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Zero-free regions for derivatives of the Selberg zeta-function

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Abstract. Let $Z(s)$ be the Selberg zeta-function associated with a compact Riemann surface. We prove that, for any positive integer k, there is a constant t_0 such that $Z^{(k)}(s)$ has no zeros in $\sigma < 1/2$, $t > t_0$. Moreover, we show that the curve $Z(1/2+it)$ spirals in the clockwise direction for all sufficiently large t, in the sense that its curvature is negative.

1. Introduction

Let $s = \sigma + it$ be a complex variable, and X a compact Riemann surface of constant negative curvature -1 with genus $g \geq 2$. The notations $f(x) = O(g(x))$ and $f(x) \ll g(x)$, as $x \to \infty$, both mean that $\limsup_{x\to\infty}(|f(x)|/g(x))$ is finite, here $g(x) > 0$. Let k be a positive integer. In this paper, all implied constants may depend on X and k, which is the number of derivatives of the Selberg zeta-function. The surface X can be regarded as a quotient $\Gamma\backslash H$, where $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a strictly hyperbolic Fuchsian group, and H is the upper half-plane of $\mathbb{C}$. Then the Selberg zeta-function associated with $X = \Gamma\backslash H$ is defined by (see Hejhal [13, §2.4, Definition 4.1])

$$Z(s) = \prod_{\{P_0\}} \prod_{l=0}^{\infty} (1 - N(P_0)^{-s-l}). \tag{1}$$

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Here $\{P_0\}$ is the primitive hyperbolic conjugacy class of Γ and $N(P_0) = \alpha^2$ if the eigenvalues of P_0 are α and α^{-1} with $|\alpha| > 1$. Equation (1) defines the Selberg zeta-function in the half-plane $\sigma > 1$. The function $Z(s)$ can be extended to an entire function of order two ([13, §2.4, Theorem 4.25]). The Selberg zeta-function has trivial zeros at integers less than two, i.e., $s = -n$, $n \geq 1$, with multiplicity $(2g-2)(2n+1)$; at $s = 0$ with multiplicity $2g-1$; and at $s = 1$ with multiplicity 1 and nontrivial zeros on the critical line $\sigma = 1/2$ with at most finitely many exceptions of zeros on the real segment $0 < s < 1$ ([13, §2.4, Theorem 4.11] and Randol [22]). By the Gauss–Bonnet formula, $\text{area}(X) = 4\pi(g-1)$. Moreover, the Selberg zeta-function satisfies the functional equation ([13, §2.4, Theorem 4.12])

$$Z(s) = f(s)Z(1-s), \tag{2}$$

where

$$f(s) = \exp\left(\text{area}(X)\int_0^{s-1/2} v\tan(\pi v)dv\right). \tag{3}$$

In this paper, we consider the zero distribution of the derivatives of $Z(s)$. The Selberg zeta-function is an interesting example of a zeta-function for which an analogue of the Riemann hypothesis (RH) is true. For the Riemann zeta-function $\zeta(s)$, it is known that RH is equivalent to $\zeta'(s)$ having no zeros in $0 < \sigma < 1/2$, moreover, RH implies that any derivative of the Riemann zeta-function has at most a finite number of non-real zeros in the half-plane $\sigma < 1/2$ (Speiser [25], Levinson and Montgomery [16], and Yildirim [32]). The Speiser–Levinson–Montgomery type relation between the zeros of the zeta-function and the zeros of its derivative was extended to Dirichlet L-functions (Yildirim [33]), to the Selberg class (Šleževičienė [24]), to the extended Selberg class (Garunkštis and Šimėnas [10]), to the Selberg zeta-function $Z(s)$ (Luo [18], Garunkštis [6]). See also Minamide [19], [20], [21], Jorgenson and Smajlović [14]. Note that the extended Selberg class contains zeta-functions (for example, the Davenport–Heilbronn zeta-function) for which an analog of RH is not true. Our main aim here is to show that any derivative of the Selberg zeta-function has a finite number of zeros in the strip $\sigma' < \sigma < 1/2$, where $\sigma' < 1/2$ is any fixed number.

In the next theorem, we obtain zero-free regions in the right and left half-planes. We also describe the locations of the zeros in the left half-plane. These results follow because of the functional equation and the expression of $Z(s)$ by the Dirichlet series.

Theorem 1. *Let k be a positive integer. Then*

(i) *there is $\sigma_0 = \sigma_0(k) \geq 1$ such that $Z^{(k)}(s) \neq 0$ in $\sigma \geq \sigma_0$;*

(ii) *$Z^{(k)}(s)$ has zeros at $s=-n$ with multiplicity $(2g-2)(2n+1)-k$, for any $n>\max\left(\frac{k}{4g-4}-\frac{1}{2},0\right)$; and at $s=0$ with multiplicity $2g-1-k$ if $k<2g-1$.*

Moreover, for any $0<\varepsilon<1/2$, there is a constant $n_0=n_0(k,\varepsilon)\le -k$ such that

(iii) *$Z^{(k)}(s)$ has k zeros, counted with multiplicities, in the disc $|s+n+1/2|\le\varepsilon$ for any $-n\le n_0$;*

(iv) *$Z^{(k)}(s)$ has no other zeros in $\sigma\le n_0$ except those mentioned in* (ii) *and* (iii).

We compare Theorem 1 to the case of the Riemann zeta-function, which has a simple zero at each even negative integer and no other zeros in $\sigma\le 0$. Spira [26], [27] proved that, for $k\ge 1$, there is an α_k so that $\zeta^{(k)}(s)$ has only real zeros for $\sigma\le\alpha_k$, and exactly one real zero in each open interval $(-1-2n,1-2n)$ for $1-2n\le\alpha_k$. Verma and Kaur [30] showed that the strip $\alpha_k<\sigma\le 0$ contains at most finitely many zeros of $\zeta^{(k)}(s)$.

A Riemann–von Mangoldt-type formula for non-real zeros of $\zeta^{(k)}(s)$ was obtained by Berndt [1]. By Theorem 1 and the equality $\overline{Z'(s)}=Z'(\overline{s})$, we see that non-real zeros of $Z'(s)$ are located in a strip of finite width. Let $N_1(T)$ denote the number of non-real zeros, counted with multiplicities, of $Z'(s)$ in $0<t\le T$. Let $N(P_{00})=\min_{P_0}\{N(P_0)\}$. Then

$$N_1(T)=\frac{\operatorname{area}(X)}{4\pi}T^2-\frac{\log N(P_{00})}{2\pi}T+o\,(T)\qquad (T\to\infty). \tag{4}$$

This formula was proved by Luo [18] with the error term $O(T)$. Later the error term was improved in [6]. The zero distribution of the derivative of $Z(s)$ is related to the multiplicity problem of the Laplacian eigenvalues (see the discussion below Theorem 1 and Theorem 4 in Luo [18]). All the nontrivial zeros $s_j=1/2\pm it_j$ of $Z(s)$ correspond to eigenvalues

$$0<\lambda_j=s_j(1-s_j)=1/4+t_j^2 \tag{5}$$

of the hyperbolic Laplacian Δ on $X=\Gamma\backslash H$ (Hejhal [13, §2.4, Theorem 4.11]). Then ([18], [6])

$$\#\{t_j:0<t_j\le T\}=N_1\left(\frac{1}{2},T\right)+\frac{T}{2\pi}\log N(P_{00})+o(T),$$

where ordinates t_j are counted without multiplicities.

The distribution of the zeros of $Z^{(k)}(s)$ outside the critical strip $n_0<\sigma<\sigma_0$ considered in Theorem 1 is important in the proof of the following Speiser–Levinson–Montgomery type result.

Theorem 2. *Let k be a positive integer. Then there is $t_0 \geq 0$ such that $Z^{(k)}(s) \neq 0$ in $\sigma < 1/2$, $t > t_0$.*

For $k = 1$, Theorem 2 was proved by Luo [18]. Later Minamide [20] showed that $t_0 = 0$, if $k = 1$. Moreover, in [20] it is obtained that the derivative of the Selberg zeta-function $Z_{\Gamma/\mathbb{H}^3}(s)$ associated with three-dimensional compact hyperbolic space $\Gamma/\mathbb{H}^3$ (for definitions, see [20] and Elstrodt, Grunewald and Mennicke [5]) is zero-free to the left of the critical line. Note that all zeros of $Z_{\Gamma/\mathbb{H}^3}(s)$ lie on the critical line except for finitely many zeros on the real line ([5, Section 5.4]). In other words, $Z_{\Gamma/\mathbb{H}^3}(s)$ has no "trivial" zeros. From the functional equation ([5, Section 5.4, formula (4.22)]) we see that all zeros of the k-th derivative of $Z_{\Gamma/\mathbb{H}^3}(s)$ are in a strip of fixed width, i.e., this function has no "trivial" zeros similar to those described in Theorem 1. As we already mentioned, the proof of Theorem 2 (and the proof of Theorem 7 in Levinson and Montgomery [16] related to $\zeta(s)$) depends heavily on the existence of many trivial zeros. Another difference from $Z(s)$ is that $Z_{\Gamma/\mathbb{H}^3}(s)$ is an entire function of order three ([5, Section 5.5, Theorem 5.8]).

From the proof of Theorem 2, the following corollary follows.

Corollary 3. *Let k be a positive integer. There is $t_1 \geq 0$ such that if $Z^{(k)}(1/2 + i\gamma) = 0$ and $\gamma > t_1$, then $Z^{(n)}(1/2 + i\gamma) = 0$ for $n = 0, 1, \ldots, k-1$.*

Further, we use the results obtained about the derivatives to study the curve $\{Z(\sigma + it) : t > 0\}$ for fixed $\sigma \leq 1/2$. We are motivated by the Riemann zeta-function, where the behavior of the curve $\zeta(1/2 + it)$ is quite mysterious.

Bohr and Courant [3] (or see Titchmarsh [29, Theorem 11.9]) proved that, for fixed $1/2 < \sigma \leq 1$, the curve $\{\zeta(\sigma + it) : t > 0\}$ is dense in the plane of complex numbers. If the Riemann hypothesis is true, then the values $\zeta(\sigma + it)$ for $t \in \mathbb{R}$ are not dense in $\mathbb{C}$ for any fixed $\sigma < \frac{1}{2}$ (Garunkštis and Steuding [9, Proposition 5]). The question whether the values $\zeta(1/2+it)$ for $t \in \mathbb{R}$ are dense in the complex plane remains open. However, as to this problem, see, for example, the work of Kalpokas, Korolev, and Steuding [15]. Moreover, Voronin [31] obtained that the set

$$\{(\zeta(\sigma + it), \zeta'(\sigma + it), \ldots, \zeta^{(n-1)}(\sigma + it)) : t > 0\} \tag{6}$$

is dense in $\mathbb{C}^n$ for all positive integers n for every fixed $\sigma \in \left(\frac{1}{2}, 1\right)$. But the set

$$\left\{\left(\zeta\left(\tfrac{1}{2} + it\right), \zeta'\left(\tfrac{1}{2} + it\right)\right) : t \in \mathbb{R}\right\}$$

is not dense in $\mathbb{C}^2$ (see [9, Theorem 1]).

Assuming RH, Gonek and Montgomery [11] showed that the curve $\{\zeta(1/2 + it) : t > 0\}$ spirals in the clockwise direction for all sufficiently large t, in the sense that its curvature is negative. The curvature of the curve is defined by $\kappa = d\phi/ds$, where ϕ is the tangential angle, and s is the arc length of the curve (Casey [4, formula (10.3)]). We have that, for fixed σ, the curvature of the curve $\{\zeta(\sigma + it) : t > 0\}$ at height t is

$$\kappa_{\zeta,\sigma}(t) = \frac{\Re\frac{\zeta''}{\zeta'}(\sigma + it)}{|\zeta'(\sigma + it)|}. \tag{7}$$

For $\sigma = 1/2$, the last formula is proved in Gonek and Montgomery [11, formula (1)]. For general σ, the proof is the same. Then, for fixed $\sigma < 1/2$, we have also $\kappa_{\zeta,\sigma}(t) < 0$ if t is large, since $\Re\frac{\zeta''}{\zeta'}(\sigma + it) < 0$ (Levinson and Montgomery [16, Section 6]). The denseness of the set (6) together with (7) gives that, for fixed $1/2 < \sigma < 1$, the curve $\{\zeta(\sigma + it) : t > 0\}$ changes sign of the curvature infinitely often.

Following the proof of formula (1) in Gonek and Montgomery [11], we see that, for fixed σ, the curvature of the curve $\{Z(\sigma + it) : t > 0\}$ is

$$\kappa_{Z,\sigma}(t) = \frac{\Re\frac{Z''}{Z'}(\sigma + it)}{|Z'(\sigma + it)|}. \tag{8}$$

For the Selberg zeta-function, we have the following unconditional result.

Theorem 4. *Let $\sigma \leq 1/2$. Then there is t_1 such that $\kappa_{Z,\sigma}(t) < 0$ for all $t > t_1$.*

In the next section, we prove Theorem 1. Section 3 is devoted to the proofs of Theorems 2, 4, and Corollary 3.

2. Proof of Theorem 1

The proof is based on the expression of $Z(s)$ by the Dirichlet series, and on the functional equation of the Selberg zeta-function. We will need several lemmas. The following two lemmas deal with the factor $f(s)$ from the functional equation (2). Recall that k is always a positive integer.

Lemma 5. *Let $f(s)$ be defined by* (3). *We have, for $t \to \infty$,*

$$f^{(k)}(s) = \operatorname{area}(X)^k \,(1/2 - s)^k \,(-i)^k f(s) \left(1 + O\left(\frac{1}{t^2}\right)\right), \tag{9}$$

uniformly in σ. Moreover, if $\varepsilon > 0$, then

$$f^{(k)}(s) = \mathrm{area}(X)^k \,(1/2-s)^k \cot^k(\pi s) f(s)\left(1 + O_\varepsilon\left(\frac{1}{|s|}\right)\right), \tag{10}$$

as $|s| \to \infty$ and $\sigma \le -k$, $|s-n-1/2| \ge \varepsilon$ for any $n \in \mathbb{Z}$.

Proof. Note that $\mathrm{area}(X) > 0$, since genus $g \ge 2$. We define $p_k(x,y) \in \mathbb{C}[x,y]$ recursively by

$$\begin{aligned} p_1(x,y) &= \mathrm{area}(X)xy,\\ p_{k+1}(x,y) &= -\frac{\partial p_k}{\partial x}(x,y) - \pi(1+y^2)\frac{\partial p_k}{\partial y}(x,y) + \mathrm{area}(X)xyp_k(x,y). \end{aligned} \tag{11}$$

Then induction together with equality (3), the multivariable chain rule, and $(\cot z)' = -1-\cot^2 z$ give

$$f^{(k)}(s) = p_k(1/2-s, \cot(\pi s))f(s).$$

We write $p_k(x,y)$ as

$$p_k(x,y) = \sum_{n=0}^{\infty} a_{k,n}(y)x^n,$$

where $a_{k,n}(y) \in \mathbb{C}[y]$. Then we see that $a_{1,1}(y) = \mathrm{area}(X)y$ and $a_{1,n}(y) = 0$ if $n \ne 1$. Comparing the coefficients of x^n on both sides of (11), we find

$$a_{k+1,n}(y) = -(n+1)a_{k,n+1}(y) - \pi(1+y^2)\frac{da_{k,n}}{dy}(y) + \mathrm{area}(X)ya_{k,n-1}(y),$$

for any $n = 0, 1, \ldots$. Here we regard $a_{k,-1}(y) = 0$. We check from the above that

- $a_{k,n}(y) = 0$ if $n > k$;
- $a_{k,k}(y) = (\mathrm{area}(X)y)^k$;
- $a_{k,k-1}(y) = -\pi\,\mathrm{area}(X)^{k-1}y^{k-2}(1+y^2)k(k-1)/2$;
- $\deg a_{k,n}(y) \le k$ for any n.

Considering these as well as the formulas

$$\cot(\sigma+it) = -i + O(e^{-2t}) \quad (t \to \infty,\ \text{uniformly in } \sigma)$$

and $\tan(\pi s) \ll_\varepsilon 1$ on $\{s \in \mathbb{C} : |s-n-1/2| \ge \varepsilon \text{ for any } n \in \mathbb{Z}\}$, we conclude Lemma 5. □

Let

$$F(s)=\operatorname{area}(X)\int_0^s z\tan(\pi z)dz, \tag{12}$$

where the integration is along the straight line segment joining the origin to s, if s is not on the real line. If s is on the real line, and not one of the points $\pm 1/2$, $\pm 3/2$, $\pm 5/2,\dots$, we define $F(s)$ by the requirement of continuity as s is approached from the upper half-plane (compare to the definition of the function $\Phi(s)$ in Randol [23, Proof of Lemma 2]).

The dilogarithm function is the function defined by the power series

$$\operatorname{Li}_2(s)=\sum_{n=1}^{\infty}\frac{s^n}{n^2}, \quad \text{for } |s|\le 1. \tag{13}$$

The analytic continuation of the dilogarithm is given by

$$\operatorname{Li}_2(s)=-\int_0^s \log(1-z)\frac{dz}{z}, \quad \text{for } s\in\mathbb{C}\setminus[1,\infty). \tag{14}$$

The following lemma can be compared to [7, Lemma 1].

Lemma 6. *For $t>0$,*

$$\begin{aligned} F(s)&=\operatorname{area}(X)\left(\frac{is^2}{2}-\frac{s}{\pi}\log\left(1+e^{2i\pi s}\right)+\frac{i}{2\pi^2}\operatorname{Li}_2(-e^{2i\pi s})+\frac{i}{24}\right)\\ &=\frac{i\operatorname{area}(X)}{2}s^2+\frac{i\operatorname{area}(X)}{24}+O\left(\frac{1+|s|}{e^{2\pi t}}\right)\quad (t\to\infty)\end{aligned}$$

uniformly in σ. Here the principal branch of the logarithm is chosen.

Proof. Let, for $t>0$,

$$P(s)=\operatorname{area}(X)\left(\frac{is^2}{2}-\frac{s}{\pi}\log\left(1+e^{2i\pi s}\right)+\frac{i}{2\pi^2}\operatorname{Li}_2(-e^{2i\pi s})+\frac{i}{24}\right).$$

By the equality $\operatorname{Li}_2(-1)=-\pi^2/12$ (see formulas (1.8) and (1.9) in Lewin [17]), we have that $\lim_{s\to 0}P(s)=0=F(0)$. In view of expressions (12) and (14), we obtain that $P'(s)=F'(s)$. This proves the first part of the lemma. The second part of the lemma follows by the power series (13). Lemma 6 is proved. □

Proof of Theorem 1. In Garunkštis, Šimėnas and Steuding [8, Lemma 7], it is proved that there is an unbounded sequence $1 < x_2 < x_3 \dots$ of real numbers and real numbers a_n, $n = 2, 3, \dots$, such that

$$Z(s) = 1 + \sum_{n=2}^{\infty} \frac{a_n}{x_n^s}, \tag{15}$$

where the Dirichlet series converges absolutely for $\sigma > 1$. By this and by Theorem 4 in Hardy and Riesz [12], for $k \geq 1$, we have

$$Z^{(k)}(s) = \sum_{n=2}^{\infty} \frac{(-1)^k a_n \log^k x_n}{x_n^s} \quad (\sigma > 1). \tag{16}$$

Without loss of generality, we can assume that $a_2 \neq 0$. Then

$$Z^{(k)}(s) = \frac{(-1)^k a_2 \log^k x_2}{x_2^s} \left(1 + O\left(\left(\frac{x_2}{x_3}\right)^{\sigma}\right)\right) \quad (\sigma \to \infty),$$

uniformly in t. Thus we see that there is $\sigma_0 > 1$ such that $Z^{(k)}(s) \neq 0$ in $\sigma \geq \sigma_0$ (compare Berndt [1, p. 577], Spira [26, p. 677], Luo [18, p. 1143]). We proved the first part of the theorem.

The second part follows by the location of real zeros of $Z(s)$. Note that each derivative decreases the multiplicity of such a zero.

Next, we investigate the zero-free regions of $Z^{(k)}(s)$ in the left-half complex plane when σ is negative with large absolute value. Differentiating the functional equation (2), we get

$$Z^{(k)}(s) = \sum_{j=0}^{k} \binom{k}{j} f^{(j)}(s) \left(Z(1-s)\right)^{(k-j)}. \tag{17}$$

By the absolute convergence of series (15) and (16), we see that, for $k \geq 1$,

$$Z(s) = 1 + O(x_2^{-\sigma}) \quad \text{and} \quad Z^{(k)}(s) \ll x_2^{-\sigma}, \tag{18}$$

uniformly in t as $\sigma \to \infty$. In view of (17), (18), and Lemma 5, we write

$$Z^{(k)}(s) = f_1(s) + f_2(s),$$

where

$$f_1(s) = \operatorname{area}(X)^k \left(1/2 - s\right)^k \cot^k(\pi s) f(s).$$

From the distribution of trivial zeros of $Z(s)$ (see the Introduction), and from the functional equation (2), we have that in $\sigma < -2$, the function $f(s)$ has zeros at $s = -n$ with multiplicity $(2g-2)(2n+1)$, where $n \geq 3$ is an integer. By this and the properties of $\cot(\pi s)$, we see that in $\sigma < -2$, the function $f_1(s)$ has zeros only at $s = 1/2 - n$ and $s = -n$, where $n \geq 3$. If s is such that, for any integer n, we have

$$|s + n - 1/2| \geq \varepsilon \quad \text{and} \quad |s + n| \geq \varepsilon, \tag{19}$$

then Lemma 5 yields

$$f_2(s) \ll \frac{|f_1(s)|}{|1/2 - s|} = |\,(1/2 - s)^{k-1} \cot^k(\pi s) f(s)| \quad (\sigma \to -\infty), \tag{20}$$

uniformly in t. Therefore, if s satisfies (19) and $-\sigma$ is sufficiently large, then

$$Z^{(k)}(s) \neq 0. \tag{21}$$

Finally, we consider the locations of the zeros of $Z^{(k)}(s)$ in the left-half complex plane. Recall that for $\sigma < -2$, the function $f(s)$ has zeros at $s = -n$ with multiplicity $(2g-2)(2n+1)$, where $n \geq 3$ is an integer. Therefore, in the half plane $\sigma < -k$, the function $f_1(s)$ is analytic. Its zeros are at $s = 1/2 - n$ with multiplicity k, and at $s = -n$ with multiplicity $(2g-2)(2n+1) - k$, where $n \geq k+1$. In $\sigma < -k$, the function $f_2(s)$ is also analytic, since $Z^{(k)}(s)$ is analytic. In view of (20), Rouché's theorem yields that, for sufficiently large positive n, the functions $Z^{(k)}(s)$ and $f_1(s)$ has the same number of zeros in the discs $|s+n-1/2| \leq \varepsilon$ and $|s+n| \leq \varepsilon$, where $n \geq k+1$ (compare Spira [27]). Since $Z(s)$ has a zero at $s = -n$ with multiplicity $(2g-2)(2n+1)$ for any $n \geq 1$, the function $Z^{(k)}(s)$ has the zero at $s = -n$ with multiplicity $(2g-2)(2n+1) - k$ for any $n \geq k$. This zero corresponds to $(2g-2)(2n+1) - k$ zeros in the disc $|s+n| \leq \varepsilon$ if n is large enough. This proves part (iii) and, by (21), part (iv) of our statement. Theorem 1 is concluded. □

3. Proofs of Theorems 2, 4, and Corollary 3

Let $n_0(n)$ and $\sigma_0(n)$, where $n = 1, \dots, k$, be constants from Theorem 1. In the proof of Theorem 2, we will use the following Proposition 7 with $h(s) = Z^{(k-1)}(s)$,

$$\sigma_1 = \min\{n_0(1), \dots, n_0(k)\}, \quad \sigma_2 = 1/2, \quad \sigma_3 = \max\{\sigma_0(1), \dots, \sigma_0(k)\}, \tag{22}$$

and a_j are zeros of $Z^{(k-1)}(s)$ in the left-half complex plane.

Proposition 7. *Let $h(s) \not\equiv 0$ be an entire function of order two, real for real s. Let $\sigma_1 < \sigma_2 < \sigma_3$, let $h(s) \neq 0$ for $\sigma \leq \sigma_1$ and $\sigma \geq \sigma_3$ except the zeros $z_j = a_j + ib_j$, $j = 1, 2, \dots$, such that $a_j < \sigma_1$, $b_j \ll 1$, as $j \to \infty$. Moreover, let the number of zeros z_j counted with multiplicities and satisfying a condition $-x \leq a_j \leq \sigma_1$ be $\gg x^2$, as $x \to \infty$.*

If $h(s)$ has a finite number of zeros in the strip $\sigma_1 < \sigma < \sigma_2$, then $h'(s)$ also has a finite number of zeros in the same strip.

Proof. Without loss of generality, let us assume that $\sigma_1 = 0$. Then, by the conditions of the proposition, we see that $h(0) \neq 0$. Denote the zeros in $\sigma_1 < \sigma < \sigma_3$ by $\rho_1 = \beta_1 + i\gamma_1, \rho_2 = \beta_2 + i\gamma_2, \dots$. We will use Hadamard's factorization theorem (Titchmarsh [28, Section 8.24]). If $h(s)$ is of order two and if p is the smallest non-negative integer such that the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{|\rho_n|^{p+1}} + \frac{1}{|z_n|^{p+1}} \right) \tag{23}$$

converges, then $h(s)$ has Hadamard's canonical representation

$$h(s) = e^{Q(s)} \prod_{n=1}^{\infty} \left(1 - \frac{s}{z_n}\right) \exp\left(\frac{s}{z_n} + \dots + \frac{1}{p}\frac{s^p}{z_n^p}\right)$$
$$\times \left(1 - \frac{s}{\rho_n}\right) \exp\left(\frac{s}{\rho_n} + \dots + \frac{1}{p}\frac{s^p}{\rho_n^p}\right),$$

where the product converges for any s, and $Q(s) = as^2 + bs + c$ is a polynomial of degree not greater than 2. Moreover, in (23), we have that p is not greater than the order of $h(s)$ (Titchmarsh [28, Section 8.23]). From the other side by conditions of Proposition 7 we see that, for some positive constants c_1 and c_2,

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{p+1}} \geq c_1 \sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} \geq c_2 \int_1^{\infty} \frac{dx}{x^p}.$$

Thus $p = 2$.

We consider the function

$$\frac{h'}{h}(s) = Q'(s) + \sum_{n=1}^{\infty} \left(\frac{1}{s - z_n} + \frac{1}{z_n} + \frac{s}{z_n^2} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{s - \rho_n} + \frac{1}{\rho_n} + \frac{s}{\rho_n^2} \right). \tag{24}$$

Further in this proof, we always assume that $\sigma_1 < \sigma < \sigma_2$ and t is sufficiently large. The aim is to prove that $\Re(h'/h(s)) < 0$. This implies Proposition 7.

First, we show that $\Re(Q'(s))$ is bounded in $\sigma_1 \le \sigma < \sigma_2$. All the zeros of $h(s)$ are distributed symmetrically with respect to the real line, since $h(s)$ is real for real s. Therefore, the function

$$\sum_{n=1}^{\infty}\left(\frac{1}{s-z_n}+\frac{1}{z_n}+\frac{s}{z_n^2}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{s-\rho_n}+\frac{1}{\rho_n}+\frac{s}{\rho_n^2}\right)-\frac{h'}{h}(s)$$

is real for real s. By this and (24), we get that $Q'(s)=2as+b$ has real coefficients and

$$\Re(Q'(s)) \ll 1. \tag{25}$$

Here and further, we always will understand that the notations $\ll$ and big O are valid as $t\to\infty$, uniformly in $\sigma_1<\sigma<\sigma_2$, without mentioning it.

In view of (24), we write

$$\frac{h'}{h}(s)=Q'(s)+I+J, \tag{26}$$

where

$$I=\sum_{n=1}^{\infty}\left(\frac{1}{s-z_n}+\frac{1}{z_n}+\frac{s}{z_n^2}\right)=\sum_{n=1}^{\infty}\left(\frac{\overline{s}-\overline{z_n}}{|s-z_n|^2}+\frac{\overline{z_n}}{|z_n|^2}+\frac{s\overline{z_n}^2}{|z_n|^4}\right)$$

and

$$J=\sum_{n=1}^{\infty}\left(\frac{1}{s-\rho_n}+\frac{1}{\rho_n}+\frac{s}{\rho_n^2}\right)=\sum_{n=1}^{\infty}\left(\frac{\overline{s}-\overline{\rho_n}}{|s-\rho_n|^2}+\frac{\overline{\rho_n}}{|\rho_n|^2}+\frac{s\overline{\rho_n}^2}{|\rho_n|^4}\right).$$

We will obtain that there is a positive constant c such that, for $\sigma_1<\sigma<\sigma_2$ and large t,

$$\Re I=\sum_{n=1}^{\infty}\left(\frac{\sigma-a_n}{|s-z_n|^2}+\frac{a_n}{|z_n|^2}+\frac{a_n^2\sigma}{|z_n|^4}+\frac{2a_nb_nt-b_n^2\sigma}{|z_n|^4}\right) \tag{27}$$

is $<-c\log t$. To prove this inequality, we will need several bounds.

By the symmetrical distribution of the zeros of $h(s)$ with respect to the real line and by the absolute convergence of the series (23), we have

$$\sum_{n=1}^{\infty}\frac{2a_nb_nt-b_n^2\sigma}{|z_n|^4}=-\sum_{n=1}^{\infty}\frac{b_n^2\sigma}{|z_n|^4}\ll 1. \tag{28}$$

Further, there is a positive constant c such that

$$\left|\frac{a_n}{|z_n|^2} - \frac{1}{a_n}\right| \leq \frac{c}{|a_n|^3} \quad \text{and} \quad \left|\frac{a_n^2}{|z_n|^4} - \frac{1}{a_n^2}\right| \leq \frac{c}{|a_n|^4}.$$

Thus

$$\sum_{n=1}^{\infty}\left(\frac{a_n}{|z_n|^2} + \frac{a_n^2\sigma}{|z_n|^4} - \frac{1}{a_n} - \frac{\sigma}{a_n^2}\right) \ll 1. \tag{29}$$

For $\sigma_1 < \sigma < \sigma_2$, any t, and any n, we have that (recall that we assumed $\sigma_1 = 0$)

$$|s - z_n| \geq |a_n|, \quad |s - \overline{z_n}| \geq |a_n|, \quad |s - a_n| \geq |a_n|, \quad |s - a_n| \geq |t|. \tag{30}$$

Therefore, there is a positive constant c such that

$$\left|\frac{\sigma - a_n}{|s - z_n|^2} - \frac{\sigma - a_n}{|s - a_n|^2} + \frac{2a_n b_n t}{|s - z_n|^2 |s - a_n|^2}\right| \leq \frac{c}{|a_n|^3}.$$

By this, we obtain

$$\sum_{n=1}^{\infty}\left(\frac{\sigma - a_n}{|s - z_n|^2} - \frac{\sigma - a_n}{|s - a_n|^2}\right) = -2\sum_{n=1}^{\infty}\frac{a_n b_n t}{|s - z_n|^2 |s - a_n|^2} + O(1). \tag{31}$$

The symmetrical distribution of zeros z_n, inequalities (30), and the convergence of the series (23) give

$$\begin{aligned} -\sum_{n=1}^{\infty}\frac{a_n b_n t}{|s - z_n|^2 |s - a_n|^2} &= \sum_{\substack{n=1 \\ b_n > 0}}^{\infty}\left(\frac{a_n b_n t}{|s - \overline{z_n}|^2 |s - a_n|^2} - \frac{a_n b_n t}{|s - z_n|^2 |s - a_n|^2}\right) \\ &= -\sum_{\substack{n=1 \\ b_n > 0}}^{\infty}\frac{4a_n b_n^2 t^2}{|s - z_n|^2 |s - \overline{z_n}|^2 |s - a_n|^2} \leq \sum_{\substack{n=1 \\ b_n > 0}}^{\infty}\frac{4b_n}{|a_n|^3} \ll 1. \end{aligned} \tag{32}$$

By (31) and (32), we have

$$\sum_{n=1}^{\infty}\left(\frac{\sigma - a_n}{|s - z_n|^2} - \frac{\sigma - a_n}{|s - a_n|^2}\right) \ll 1. \tag{33}$$

Formulas (27), (28), (29), and (33) yield the equality

$$\Re I = \sum_{n=1}^{\infty} \left(\frac{\sigma - a_n}{|s - a_n|^2} + \frac{1}{a_n} + \frac{\sigma}{a_n^2} \right) + O(1). \tag{34}$$

Then, by the relation

$$\frac{\sigma - a_n}{|s - a_n|^2} + \frac{1}{a_n} + \frac{\sigma}{a_n^2} = \frac{t^2(a_n + \sigma) + \sigma^3 - a_n\sigma^2}{a_n^2 |s - a_n|^2},$$

by inequalities (30), and by the convergence of (23), we see that

$$\Re I = t^2 \sum_{n=1}^{\infty} \frac{1 + \sigma/a_n}{a_n |s - a_n|^2} + O(1) = t^2 \sum_{|a_n| > 2\sigma_2} \frac{1 + \sigma/a_n}{a_n |s - a_n|^2} + O(1). \tag{35}$$

If $t \geq 2\sigma_2$ and $|a_n| \leq t$, then we get that $|s - a_n| \leq 2t$. Thus

$$t^2 \sum_{|a_n| > 2\sigma_2} \frac{1 + \sigma/a_n}{a_n |s - a_n|^2} \leq -\frac{1}{16} \sum_{2\sigma_2 < |a_n| \leq t} \frac{1}{|a_n|}. \tag{36}$$

By the condition of the proposition, we have that the number of zeros z_n, satisfying the condition $2\sigma_2 < |a_n| \leq t$, is $\gg t^2$. This and (35) yield the existence of a positive constant c such that, for $\sigma_1 < \sigma < \sigma_2$ and large t,

$$\Re I < -ct. \tag{37}$$

Next, we will show that, for $\sigma_1 < \sigma < \sigma_2$, the real part of a function

$$J = \sum_{n=1}^{\infty} \left(\frac{1}{s - \rho_n} + \frac{1}{\rho_n} + \frac{s}{\rho_n^2} \right) = \sum_{n=1}^{\infty} \left(\frac{\overline{s} - \overline{\rho_n}}{|s - \rho_n|^2} + \frac{\overline{\rho_n}}{|\rho_n|^2} + \frac{s\overline{\rho_n}^2}{|\rho_n|^4} \right)$$

is $< o(t)$. We have

$$\Re J = \sum_{n=1}^{\infty} \left(\frac{\sigma - \beta_n}{|s - \rho_n|^2} + \frac{\beta_n}{|\rho_n|^2} + \frac{\beta_n^2 \sigma}{|\rho_n|^4} + \frac{2\beta_n \gamma_n t - \gamma_n^2 \sigma}{|\rho_n|^4} \right).$$

In view of the symmetrical distribution of zeros of $h(s)$ with respect to the real line, we see that

$$\sum_{n=1}^{\infty} \frac{2\beta_n \gamma_n t}{|\rho_n|^4} = 0.$$

By the convergence of the series (23), we get

$$\sum_{n=1}^{\infty}\frac{\beta_n^2\sigma}{|\rho_n|^4}\ll 1\quad\text{and}\quad\sum_{n=1}^{\infty}\left(\frac{\gamma_n^2\sigma}{|\rho_n|^4}-\frac{\sigma}{|\rho_n|^2}\right)\ll 1.$$

This gives

$$\begin{aligned}\Re J&=\sum_{n=1}^{\infty}\left(\frac{\sigma-\beta_n}{|s-\rho_n|^2}+\frac{\beta_n}{|\rho_n|^2}-\frac{\sigma}{|\rho_n|^2}\right)+O(1)\\&=\sum_{n=1}^{\infty}\frac{(\sigma-\beta_n)(2\beta_n\sigma-\sigma^2+2\gamma_n t-t^2)}{|s-\rho_n|^2|\rho_n|^2}+O(1).\end{aligned}$$

We have that $(\sigma-\beta_n)(2\sigma\beta_n-\sigma^2)>0$ if and only if $\sigma/2<\beta_n<\sigma$. By the condition of the proposition, we see that there are only finitely many zeros ρ_n which satisfy the inequality $\sigma/2<\beta_n<\sigma\le\sigma_2$. Thus we have

$$\sum_{n=1}^{\infty}\frac{(\sigma-\beta_n)(2\beta_n\sigma-\sigma^2)}{|s-\rho_n|^2|\rho_n|^2}<O(1).$$

Therefore,

$$\Re J\le\sum_{n=1}^{\infty}\frac{(\sigma-\beta_n)(2\gamma_n t-t^2)}{|s-\rho_n|^2|\rho_n|^2}+O(1).$$

We obtain that $(\sigma-\beta_n)(2t\gamma_n-t^2)>0$ if and only if (i) $\beta_n>\sigma$ and $\gamma_n<t/2$ or (ii) $\beta_n<\sigma$ and $\gamma_n>t/2$. If t is sufficiently large, then by the condition of the proposition, there are no zeros ρ_n which satisfy the condition $\beta_n<\sigma$ and $\gamma_n>t/2$. Therefore, for sufficiently large t,

$$\begin{aligned}\Re J\le&(\sigma_1-\sigma_2)\sum_{\gamma_n\le t/2}\frac{2\gamma_n t-t^2}{|s-\rho_n|^2|\rho_n|^2}+O(1)\\\le&(\sigma_1-\sigma_2)t\sum_{\substack{\gamma_n\le t/2\\\gamma_n\ne 0}}\frac{2\gamma_n-t}{(t-\gamma_n)^2\gamma_n^2}+O(1)\\\le&2(\sigma_1-\sigma_2)t\sum_{\substack{\gamma_n\le t/2\\\gamma_n\ne 0}}\frac{1}{(\gamma_n-t)\gamma_n^2}+O(1).\end{aligned}$$

Let $N(T)$ be the number of zeros of $h(s)$ in the region $\sigma_1<\sigma<\sigma_3$, $0<t\le T$. By the convergence of the series (23), we have that $N(T)=o(T^3)$ as

$T \to \infty$. Here and later, the notation $f(x) = o(g(x))$, as $x \to \infty$, means that $\lim_{x\to\infty} f(x)/g(x) = 0$, where $g(x) > 0$. Thus

$$\sum_{0<\gamma_n\le t/2} \frac{1}{(t-\gamma_n)\gamma_n^2} = \sum_{j=1}^{\infty} \sum_{t/2^{j+1}<\gamma_n\le t/2^j} \frac{1}{(t-\gamma_n)\gamma_n^2} \le \sum_{j=2}^{\infty} \frac{N(t/2^{j-1})}{(t/2)(t^2/2^{2j+2})} = o(1).$$

This bound gives that

$$\sum_{-t/2<\gamma_n<0} \frac{1}{(t-\gamma_n)\gamma_n^2} = \sum_{0<\gamma_n\le t/2} \frac{1}{(t+\gamma_n)\gamma_n^2} \le \sum_{0<\gamma_n\le t/2} \frac{1}{(t-\gamma_n)\gamma_n^2} = o(1).$$

Clearly,

$$\sum_{\gamma_n\le -t/2} \frac{1}{(t-\gamma_n)\gamma_n^2} = \sum_{\gamma_n\ge t/2} \frac{1}{(t+\gamma_n)\gamma_n^2} \le \sum_{\gamma_n\ge t/2} \frac{1}{\gamma_n^3} = o(1).$$

Consequently,

$$\Re J < o(t). \tag{38}$$

In view of (25), (37), and (38), formula (26) yields, for some positive number c,

$$\Re\left(\frac{h'}{h}(s)\right) < -ct. \tag{39}$$

Thus, for $\sigma_1 < \sigma < \sigma_2$ and large t, we have $h'(s) \ne 0$, if $h(s) \ne 0$. Proposition 7 is proved. □

Proof of Theorem 2. We apply Proposition 7 repeatedly k times to the Selberg zeta-function $Z(s)$. Next, we check that each function $Z^{(n)}(s)$, $n = 0, \ldots, k-1$, satisfies conditions of Proposition 7.

The Selberg zeta-function $Z(s)$ is an entire function of order 2 (see the Introduction). Entire functions $h(s)$ and $h'(s)$ are of the same order (Boas [2, Subsection 2.4.1.]). Thus any derivative of $Z(s)$ is an entire function of order 2.

In Proposition 7, we choose σ_1, σ_2, and σ_3 as indicated in equalities (22). Theorem 1 implies that for each function $Z^{(n)}(s)$, $n = 0, \ldots, k-1$, the number of zeros in $-x \le \sigma < \sigma_1$ is $\gg x^2$ as $x \to \infty$, and absolute values of imaginary parts of these zeros are bounded by an absolute constant.

It is known that $Z(s) \ne 0$, if $\sigma < 1/2$ and $t \ne 0$ (see the Introduction). Then Proposition 7 implies Theorem 2. □

Proof of Corollary 3. In light of the proof of Theorem 2, we see that each function $Z^{(n)}(s)$, $n = 0, \dots, k-1$, satisfies conditions of Proposition 7. Then, by inequality (39), we have that, for $n = 0, \dots, k-1$ and $\sigma_1 < \sigma < 1/2$, there is a positive c such that

$$\Re\left(\frac{Z^{(n+1)}}{Z^{(n)}}(s)\right) < -ct, \tag{40}$$

if t is sufficiently large. This inequality proves Corollary 3. $\square$

Proof of Theorem 4. We apply inequality (40) for $n = 1$. From the formulation of Proposition 7, it follows that σ_1 can be chosen as small as we like. Then the theorem follows by expression (8) of the curvature $\kappa_{Z,\sigma}(t)$. $\square$

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