

On additive representation functions

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Abstract. Let A be an infinite set of natural numbers. For $n \in \mathbb{N}$, let $r(A, n)$ denote the number of solutions of the equation $n = a + b$ with $a, b \in A$, $a \leq b$. Let $|A(x)|$ be the number of integers in A which are less than or equal to x . In this paper, we prove that if $r(A, n) \neq 1$ for all sufficiently large integers n , then $|A(x)| > \frac{1}{2}(\log x / \log \log x)^2$ for all sufficiently large x .

Let $\mathbb{N}$ be the set of all natural numbers, and let A be an infinite set of $\mathbb{N}$. For $n \in \mathbb{N}$, let $r(A, n)$ denote the number of solutions of the equation $n = a + b$ with $a, b \in A$, $a \leq b$. Let $A(x)$ be the set of integers in A which are less than or equal to x . In 1998, NICOLAS, RUZSA and SÁRKÖZY [3] proved that there exist an infinite set A of $\mathbb{N}$ and a positive constant c such that $r(A, n) \neq 1$ for all sufficiently large integers n and $|A(x)| \leq c(\log x)^2$ for all $x \geq 2$. In [3], it was also proved that if A is an infinite set of $\mathbb{N}$ such that $r(A, n) \neq 1$ for all sufficiently large integers n , then

$$\limsup |A(x)| \left(\frac{\log \log x}{\log x} \right)^{3/2} \geq \frac{1}{20}.$$

In 2001, SÁNDOR [4] disproved a conjecture of ERDŐS and FREUD [2] by constructing an A such that $r(A, n) \leq 3$ for all n , but $r(A, n) = 1$ holds only for finitely many values of n . In 2004, BALASUBRAMANIAN and PRAKASH [1] showed

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that there exists an absolute constant $c > 0$ with the following property: for any infinite set A of $\mathbb{N}$ such that $r(A, n) \neq 1$ for all sufficiently large integers n ,

$$|A(x)| \geq c \left(\frac{\log x}{\log \log x} \right)^2$$

for all sufficiently large x . One can obtain $c = \frac{1}{2904}$ from the proof of [1].

In this paper, the following result is proved.

Theorem 1. *If A is an infinite subset of $\mathbb{N}$ such that $r(A, n) \neq 1$ for all sufficiently large integers n , then*

$$|A(x)| > \frac{1}{2} \left(\frac{\log x}{\log \log x} \right)^2$$

for all sufficiently large x .

The key points in this paper are Lemmas 2 and 3. We believe that Lemma 3 will be useful in the future in Graph Theory.

1. Proofs

In the following, we always assume that A is an infinite subset of $\mathbb{N}$, and $r(A, n) \neq 1$ for all $n \geq n_0$ and $a_0 \in A$ with $a_0 \geq n_0$.

Firstly, we give some lemmas.

Lemma 1 ([1, Lemma 1]). *For every real number $t \geq a_0$, the interval $(t, 2t]$ contains an element of the set A .*

Lemma 2. *If x is a large number with*

$$|A(x)| \leq \left(\frac{\log x}{\log \log x} \right)^2$$

and

$$a_0 \leq b \leq \frac{x}{(\log x)^2},$$

then there exists $a \in A$ with $a > 3b$ and $a + b < x$ such that

$$[a - b, a) \cap A = \emptyset, \quad |(b, a + b] \cap A| \geq \frac{a + b}{2b} - 1.$$

PROOF. By Lemma 1, $(b, 2b] \cap A \neq \emptyset$. Since

$$(|A(x)| + 2)b \leq \left(\frac{\log x}{\log \log x} \right)^2 \frac{x}{(\log x)^2} + 2 \frac{x}{(\log x)^2} < x,$$

$a_0 \leq b$ and $a_0 \in A$, it follows that

$$|(b, (|A(x)| + 2)b] \cap A| < |A(x)|.$$

So there exists an integer $1 \leq k \leq |A(x)|$ such that

$$(ib, (i+1)b] \cap A \neq \emptyset, \quad i = 1, 2, \dots, k$$

and $((k+1)b, (k+2)b] \cap A = \emptyset$. By Lemma 1, $((k+1)b, 2(k+1)b] \cap A \neq \emptyset$. Now we take a to be the least integer in $((k+1)b, 2(k+1)b] \cap A$. Noting that $((k+1)b, (k+2)b] \cap A = \emptyset$, we have $a > (k+2)b \geq 3b$ and $(k+1)b < a - b < a$. It follows that $[a - b, a) \cap A = \emptyset$. It is clear that

$$a + b \leq 2(k+1)b + b \leq 5kb \leq 5|A(x)|b \leq 5 \left(\frac{\log x}{\log \log x} \right)^2 \frac{x}{(\log x)^2} < x,$$

and

$$\begin{aligned} |(b, a + b] \cap A| &= \sum_{i=1}^k |(ib, (i+1)b] \cap A| + |((k+1)b, a + b]| \\ &\geq k + 1 = \frac{2(k+1)b + b}{2b} - \frac{1}{2} > \frac{a + b}{2b} - 1. \end{aligned}$$

This completes the proof of Lemma 2. $\square$

PROOF OF THEOREM 1. We assume that x is a large number. If

$$|A(x)| > \left(\frac{\log x}{\log \log x} \right)^2,$$

then we are done. In the following, we assume that

$$|A(x)| \leq \left(\frac{\log x}{\log \log x} \right)^2. \quad (1)$$

We will prove that

$$|A(x)| > \frac{1}{2} \left(\frac{\log x}{\log \log x} \right)^2.$$

Let $b_1 = a_0$. By Lemma 2, there exists $a_1 \in A$ with $a_1 > 3b_1$ and $a_1 + b_1 < x$ such that

$$[a_1 - b_1, a_1) \cap A = \emptyset, \quad |(b_1, a_1 + b_1] \cap A| \geq \frac{a_1 + b_1}{2b_1} - 1.$$

Let $b_2 = a_1 + b_1$. Continuing this procedure, we obtain two sequences $b_1 < b_2 < \dots < b_m$ and $a_1 < a_2 < \dots < a_m$ with $a_k > 3b_k$, $a_k + b_k < x$ ($1 \leq k \leq m$) and $b_k = a_{k-1} + b_{k-1}$ ($2 \leq k \leq m$) such that

$$[a_k - b_k, a_k) \cap A = \emptyset, \quad |(b_k, a_k + b_k] \cap A| \geq \frac{a_k + b_k}{2b_k} - 1, \quad k = 1, 2, \dots, m,$$

where

$$a_m + b_m > \frac{x}{(\log x)^2}, \quad b_m = a_{m-1} + b_{m-1} \leq \frac{x}{(\log x)^2}.$$

For any $1 \leq i < j \leq m$, by $r(A, a_i + a_j) \neq 1$, we may choose one pair $c_{i,j}, d_{i,j} \in A$ with $d_{i,j} \neq a_j$ and $c_{i,j} \leq d_{i,j}$ such that

$$a_i + a_j = c_{i,j} + d_{i,j}.$$

Let

$$\begin{aligned} S_k &= \{c_{i,k} \mid i < k, d_{i,k} < a_k\} \cup \{d_{i,k} \mid i < k, d_{i,k} < a_k\}, \\ M_k &= \{i \mid i < k, d_{i,k} < a_k\}, \\ T_k &= \{d_{i,k} \mid i < k, d_{i,k} > a_k\}, \end{aligned}$$

and

$$N_k = \{i \mid i < k, d_{i,k} > a_k\}.$$

We will prove that

$$S_k \subseteq A \cap (b_k, a_k), \quad |S_k| \geq |M_k|, \quad (2)$$

and

$$T_k \subseteq A \cap (a_k, a_k + b_k], \quad |T_k| = |N_k|. \quad (3)$$

For $k = 1$, we have $S_k = T_k = \emptyset$ and $M_k = N_k = \emptyset$. So (2) and (3) hold for $k = 1$. Now we assume that $k \geq 2$.

It is clear that

$$d_{i,k} = a_i + a_k - c_{i,k} \leq a_i + a_k \leq a_{k-1} + a_k \leq b_k + a_k.$$

This implies that $T_k \subseteq A \cap (a_k, a_k + b_k]$. If $d_{u,k} = d_{v,k} \in T_k$ for some pairs $1 \leq u < v < k$, then, by

$$a_u + a_k = c_{u,k} + d_{u,k}, \quad a_v + a_k = c_{v,k} + d_{v,k},$$

we have

$$\begin{aligned} a_v &= c_{v,k} + d_{v,k} - a_k > c_{v,k} \geq c_{v,k} - c_{u,k} = a_v - a_u \\ &\geq a_v - a_{v-1} \geq a_v - a_{v-1} - b_{v-1} = a_v - b_v. \end{aligned}$$

This contradicts $[a_v - b_v, a_v) \cap A = \emptyset$. Thus, if $d_{u,k}, d_{v,k} \in T_k$ with $1 \leq u < v < k$, then $d_{u,k} \neq d_{v,k}$. Hence $|T_k| = |N_k|$. Now we have proved that (3) holds.

If $i < k$ and $d_{i,k} < a_k$, then by $[a_k - b_k, a_k) \cap A = \emptyset$, we have $d_{i,k} < a_k - b_k$. Thus

$$c_{i,k} = a_i + a_k - d_{i,k} > a_k - (a_k - b_k) = b_k.$$

It follows that $S_k \subseteq (b_k, a_k) \cap A$.

To prove $|S_k| \geq |M_k|$, it is convenient to use the language from graph theory.

A graph G consists of two parts: $V = V(G)$ of its vertices and $E = E(G)$ of its edges, where $E(G)$ is a subset of $\{\{u, v\} \mid u, v \in V\}$. Here we allow G contains loops (i.e., $\{v, v\} \in E(G)$) and G is an undirected graph. A nontrivial closed walk is an alternating sequence of vertices and edges $v_1, e_1, v_2, \dots, v_{n-1}, e_{n-1}, v_n, e_n, v_1$ such that at least one of the edges appears exactly one time and each edge repeats at most two times. Furthermore, if n is even, then the nontrivial closed walk is called a *nontrivial even closed walk*, otherwise, a *nontrivial odd closed walk*. A nontrivial closed walk $v_1, e_1, v_2, \dots, v_{n-1}, e_{n-1}, v_n, e_n, v_1$ is called a closed trail if $v_1, v_2, \dots, v_n$ are distinct. Furthermore, if n is even, then the closed trail is called an *even closed trail*, otherwise, an *odd closed trail*. In these definitions, we allow $n = 1$.

Lemma 3. *If a graph G has no nontrivial even closed walk, then*

$$|E(G)| \leq |V(G)|.$$

PROOF. It is enough to prove the lemma when G is connected. Since G has no nontrivial even closed walk, it follows that G has no even closed trail.

Suppose that K and L were two distinct odd closed trails of G .

If K and L have at least one common vertex v , then K and L can be written as

$$K : v, e, u_1, e_1, \dots, u_m, e_m, v,$$

and

$$L : v, e', v_1, e'_1, \dots, v_n, e'_n, v.$$

Thus

$$K \cup L : v, e, u_1, e_1, \dots, u_m, e_m, v, e', v_1, e'_1, \dots, v_n, e'_n, v$$

is a nontrivial even closed walk of G , a contradiction.

If K and L have no common vertex, then there is a walk W which connects K and L , since G is connected. Let W_0 be the shortest walk which connects K and L . Now K , L and W_0 can be written as

$$K : u, e, u_1, e_1, \dots, u_m, e_m, u,$$

$$L : v, e', v_1, e'_1, \dots, v_n, e'_n, v,$$

and

$$W_0 : u, e'', w_1, e''_1, \dots, w_t, e''_t, v.$$

Thus

$$K \cup W_0 \cup L \cup W_0 : u, e, \dots, e_m, u, e'', \dots, e''_t, v, e', \dots, e'_n, v, e''_t, \dots, e'', u$$

is a nontrivial even closed walk of G , a contradiction.

Now we have proved that G has at most one odd closed trail (includes loops). For any subgraph H of G , let $\mu(H) = |E(H)| - |V(H)|$. Let H_1 be a connected subgraph of G with the least $|V(H_1)|$ such that $\mu(H_1) = \mu(G)$. Since G has at most one odd closed trail, it follows that H_1 has at most one odd closed trail. Thus H_1 contains only one vertex or H_1 is an odd closed trail. So $\mu(H_1) = -1$ or 0 . That is, $\mu(G) = -1$ or 0 . Therefore, $|E(G)| \leq |V(G)|$. This completes the proof of Lemma 3. $\square$

Now we return to the proof of Theorem 1. If $S_k = \emptyset$, then $M_k = \emptyset$. In this case, $|S_k| = |M_k|$. Now we assume that $S_k \neq \emptyset$, and define a graph G_k such that $V(G_k) = S_k$ and

$$E(G_k) = \{\{c_{i,k}, d_{i,k}\} \mid i < k, d_{i,k} < a_k\}.$$

Now we show that G_k has no nontrivial even closed walk.

Suppose that G_k has a nontrivial even closed walk:

$$v_1, e_1, v_2, \dots, v_{2n-1}, e_{2n-1}, v_{2n}, e_{2n}, v_1.$$

Since $\{v_i, v_{i+1}\} \in E(G_k)$, there exists $\ell_i < k$ such that

$$v_i + v_{i+1} = a_{\ell_i} + a_k,$$

where $v_{2n+1} = v_1$. Thus

$$\sum_{i=1}^{2n} (-1)^i (a_{\ell_i} + a_k) = \sum_{i=1}^{2n} (-1)^i (v_i + v_{i+1}) = 0.$$

It follows that

$$\sum_{i=1}^{2n} (-1)^i a_{\ell_i} = 0.$$

We rewrite this as

$$\sum_{i=1}^{k-1} x_i a_i = 0.$$

Since at least one of edges appears exactly one time and each edge repeats at most two times in $e_1, e_2, \dots, e_{2n}$, it follows that $x_i \in \{-2, -1, 0, 1, 2\}$ ($1 \leq i \leq k-1$), and at least one of x_i is nonzero. Let j be the largest index such that $x_j \neq 0$. Noting that

$$a_{i+1} > 3b_{i+1} = 3(a_i + b_i) > 3a_i,$$

we have

$$a_j \leq |x_j a_j| = \left| -\sum_{i=1}^{j-1} x_i a_i \right| \leq 2 \sum_{i=1}^{j-1} a_i < 2 \left(\frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{j-1}} \right) a_j < a_j,$$

a contradiction. Hence G_k has no nontrivial even closed walk. By Lemma 3, we have

$$|M_k| = |E(G_k)| \leq |V(G_k)| = |S_k|.$$

Thus we have proved that (2) holds. By (2) and (3), we have

$$\begin{aligned} |A \cap (b_k, a_k + b_k]| &= |A \cap (b_k, a_k)| + |A \cap (a_k, a_k + b_k]| + |\{a_k\}| \\ &\geq |S_k| + |T_k| + 1 \geq |M_k| + |N_k| + 1 = k. \end{aligned}$$

Noting that $b_{k+1} = a_k + b_k$ for $k = 1, 2, \dots, m-1$ and $a_m + b_m < x$, we have

$$\begin{aligned} |A(x)| &\geq \sum_{k=1}^{m-1} |A \cap (b_k, b_{k+1})| + |A \cap (b_m, a_m + b_m)| \\ &= \sum_{k=1}^m |A \cap (b_k, a_k + b_k]| \geq 1 + 2 + \dots + m = \frac{1}{2}m(m+1). \end{aligned}$$

On the other hand,

$$\begin{aligned}
 |A(x)| &\geq \sum_{k=1}^m |(b_k, a_k + b_k] \cap A| \geq \sum_{k=1}^m \left(\frac{a_k + b_k}{2b_k} - 1 \right) \\
 &= \sum_{k=1}^{m-1} \frac{b_{k+1}}{2b_k} + \frac{a_m + b_m}{2b_m} - m \geq \sum_{k=1}^{m-1} \frac{b_{k+1}}{2b_k} + \frac{x}{2b_m(\log x)^2} - m \\
 &\geq m \left(\frac{x}{2b_m(\log x)^2} \prod_{k=1}^{m-1} \frac{b_{k+1}}{2b_k} \right)^{1/m} - m = \frac{1}{2} m \left(\frac{x}{b_1(\log x)^2} \right)^{1/m} - m.
 \end{aligned}$$

If

$$m < \frac{1}{4} \frac{\log x}{\log \log x},$$

then

$$\begin{aligned}
 |A(x)| &\geq \frac{1}{2} m \left(\frac{x}{b_1(\log x)^2} \right)^{1/m} - m \geq \frac{1}{2} e^{(\log x - 2 \log \log x - \log b_1)/m} - m \\
 &> \frac{1}{2} e^{3 \log \log x} - \frac{1}{4} \frac{\log x}{\log \log x} = \frac{1}{2} (\log x)^3 - \frac{1}{4} \frac{\log x}{\log \log x} > (\log x)^2,
 \end{aligned}$$

a contradiction with (1). So

$$m \geq \frac{1}{4} \frac{\log x}{\log \log x}.$$

If

$$m < \frac{\log x}{\log \log x},$$

then

$$\begin{aligned}
 |A(x)| &\geq \frac{1}{2} m \left(\frac{x}{b_1(\log x)^2} \right)^{1/m} - m \\
 &\geq \frac{1}{8} \frac{\log x}{\log \log x} \exp \left(\frac{(\log x - 2 \log \log x - \log b_1) \log \log x}{\log x} \right) - m \\
 &= \frac{1}{8} \frac{\log x}{\log \log x} \exp \left(\log \log x + \frac{(-2 \log \log x - \log b_1) \log \log x}{\log x} \right) - m \\
 &= \frac{1}{8} \frac{(\log x)^2}{\log \log x} (1 + o(1)) > \left(\frac{\log x}{\log \log x} \right)^2,
 \end{aligned}$$

a contradiction with (1). Hence

$$m \geq \frac{\log x}{\log \log x}.$$

Therefore,

$$|A(x)| \geq \frac{1}{2}m(m+1) > \frac{1}{2} \left(\frac{\log x}{\log \log x} \right)^2.$$

This completes the proof. $\square$

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