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On Riemannian manifolds with $Spin(7)$-structure

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Abstract. We describe explicitly the space of covariant derivatives of the four-form of a $Spin(7)$-structure and also prove that $Spin(7)$-structures of type W_2 are locally conformal parallel. Finally, we give an example of $Spin(7)$-structure of type $\mathcal{W}_2$ not globally conformal parallel.

1. Introduction

An eight-dimensional Riemannian manifold M has a $Spin(7)$-structure, if M admits a reduction of the structure group of the tangent bundle to $Spin(7)$. This can be described geometrically by saying that there is a three-fold vector cross product P defined on M. Associated with P there is a four-form φ invariant under the action of $Spin(7)$. Under this action, the covariant derivative of φ can be decomposed into two components. This decomposition is used to classify $Spin(7)$-structures. Such a classification was shown by FERNÁNDEZ ([F1]).

In [F1] it is proved that the space W of tensors having the same symmetries as the covariant derivative of φ has two $Spin(7)$-irreducible components, W_1 and W_2. Thus there are four classes of $Spin(7)$-structure, namely, $\mathcal{P}$, $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}$. The list of known examples of $Spin(7)$-structures is relatively short (see [FG], [F1], [F2], [Br], [BS], [Z]). Moreover, there are not known examples for the class $\mathcal{W}_2$ not globally conformal to a $Spin(7)$-structure of type $\mathcal{P}$

In this paper we describe explicitly the space W of covariant derivative of φ. By another hand, from the defining conditions for classes of $Spin(7)$-structures by means of the exterior algebra, we derive that $Spin(7)$-structures of type $\mathcal{W}_2$ can be considered as locally conformal to $Spin(7)$-structures of type $\mathcal{P}$. Finally, we give examples of compact manifolds of type $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}$. In particular, we show that there are two

$Spin(7)$-structures of type $\mathcal{W}_2$ defined in $S^7 \times S^1$. Moreover, by topological obstructions, the manifold $S^7 \times S^1$ does not admit any $Spin(7)$-structure of type $\mathcal{P}$.

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2. Preliminaries

Let V be an eight dimensional real vector space with a inner product $\langle\, ,\rangle$. We denote by $\Lambda^k(V)$ the k-th Grassman space V (i.e., the space generated by the skew-symmetric products $v_1 \wedge v_2 \ldots \wedge v_k$). The inner product $\langle\, ,\rangle$ can be extended to $\Lambda^k(V)$ by the formula

$$\langle v_1 \wedge v_2 \wedge \ldots \wedge v_k \,,\, w_1 \wedge w_2 \wedge \ldots \wedge w_k \rangle = \det(\langle v_i, w_j\rangle),$$

for $v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_k \in V$. A *three-fold vector cross product* over V ([E], [BG]) is a trilinear map $P : V \times V \times V \to V$ satisfying the axioms:

$$\langle P(x,y,z), x\rangle = \langle P(x,y,z), y\rangle = \langle P(x,y,z), z\rangle = 0, \tag{2.1}$$

$$\|P(x,y,z)\|^2 = \|x \wedge y \wedge z\|^2. \tag{2.2}$$

for $x, y, z \in V$. It follows from (2.1) that P is skew-symmetric. Associated with P there is a skew-symmetric four-form φ, called the *fundamental four-form*, given by

$$\varphi(x,y,z,w) = \langle P(x,y,z), w\rangle,$$

for $x, y, z, w \in V$. Next lemma follows from definition of P.

Lemma 2.1 ([F1]).

$$\langle P(x,y,z), P(x,y,u)\rangle = \langle x \wedge y \wedge z, x \wedge y \wedge u\rangle, \tag{2.3}$$

$$\begin{aligned} &P(x,y,P(x,y,z)) \\ &= -\|x \wedge y\|^2 z + \langle x \wedge y, x \wedge z\rangle y + \langle y \wedge x, y \wedge z\rangle x, \end{aligned} \tag{2.4}$$

for $x, y, z, w \in V$.

We will need the following consequence of the last lemma.

Corollary 2.2. *For* $x, y, z, u \in V$, *we have*

$$\begin{aligned} P(x,y,P(x,z,u)) + P(x,z,P(x,y,u)) = -2\langle x \wedge y, x \wedge z\rangle u + \\ +\langle x \wedge z, x \wedge u\rangle y + \langle x \wedge y, x \wedge u\rangle z - \langle x \wedge y, z \wedge u\rangle x - \langle x \wedge z, y \wedge u\rangle x, \end{aligned}$$

PROOF. It follows by replacing in (2.4) y, z by $y + z, u$, respectively. $\square$

In an eight-dimensional vector space V with a three-fold vector cross product P, a *Cayley basis* for V is an orthonormal basis $\{e_{-1}, e_0, \dots , e_6\}$ such that

$$P(e_{-1}, e_i, e_{i+1}) = e_{i+3},$$

for all $i \in \mathbb{Z}_7$. For such a basis next lemma gives two alternatives.

Lemma 2.3. *Let V be an eight dimensional real vector space with an inner product $\langle\, ,\rangle$ and a three-fold vector cross product P.*
If $\{e_{-1}, e_0, \dots, e_6\}$ is a Cayley basis for V, then we have two alternatives:
(a) $P(e_{i+4}, e_{i+5}, e_{i+6}) = e_{i+2}$, *for all $i \in \mathbb{Z}_7$. In this case we say that $\{e_{-1}, e_0, \dots , e_6\}$ is a Cayley$_+$ basis.*
(b) $P(e_{i+4}, e_{i+5}, e_{i+6}) = -e_{i+2}$, *for all $i \in \mathbb{Z}_7$. In this case we say that $\{e_{-1}, e_0, \dots , e_6\}$ is a Cayley$_-$ basis.*

PROOF. By (2.2), we have

$$\begin{aligned} P(e_{i+4}, e_{i+5}, e_{i+6}) &= -P(e_{i+4}, e_{i+5}, P(e_{i+4}, e_{i+3}, e_{-1})) \\ &= P(e_{i+4}, e_{i+3}, P(e_{i+4}, e_{i+5}, e_{-1})) = -P(e_i, e_{i+3}, e_{i+4}), \end{aligned}$$

for all $i \in \mathbb{Z}_7$. On the other hand, since

$$P(e_{-1}, e_i, e_{i+1}) = e_{i+3} \quad \text{and} \quad \|P(e_{i+4}, e_{i+5}, e_{i+6})\| = 1,$$

for all $i \in \mathbb{Z}_7$, we obtain

$$P(e_{i+4}, e_{i+5}, e_{i+6}) = \pm e_{i+2}.$$

If $P(e_{j+4}, e_{j+5}, e_{j+6}) = e_{j+2}$ for some $j \in \mathbb{Z}_7$, then $e_{j+2} = -P(e_j, e_{j+3}, e_{j+4})$. Hence $P(e_{j+2}, e_{j+3}, e_{j+4}) = e_j$. Similar arguments show that

$$\begin{aligned} P(e_j, e_{j+1}, e_{j+2}) &= e_{j+5}, & P(e_{j+5}, e_{j+6}, e_j) &= e_{j+3}, \\ P(e_{j+3}, e_{j+4}, e_{j+5}) &= e_{j+1}, & P(e_{j+1}, e_{j+2}, e_{j+3}) &= e_{j+6}, \\ P(e_{j+6}, e_j, e_{j+1}) &= e_{j+4}. \end{aligned}$$

The case (b) can be deduced in a similar way. $\square$

If $\{\alpha_{-1}, \alpha_0, \alpha_1, \dots , \alpha_6\}$ is the dual basis of a Cayley$_\pm$ basis $\{e_{-1}, e_0, \dots , e_6\}$ of V, then the fundamental four-form φ can be expressed in one the following ways

$$\varphi = \sum_{i\in\mathbb{Z}_7} \alpha_{-1} \wedge \alpha_i \wedge \alpha_{i+1} \wedge \alpha_{i+3} - \sum_{i\in\mathbb{Z}_7} \alpha_{i+2} \wedge \alpha_{i+4} \wedge \alpha_{i+5} \wedge \alpha_{i+6}, \tag{2.5}$$

or

$$\varphi = \sum_{i\in\mathbb{Z}_7} \alpha_{-1} \wedge \alpha_i \wedge \alpha_{i+1} \wedge \alpha_{i+3} + \sum_{i\in\mathbb{Z}_7} \alpha_{i+2} \wedge \alpha_{i+4} \wedge \alpha_{i+5} \wedge \alpha_{i+6}. \tag{2.6}$$

3. The space of covariant derivatives of the fundamental four-form

We consider an eight-dimensional real vector space V with a three-fold vector cross product P. In [F1] it is defined the vector space W that consists of those tensor fields having the same symmetries as the covariant derivative of φ, i.e.,

$$W = \{\alpha \in V^* \otimes \Lambda^4(V^*) \mid \alpha(w,x,y,z,P(x,y,z)) = 0, \quad \text{for all} \quad w,x,y,z \in V\}, \tag{3.1}$$

where $\Lambda^4(V^*)$ denotes the set of skew-symmetric four-forms on V. If $\alpha \in W$, by polarization on z we have

$$\alpha(w,x,y,P(x,y,z),u) = \alpha(w,x,y,z,P(x,y,u)), \tag{3.2}$$

for all $w,x,y,z,u \in V$. Next lemma gives another way to describe W.

Lemma 3.1. *If $\alpha \in V^* \otimes \Lambda^4(V^*)$, the following conditions are equivalent:*

(a) $\alpha \in W$.

(b) $\alpha(w,x,y,P(x,y,z),P(x,y,u)) = -\|x \wedge y\|^2 \alpha(w,x,y,z,u)$, *for all $w,x,y,z,u \in V$.*

Proof. If $\alpha \in W$, condition (b) follows easily by using (3.2) and (2.4). Conversely, taking $P(x,y,z)$ instead of u in (b), and using (3.2), we will deduce (a). $\square$

We give an explicit description for $\alpha \in W$ in next lemma.

Lemma 3.2. *Let $\{e_{-1}, e_0, e_1, \dots, e_6\}$ be a Cayley$_\pm$ basis of V and $\{\alpha_{-1}, \alpha_0, \alpha_1, \dots, \alpha_6\}$ its dual basis. Then each $\alpha \in W$ is given by*

$$\begin{aligned}
\alpha = \pm & \sum_{i \in \mathbb{Z}_7 \cup \{-1\},\, j \in \mathbb{Z}_7} a_{ij} \alpha_i \otimes (\alpha_{-1} \wedge e_j \lrcorner \varphi - \alpha_j \wedge e_{-1} \lrcorner \varphi) \\
= & \sum_{i \in \mathbb{Z}_7 \cup \{-1\},\, j \in \mathbb{Z}_7} a_{ij} \alpha_i \otimes (\alpha_{j+4} \wedge e_{j+5} \lrcorner \varphi - \alpha_{j+5} \wedge e_{j+4} \lrcorner \varphi) \\
= & \sum_{i \in \mathbb{Z}_7 \cup \{-1\},\, j \in \mathbb{Z}_7} a_{ij} \alpha_i \otimes (\alpha_{j+1} \wedge e_{j+3} \lrcorner \varphi - \alpha_{j+3} \wedge e_{j+1} \lrcorner \varphi) \\
= & \sum_{i \in \mathbb{Z}_7 \cup \{-1\},\, j \in \mathbb{Z}_7} a_{ij} \alpha_i \otimes (\alpha_{j+2} \wedge e_{j+6} \lrcorner \varphi - \alpha_{j+6} \wedge e_{j+2} \lrcorner \varphi),
\end{aligned}$$

where $\lrcorner$ denotes the interior product.

Proof. From (2.5) and (2.6) it can be checked that

$$\begin{aligned}
&\pm \alpha_i \otimes (\alpha_{-1} \wedge e_j \lrcorner \varphi - \alpha_j \wedge e_{-1} \lrcorner \varphi) \\
&\quad = \alpha_i \otimes (\alpha_{j+4} \wedge e_{j+5} \lrcorner \varphi - \alpha_{j+5} \wedge e_{j+4} \lrcorner \varphi) \\
&\quad = \alpha_i \otimes (\alpha_{j+1} \wedge e_{j+3} \lrcorner \varphi - \alpha_{j+3} \wedge e_{j+1} \lrcorner \varphi) \\
&\quad = \alpha_i \otimes (\alpha_{j+2} \wedge e_{j+6} \lrcorner \varphi - \alpha_{j+6} \wedge e_{j+2} \lrcorner \varphi).
\end{aligned}$$

From the definition of W we have

$$\alpha(e_i, e_{-1}, e_j, e_{j+1}, e_{j+3}) = \alpha(e_i, e_{j+2}, e_{j+4}, e_{j+5}, e_{j+6}) = 0,$$

for each $i \in \mathbb{Z}_7 \cup \{-1\}$ and $j \in \mathbb{Z}_7$. Using Lemma 3.1, we obtain

$$\begin{aligned}
\alpha(e_i, e_{-1}, e_j, e_{j+1}, e_{j+2}) &= -\alpha(e_i, e_{-1}, e_j, e_{j+3}, e_{j+6}) \\
= -\,\alpha(e_i, e_{-1}, e_{j+1}, e_{j+3}, e_{j+4}) &= \alpha(e_i, e_{-1}, e_{j+2}, e_{j+4}, e_{j+6}) \\
= \mp\,\alpha(e_i, e_j, e_{j+1}, e_{j+3}, e_{j+5}) &= \pm\alpha(e_i, e_j, e_{j+2}, e_{j+5}, e_{j+6}) \\
= \mp\,\alpha(e_i, e_{j+1}, e_{j+2}, e_{j+4}, e_{j+5}) &= \pm\alpha(e_i, e_{j+3}, e_{j+4}, e_{j+5}, e_{j+6}),
\end{aligned}$$

for all $i \in \mathbb{Z}_7 \cup \{-1\}$ and $j \in \mathbb{Z}_7$. Taking (2.5) and (2.6) into account, we will have the required expressions. $\square$

Remark 3.3. Since $\dim W = 56$, the set of tensors $\alpha_i \otimes (\alpha_{j+1} \wedge e_{j+3} \lrcorner \varphi - \alpha_{j+3} \wedge e_{j+1} \lrcorner \varphi)$ is a basis for W. Moreover, by using the relationship between the interior product and the wedge product we get the following expression for $\alpha \in W$

$$\alpha = \sum_{i \in \mathbb{Z}_7 \cup \{-1\},\, j \in \mathbb{Z}_7} a_{ij}\ \alpha_i \otimes (\alpha_{-1} \wedge *(\alpha_j \wedge \varphi) - \alpha_j \wedge *(\alpha_{-1} \wedge \varphi)),$$

where $*$ denotes the Hodge $*$-operator.

Fernández ([F1]) proves that W has two irreducible components under the action of $Spin(7)$,

$$W = W_1^{(48)} \oplus W_2^{(8)},$$

the upper index indicates the corresponding dimension. Thus each $\alpha \in W$ has two components. These components can be studied by means of the exterior algebra. In fact, the irreducible components of W can be described by applying Schur's Lemma to the restrictions to W of the $Spin(7)$-equivariant map

$$\begin{gathered}
V^* \otimes \Lambda^4(V^*) \to \Lambda^5(V^*) \\
\alpha = x \otimes a \wedge b \wedge c \wedge d \to s(\alpha) = x \wedge a \wedge b \wedge c \wedge d.
\end{gathered} \tag{3.3}$$

In [Br] it can be found the descriptions of the decompositions of $\Lambda^5(V^*)$ in its irreducible components under the $Spin(7)$-action, i.e.,

$$\Lambda^5(V^*) = \Lambda^5_{(48)}(V^*) \oplus \Lambda^5_{(8)}(V^*),$$

where the subindices indicate the corresponding dimension. The above irreducible components are as follows

$$\Lambda^5_{(48)}(V^*) = \{\alpha \in \Lambda^5(V^*) \mid *\alpha \wedge \varphi = 0\}, \tag{3.4}$$

$$\Lambda^5_{(8)}(V^*) = \{\alpha \wedge \varphi \mid \alpha \in V^*\}. \tag{3.5}$$

4. $Spin(7)$-structures

An eight-dimensional C^∞ Riemannian manifold M with tensor metric field $\langle\, ,\rangle$, has a *$Spin(7)$-structure*, if the structure group of the bundle of orthonormal frames can be reduced from $O(8)$ to the spinor group $Spin(7)$. Geometrically this means that for each $m \in M$ the tangent space $T_m(M)$ is provided with a three-fold vector cross product P_m such that the map $m \to P_m$ is C^∞. This is also equivalent to the existence of a non where vanishing four-form φ such that it can be locally expressed in one of the following ways

$$\varphi = \sum_{i \in \mathbb{Z}_7} \alpha_{-1} \wedge \alpha_i \wedge \alpha_{i+1} \wedge \alpha_{i+3} \mp \sum_{i \in \mathbb{Z}_7} \alpha_{i+2} \wedge \alpha_{i+4} \wedge \alpha_{i+5} \wedge \alpha_{i+6}. \tag{4.1}$$

where $\{\alpha_{-1}, \alpha_0, \dots, \alpha_6\}$ is a local orthonormal frame of the cotangent bundle.

In each point of a Riemannian manifold with a $Spin(7)$-structure there are local orthonormal frames $\{E_{-1}, E_0, \dots, E_6\}$, called *Cayley frames*, such that

$$P(E_{-1}, E_i, E_{i+1}) = E_{i+3},$$

for all $i \in \mathbb{Z}_7$. Wether $P(E_{i+4}, E_{i+5}, E_{i+6}) = E_{i+2}$ or $P(E_{i+4}, E_{i+5}, E_{i+6}) = -E_{i+2}$, for all $i \in \mathbb{Z}_7$ we say that such a frame is a *Cayley$_+$* or *Cayley$_-$ frame*, respectively. Along this section $\{\alpha_{-1}, \alpha_0, \dots, \alpha_6\}$ will be the dual forms of a local Cayley$_\pm$ frame. On the same point there is not simultaneously a Cayley$_+$ frame and a Cayley$_-$ frame, because in that case we would have $*\varphi = \varphi$ and $*\varphi = -\varphi$. Then on each connected component of M there exists only one type ($+$ or $-$) of local Cayley frames.

Let ∇ be the Riemannian connection of $\langle\, ,\rangle$. In [F1] it is shown that in each point $m \in M$ the covariant derivative $\nabla\varphi$ belongs to $W \subset T^*_m M \otimes \Lambda^4 T^*_m M$ defined as in (3.1).

A $Spin(7)$-structure is said of type $\mathcal{P}$, $\mathcal{W}_1$, $\mathcal{W}_2$ or $\mathcal{W}$, if the covariant derivative $\nabla\varphi$ lies in $\{0\}$, W_1, W_2 or W, respectively.

Denoting by d the exterior derivative on M, it is obvious that $d\varphi = s(\nabla\varphi)$. Using Shur's Lemma, from (3.4) and (3.5) one deduces a characterization for each type of $Spin(7)$-structure. These characterizations are shown in the following table

$\mathcal{P}$	$d\varphi = 0$
$\mathcal{W}_1$	$*d\varphi \wedge \varphi = 0$
$\mathcal{W}_2 = \mathcal{L}CP$	$d\varphi = \alpha \wedge \varphi$
$\mathcal{W}$	no relation

Table 1

In agreeing with the notation used in [F1], we consider the one-form $pd^*\varphi$ on M defined by

$$pd^*\varphi = - * (*d\varphi \wedge \varphi).$$

We denote by π_8 the projection $\Lambda^5 T^*M \to \Lambda^5_{(8)} T^*M$. In the following lemma we compute $\pi_8(d\varphi)$.

Lemma. *We have:*

$$\pi_8(d\varphi) = \frac{1}{7} pd^*\varphi \wedge \varphi.$$

Proof. A straightforward computation shows

$$*(*(\alpha_i \wedge \varphi) \wedge \varphi) = -7\alpha_i,$$

for all $i \in \mathbb{Z}_7 \cup \{-1\}$. We write $\pi_8(d\varphi) = \alpha \wedge \varphi$ and $\alpha = \sum_{i\in\mathbb{Z}_7\cup\{-1\}} c_i\ \alpha_i$. Then

$$\begin{aligned} pd^*\varphi &= - * (*d\varphi \wedge \varphi) \;=\; - * (*\pi_8(d\varphi) \wedge \varphi) \\ &= - \sum_{i\in\mathbb{Z}_7\cup\{-1\}} c_i * (*(\alpha_i \wedge \varphi) \wedge \varphi) \;=\; \sum_{i\in\mathbb{Z}_7\cup\{-1\}} 7c_i\alpha_i \;=\; 7\alpha. \quad \square \end{aligned}$$

From the previous lemma it follows immediately the following corollary.

Corollary 4.2. *The form α appearing in Table 1 is such that $pd^*\varphi = 7\alpha$.*

From (2.5) and (2.6) it follows the following lemma proved in [Bo].

Lemma 4.3 ([Bo]). *Let α be a skew-symmetric p-form. If $p \leq 2$, $\varphi \wedge \alpha = 0$ if and only if $\alpha = 0$.*

Lemma 4.4. *If P is of type $\mathcal{W}_2$, the one-form $pd^*\varphi$ is closed.*

Proof. If P is of type $\mathcal{W}_2$, then $d\varphi = \frac{1}{7}pd^*\varphi \wedge \varphi$. By differentiating we have $0 = dpd^*\varphi \wedge \varphi$. By Lemma 4.3, we conclude $dpd^*\varphi = 0$. $\square$

Remark 4.5. $Spin(7)$-structures of type $\mathcal{P}$ are usually called parallel. If we consider a $Spin(7)$-structure of type $\mathcal{W}_2$, by Lemma 4.4 and Poincaré's Lemma, in each point m of M there exists a local function σ such that $d\sigma = -\frac{1}{28}pd^*\varphi$. Doing the conformal change of metric given by $\langle\,,\rangle_o = e^{2\sigma}\langle\,,\rangle$ in the neighborhood U of m where σ is defined, we get a $Spin(7)$-structure on U of type parallel. This argument makes reasonable to call $Spin(7)$-structures of type $\mathcal{W}_2$, locally conformal parallel ($\mathcal{LCP}$).

5. Examples

The manifold $S^7 \times S^1$

In [FG] it is shown that the sphere S^7 has a nearly parallel G_2-structure, i.e., there is a non where vanishing three-form φ on S^7 such that it can be written locally in the way

$$\varphi = \sum_{i\in\mathbb{Z}_7} \alpha_i \wedge \alpha_{i+1} \wedge \alpha_{i+3},$$

where $\{\alpha_0, \dots, \alpha_6\}$ is an local orthonormal frame of the cotangent bundle of S^7. The nearly parallel condition implies $d\varphi = k * \varphi$ and $d * \varphi = 0$ (see [C], [FG]).

We consider on the product manifold $S^7 \times S^1$ the four-forms given by

$$\overline{\varphi}_\pm = \eta \wedge \varphi \pm *\varphi,$$

where η is a non null one-form of Maurer-Cartan on S^1. For the local coframe $\{\eta, \alpha_0, \dots, \alpha_6\}$, the forms $\varphi_\pm$ are locally written

$$\overline{\varphi}_\pm = \sum_{i\in\mathbb{Z}_7} \eta \wedge \alpha_i \wedge \alpha_{i+1} \wedge \alpha_{i+3} \mp \sum_{i\in\mathbb{Z}_7} \alpha_{i+2} \wedge \alpha_{i+4} \wedge \alpha_{i+5} \wedge \alpha_{i+6}.$$

Then each one of the forms $\overline{\varphi}_\pm$ defines a $Spin(7)$-structure. By differentiating we have

$$d\overline{\varphi}_\pm = \eta \wedge k * \varphi = \pm k\eta \wedge \overline{\varphi}_\pm.$$

Hence the $Spin(7)$-structures are of type $\mathcal{W}_2$ and they are not of type $\mathcal{P}$. In fact, there are topological obstructions to the existence of $Spin(7)$-structure of type $\mathcal{P}$. In [Bo] it is proved that if a compact manifold M has a $Spin(7)$-structure of type $\mathcal{P}$, then

$$b_4(M) \neq 0 \quad \text{and} \quad b_4(M) \geq b_1(M), \tag{5.1}$$

where b_i denote the Betti numbers. Since for $S^7\times S^1$ we have $b_1(S^7\times S^1)=1$ and $b_4(S^7\times S^1)=0$, we conclude that $S^7\times S^1$ does not admit any $Spin(7)$-structure of type $\mathcal{P}$. To our knowledge, these are the first known examples of $Spin(7)$-structures of type $\mathcal{W}_2$ not globally conformal to one of type $\mathcal{P}$.

However, for each point of $S^7\times S^1$ there exists a neighborhood $S^7\times U$ where the fucntions σ such that $d\sigma = \mp\frac{k}{4}\eta$ are defined. Doing the conformal changes of metric given by $\langle\,,\rangle_0 = e^{2\sigma}\langle\,,\rangle$ in $S^7\times U$, we get two $Spin(7)$-structures of type $\mathcal{P}$ defined on $S^7\times U$.

The manifolds $\boldsymbol{M}(k)\times\mathbb{T}^5$

Let us consider the manifolds $M(k)$ described in [CFG] as follows. For a fixed $k\in\mathbb{R},\ k\neq 0$, let $G(k)$ be the three-dimensional connected and solvable (non-nilpotent) Lie group consisting of the matrices

$$\boldsymbol{a} = \begin{pmatrix} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x,y,z\in\mathbb{R}$. Then, a global coordinate system $\{x,y,z\}$ for $G(k)$ is given by $x(\boldsymbol{a})=x,\ \ y(\boldsymbol{a})=y,\ \ z(\boldsymbol{a})=z$. A straightforward computation proves that a basis of right invariant 1-forms on $G(k)$ is $\{dx-kxdz, dy+kydz, dz\}$.

The Lie group $G(k)$ can be also described as the semidirect product $\mathbb{R}\times_\phi\mathbb{R}^2$, where

$$\phi\ :\ \mathbb{R}\ \longrightarrow\ \mathrm{Aut}(\mathbb{R}^2)$$

is the representation defined by

$$\phi(t) = \begin{pmatrix} e^{kz} & 0 \\ 0 & e^{-kz} \end{pmatrix},\quad z\in\mathbb{R}.$$

Therefore $G(k)$ posesses a discrete subgroup $\Gamma(K)$ such that the quotient manifold $M(k)=G(k)/\Gamma(k)$ is compact. Moreover, the one-forms $dx-kxdz,\ dy+kydz,\ dz$ descend to $M(k)$. Let us denote by α,β,γ respectively, the induced one-forms on $M(k)$. Then, we have $d\alpha=-k\alpha\wedge\gamma,\ \ d\beta=k\beta\wedge\gamma,\ \ d\gamma=0$.

Let us consider the product manifold $M(k)\times\mathbb{T}^5$ where $\mathbb{T}^5$ is a five-dimensional torus. Let $\eta_1,\eta_2,\eta_3,\eta_4,\eta_5$ be a basis of closed one-forms on $\mathbb{T}^5$. Then in $M(k)\times\mathbb{T}^5$ we have the following basis of one-forms

$$\alpha_{-1}=\eta_5,\ \alpha_0=\gamma,\ \alpha_1=\eta_1,\ \alpha_2=\eta_2,\ \alpha_3=\alpha,\ \alpha_4=\eta_3,\ \alpha_5=\beta,\ \alpha_6=\eta_4.$$

Therefore,

$$\begin{gathered} d\alpha_{-1}=d\alpha_0=d\alpha_1=d\alpha_2=d\alpha_4=d\alpha_6=0, \\ d\alpha_3=k\alpha_0\wedge\alpha_3,\quad d\alpha_5=-k\alpha_0\wedge\alpha_5. \end{gathered}$$

Let $\langle\,,\rangle$ be a tensor metric field on $M(k)\times\mathbb{T}^5$ given by

$$\langle\,,\rangle = \alpha_{-1}\otimes\alpha_{-1} + \sum_{i\in\mathbb{Z}_7}\alpha_i\otimes\alpha_i.$$

We consider the frame $\{E_{-1},E_0,E_1,\dots,E_6\}$ of orthonormal vector fields dual of the one-forms α_i. We define the three-fold vector cross products P_+ and P_- such that $\{E_{-1},E_0,E_1,\dots,E_6\}$ is a Cayley$_+$ or Cayley$_-$ frame, respectively, i.e.,

$$P(E_{-1},E_i,E_{i+1}) = E_{i+3}\ \ ; P(E_{i+4},E_{i+5},E_{i+6}) = \pm E_{i+2},$$

for all $i\in\mathbb{Z}_7$. Then fundamental four-forms $\varphi_\pm$ are given as in (2.5) and (2.6). The exterior differential of the fundamental four-forms $\varphi_\pm$ of $P_\pm$ are given by

(5.2)
$$\begin{aligned} d\varphi_\pm = &- k\alpha_{-1}\wedge\alpha_0\wedge\alpha_3\wedge\alpha_4\wedge\alpha_6 + k\alpha_{-1}\wedge\alpha_0\wedge\alpha_5\wedge\alpha_6\wedge\alpha_1 \\ &\pm k\alpha_0\wedge\alpha_2\wedge\alpha_4\wedge\alpha_5\wedge\alpha_6 \mp k\alpha_0\wedge\alpha_6\wedge\alpha_1\wedge\alpha_2\wedge\alpha_3. \end{aligned}$$

Since $*d\varphi_\pm\wedge\varphi_\pm = 0$ and $d\varphi_\pm\neq 0$, the $Spin(7)$-structures are of type $\mathcal{W}_1$ and they are not of type $\mathcal{P}$. In summary,

$$P_+, P_- \in \mathcal{W}_1 - \mathcal{P}.$$

In [CFG] are computed the Betti numbers for $M(k)$, i.e.,

$$b_1(M(k)) = b_2(M(k)) = 1.$$

Hence for the product manifold $M(k)\times\mathbb{T}^5$ we have

$$b_1 = 6, \quad \text{and} \quad b_4 = 30.$$

Therefore, in this case the topological obstruction (5.1) does not work as in the previous example.

The manifolds $\boldsymbol{M}(k) \times H/\Gamma \times \mathbb{T}^2$

Let us consider the product of $M(k)$, the Heisenberg compact nilmanifold H/Γ and a two-dimensional torus. The manifold $M(k)$ has been considered in the previous example. Let us decribe, briefly, the manifold H/Γ (see [FI] for more details).

Let H be the Heisenberg group of dimension three, i.e., H is the connected, simply connected and nilpotent Lie group consisting of matrices:

$$\boldsymbol{a} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{R}$. A (global) coordinate system $\{x, y, z\}$ over H is given by: $x(\boldsymbol{a}) = x$, $y(\boldsymbol{a}) = y$, $z(\boldsymbol{a}) = z$. It is easy to show that $\{dx, dy, dz - xdy\}$ are linearly independent left invariant one-forms of H. Let Γ be the discrete subgroup of all matrices of H which entries x, y, z are integers. The quotient space H/Γ is called the *Heisenberg compact nilmanifold*. Since $\{dx, dy, dz - xdy\}$ are left invariant one-forms under the action of Γ, they descend respectively to the one-forms $\{\sigma, \rho, \mu\}$ on H/Γ such that $d\mu = \rho \wedge \sigma$.

Let us consider the product manifold $M(k) \times H/\Gamma \times \mathbb{T}^2$ where $\mathbb{T}^2$ is a two-dimensional torus. Let η_1, η_2 be a basis of closed one-forms on $\mathbb{T}^2$. Then for $M(k) \times H/\Gamma \times \mathbb{T}^2$ we have the next basis for one-forms

$$\alpha_{-1} = \eta_2,\ \alpha_0 = \gamma,\ \alpha_1 = \rho,\ \alpha_2 = \sigma,\ \alpha_3 = \alpha, \alpha_4 = \mu,\ \alpha_5 = \beta,\ \alpha_6 = \eta_1.$$

Therefore,

$$\begin{gathered} d\alpha_{-1} = d\alpha_0 = d\alpha_1 = d\alpha_2 = d\alpha_6 = 0, \\ d\alpha_3 = k\alpha_0 \wedge \alpha_3, \qquad d\alpha_4 = \alpha_1 \wedge \alpha_2, \qquad d\alpha_5 = -k\alpha_0 \wedge \alpha_5. \end{gathered}$$

Let $\langle\,,\rangle$ be a tensor metric field on $M(k) \times H/\Gamma \times \mathbb{T}^1$ given by $\langle\,,\rangle = \alpha_{-1} \otimes \alpha_{-1} + \sum_{i \in \mathbb{Z}_7} \alpha_i \otimes \alpha_i$. We consider the frame $\{E_{-1}, E_0, E_1, \dots, E_6\}$ of orthonormal vector fields dual to the one-forms α_i. We define the threefold vector cross products P_+ and P_- such that $\{E_{-1}, E_0, E_1, \dots, E_6\}$ is a Cayley$_+$ and Cayley$_-$ frame, respectively. The exterior differential of the fundamental four-forms $\varphi_\pm$ of $P_\pm$ are given by

$$\begin{aligned} d\varphi_\pm = & -k\alpha_{-1} \wedge \alpha_0 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_6 + k\alpha_{-1} \wedge \alpha_0 \wedge \alpha_5 \wedge \alpha_6 \wedge \alpha_1 \\ & \pm k\alpha_0 \wedge \alpha_2 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6 \mp k\alpha_0 \wedge \alpha_6 \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \\ & - \alpha_{-1} \wedge \alpha_5 \wedge \alpha_0 \wedge \alpha_1 \wedge \alpha_2 - \alpha_{-1} \wedge \alpha_6 \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3. \end{aligned}$$

Since $*d\varphi_\pm \wedge \varphi_\pm \neq 0$, the $Spin(7)$-structures are not of type $\mathcal{W}_1$.

Now we compute $pd^*\varphi_\pm = *(*d\varphi_\pm \wedge \varphi_\pm)$. We obtain

$$pd^*\varphi_\pm = \mp 2\alpha_{-1}.$$

Since $d\varphi_\pm \neq \frac{1}{7}pd^*\varphi_\pm \wedge \varphi_\pm$, the $Spin(7)$-structures are not of type $\mathcal{W}_2$. In summary,

$$P_+, P_- \in \mathcal{W} - (\mathcal{W}_1 \cup \mathcal{W}_2).$$

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