic} = \mathbf{Ric} + (n-1)c_0 F + (n-1)c^2 F^2,$$

where $c_m := c_x^m$ and $c_0 := c_m y^m$.

Now, suppose that $F = \alpha + \beta$ is a Randers metric of isotropic S -curvature, $\mathbf{S} = (n+1)cF$. Then, by Lemma 3.1 in [2], α and β satisfy

$$r_{00} + 2\beta s_0 = 2c(\alpha^2 - \beta^2), \quad (2.1)$$

that is,

$$r_{ij} = -b_i s_j - b_j s_i + 2c(a_{ij} - b_i b_j).$$

We have

$$r_i = -b^2 s_i + 2c(1 - b^2)b_i, \quad (2.2)$$

$$q_i = -b^2 t_i + 2c(1 - b^2)s_i \quad (2.3)$$

and

$$q_0 + t_0 = (1 - b^2)(t_0 + 2cs_0). \quad (2.4)$$

From Theorem 1.1 and Theorem 1.2, we obtain the following result.

Theorem 2.1. *Let $F = \alpha + \beta$ be a Randers metric on a manifold M of dimension n . Assume that F is of isotropic S -curvature, $\mathbf{S} = (n+1)cF$. Then F is a projective Ricci flat metric if and only if α and β satisfy the following equations:*

$${}^\alpha \mathbf{Ric} = t_m^m \alpha^2 + 2t_{00} - (n-1)[s_{0;0} + s_0^2 + 4c^2 \alpha^2 + 2c_0 \beta]; \quad (2.5)$$

$$s_{0;m}^m = (n-1)(t_0 + 2cs_0). \quad (2.6)$$

PROOF. By (2.2), we have

$$r_0 = -b^2 s_0 + 2c(1 - b^2)\beta.$$

Further, we have

$$r_{0;0} = -b^2 s_{0;0} + 2(1 - b^2)[-s_0^2 - 6c\beta s_0 + c_0\beta + 2c^2 \alpha^2 - 6c^2 \beta^2], \quad (2.7)$$

$$q_0 = -b^2 t_0 + 2c(1 - b^2)s_0. \quad (2.8)$$

Then, from (1.22) and (1.23), we get (2.5) and (2.6). $\square$

Actually, Theorem 2.1 is the revised version of Theorem 4.1 in [1]. We have simplified dramatically the conditions for a Randers metric with isotropic S -curvature to be a projective Ricci flat metric.

References

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(Received September 6, 2017)