

Finite groups with some subgroups of Sylow subgroups weakly $\mathcal{H}$ -embedded

By MOHAMED ASAAD (Cairo)

Abstract. Let G be a finite group and H a subgroup of G . We say that H is an $\mathcal{H}$ -subgroup in G if $N_G(H) \cap H^g \leq H$, for all $g \in G$. The subgroup H is called weakly $\mathcal{H}$ -embedded in G if G has a normal subgroup K such that $H^G = HK$ and $H \cap K$ is an $\mathcal{H}$ -subgroup in G , where H^G is the normal closure of H in G , that is, $H^G = \langle H^g : g \in G \rangle$. Using this concept, we improve and extend Theorem 1.6 and Corollary 1.9 of [3] and Theorem 3.1 of [17].

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. A class of groups $\mathfrak{F}$ is a formation if it contains all homomorphic images of a group in $\mathfrak{F}$, and if G/M and G/N are both in $\mathfrak{F}$, then $G/(M \cap N)$ is also in $\mathfrak{F}$ for any normal subgroups M and N of G . A formation $\mathfrak{F}$ is said to be saturated if $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$. For more details about saturated formations, see [8]. Let $\mathfrak{U}$ be the class of all supersolvable groups. It is known that $\mathfrak{U}$ is a saturated formation. Let $Z_{\mathfrak{U}}(G)$ denote the $\mathfrak{U}$ -hypercenter of G , where $Z_{\mathfrak{U}}(G)$ is the product of all normal subgroups H of G such that every chief factor of G below H has prime order. Clearly, $Z_{\infty}(G) \leq Z_{\mathfrak{U}}(G)$, where $Z_{\infty}(G)$ is the hypercenter of G , that is, $Z_{\infty}(G)$ is the largest term of the upper central series of G . The generalized Fitting subgroup $F^*(G)$ of G is the set of all elements x of G which induce an inner automorphism on every chief factor of G (see [16, Chapter X, Section 13]). A subgroup H of G is said to be c -normal in G if there

Mathematics Subject Classification: 20D10, 20D15, 20D20.

Key words and phrases: $\mathcal{H}$ -subgroup, weakly $\mathcal{H}$ -subgroup, weakly $\mathcal{H}$ -embedded subgroup, saturated formation.

exists a normal subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the largest normal subgroup of G contained in H (see [18]). A subgroup H of G is said to be an $\mathcal{H}$ -subgroup of G if $N_G(H) \cap H^g \leq H$ for all $g \in G$ (see [5]). The set of all $\mathcal{H}$ -subgroups of G will be denoted by $\mathcal{H}(G)$. It is easy to note that the Sylow subgroups of a normal subgroup of G belong to $\mathcal{H}(G)$. A subgroup H of G is said to be a weakly $\mathcal{H}$ -subgroup of G if G has a normal subgroup K such that $G = HK$ and $H \cap K \in \mathcal{H}(G)$ (see [2]). A subgroup H of G is said to be weakly $\mathcal{H}$ -embedded in G if G has a normal subgroup K such that $H^G = HK$ and $H \cap K \in \mathcal{H}(G)$, where H^G is the normal closure of H in G , that is, $H^G = \langle H^g : g \in G \rangle$, (see [3]). Clearly, c -normal subgroups, $\mathcal{H}$ -subgroups and weakly $\mathcal{H}$ -subgroups are weakly $\mathcal{H}$ -embedded. The converse is not true (see [3, Examples 1.3, 1.4 and 1.5]). Using the above concepts, many interesting results have been obtained. For example, LI and QIAO [17] proved the following statement: Let p be an odd prime dividing the order of G , and P a Sylow p -subgroup of G . Suppose that there exists a subgroup D of P with $1 < |D| = d < |P|$ such that every subgroup H of P with $|H| = d$ is a weakly $\mathcal{H}$ -subgroup of G and $N_G(H)$ is p -nilpotent. Then G is p -nilpotent. In [3], the authors proved that if P is a Sylow p -subgroup of G , then G is p -nilpotent if and only if $N_G(P)$ is p -nilpotent and every maximal subgroup of P is weakly $\mathcal{H}$ -embedded in G . Moreover, working within the framework of formation theory, they proved: Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$, and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If, for every Sylow subgroup P of $F^*(E)$, every maximal subgroup of P is weakly $\mathcal{H}$ -embedded in G , then $G \in \mathfrak{F}$. For more results in this direction of study, see [2]–[4], [6]–[7], [11]–[12], [17]–[19], and [21].

In this paper, we go further in studying the influence of weakly $\mathcal{H}$ -embedded subgroups on the structure of finite groups. More precisely, we prove:

Theorem 1.1. *Let G be a group, and P a Sylow p -subgroup of G . Suppose that there exists a subgroup D of P with $1 < |D| = d < |P|$ such that every subgroup H of P with $|H| = d$ is weakly $\mathcal{H}$ -embedded in G and $N_G(H)$ is p -nilpotent. Then G is p -nilpotent.*

Theorem 1.2. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$, and E a normal subgroup of G with $G/E \in \mathfrak{F}$. For every prime p dividing $|E|$ and every Sylow p -subgroup E_p of E , suppose that E_p has a subgroup D with $1 < |D| = d < |E_p|$ and every subgroup of E_p of order $p^n d$ ($n = 0, 1$) is weakly $\mathcal{H}$ -embedded in G . Then $G \in \mathfrak{F}$.*

Theorem 1.3. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$, and E a normal subgroup of G with $G/E \in \mathfrak{F}$. For every prime p dividing $|F^*(E)|$ and every Sylow p -subgroup P of $F^*(E)$, suppose that P has a subgroup D with $1 < |D| = d < |P|$ and every subgroup of P of order $p^n d$ ($n = 0, 1$) is weakly $\mathcal{H}$ -embedded in G . Then $G \in \mathfrak{F}$.*

Clearly, Theorems 1.1, 1.2 and 1.3 improve and extend the above-mentioned results and many new related results in the literature.

Remark. Theorems 1.2 and 1.3 are false if we assume $n = 0$ only as the following examples show:

Example 1.4. Write $G = SL(2, 3)$. Then G is the split extension of a quaternion group of order 8 by the cyclic group of order 3. Clearly, the center of G is a unique subgroup of order 2, and so it is weakly $\mathcal{H}$ -embedded in G . Then G/P satisfies the hypothesis of Theorems 1.2 and 1.3 when $n = 0$, but $G \notin \mathfrak{U}$.

Example 1.5. Let $H = \langle a, b : a^5 = b^5 = 1, ab = ba \rangle$ and α be an automorphism of H of order 3 satisfying that $a^\alpha = b$, $b^\alpha = a^{-1}b^{-1}$. Let $H = H_1$, $H_2 = \langle a^1, b^1 \rangle$ be two copies of H , and denote by $G = [H_1 \times H_2] \langle \alpha \rangle$ the corresponding semidirect product. Then G has at least four distinct minimal normal subgroups H_i ($i = 1, 2, 3, 4$) of G of order 25, and so G is not supersolvable and if A is any subgroup of order 25, then there exists $i \in \{1, 2, 3, 4\}$ such that $A \cap H_i = 1$ (see [15]). Now it is easy to note that every subgroup of $H_1 \times H_2$ of order 5^3 is not normal in G , so G contains nonnormal subgroups of order 5^2 . Let A be any nonnormal subgroup of G of order 25. Then $A < A^G = H_1 \times H_2 = AH_i$ for some $i \in \{1, 2, 3, 4\}$, that is, A is weakly $\mathcal{H}$ -embedded in G and, since normal subgroups of order 5^2 are weakly $\mathcal{H}$ -embedded in G , we have every subgroup of order 5^2 is weakly $\mathcal{H}$ -embedded in G , but $G \notin \mathfrak{U}$.

All unexplained notation and terminology are standard (see [10], [13]–[14]).

2. Preliminaries

Lemma 2.1 ([3, Lemma 2.2]). *Let H be a subgroup of G . Then:*

- (1) *If H is weakly $\mathcal{H}$ -embedded in G , $H \leq M \leq G$, then H is weakly $\mathcal{H}$ -embedded in M .*
- (2) *Let N be a normal subgroup of G and $N \leq H$. Then H is weakly $\mathcal{H}$ -embedded in G if and only if H/N is weakly $\mathcal{H}$ -embedded in G/N .*

- (3) Let H be a p -subgroup of G for some prime p , and N a normal p' -subgroup of G . If H is weakly $\mathcal{H}$ -embedded in G , then HN/N is weakly $\mathcal{H}$ -embedded in G/N .

Lemma 2.2 ([13, Lemma 3.6.10]). Let K be a normal subgroup of G , and P be a p -subgroup of G . If P_1 is a Sylow p -subgroup of PK , then $N_{G/K}(PK/K) = N_G(P_1)K/K$.

Lemma 2.3 ([10, Theorem 8.3.1]). If P is a Sylow p -subgroup of G , with p odd, and if $N_G(Z(J(P)))$ is p -nilpotent, then G is p -nilpotent.

Lemma 2.4 ([3, Theorem 1.6]). Let p be a prime dividing the order of G , and P a Sylow p -subgroup of G . Then G is p -nilpotent if and only if $N_G(P)$ is p -nilpotent and every maximal subgroup of P is weakly $\mathcal{H}$ -embedded in G .

Lemma 2.5 ([9, Corollary B3]). If H is a 2-subgroup of G such that $H \in \mathcal{H}(G)$ and $N_G(H)/C_G(H)$ is a 2-group, then H is a Sylow 2-subgroup of H^G .

Lemma 2.6 ([5, Theorem 6(2)]). Let H be an $\mathcal{H}$ -subgroup of G . If H is subnormal in G , then H is normal in G .

Lemma 2.7. Let P be a nontrivial normal p -subgroup of G and $L \leq P$. Then L is weakly $\mathcal{H}$ -embedded in G if and only if G has a normal subgroup K such that $L^G = LK$ and $L \cap K$ is normal in G .

PROOF. If G has a normal subgroup K such that $L^G = LK$ and $L \cap K$ is normal in G , then L is weakly $\mathcal{H}$ -embedded in G . Conversely, let L be a weakly $\mathcal{H}$ -embedded in G . Then G has a normal subgroup K such that $L^G = LK$ and $L \cap K \in \mathcal{H}(G)$. It is clear that $L \cap K$ is subnormal in G . Then, by Lemma 2.6, $L \cap K$ is normal in G . $\square$

Lemma 2.8 ([20, Theorem 7.19, Chapter 1]). Let H be a normal subgroup of G . Then $H \leq Z_{\mathfrak{U}}(G)$ if and only if $H/\Phi(H) \leq Z_{\mathfrak{U}}(G/\Phi(H))$.

Lemma 2.9 ([10, Theorem 5.3.13]). For an odd prime p , a p -group P possesses a characteristic subgroup D of class at most 2 and of exponent p such that every nontrivial p' -automorphism of P induces a nontrivial automorphism of D .

Lemma 2.10 ([6, Lemma 2.10]). Let P be a normal p -subgroup of a group G . Let D be a characteristic subgroup of P such that every nontrivial p' -automorphism of P induces a nontrivial automorphism of D . If $D \leq Z_{\mathfrak{U}}(G)$, then $P \leq Z_{\mathfrak{U}}(G)$.

Lemma 2.11 ([10, Theorem 7.6.1]). Let P be a Sylow p -subgroup of G , where p is the smallest prime dividing $|G|$. If P is cyclic, then G is p -nilpotent.

Lemma 2.12 ([4, Theorem A]). *Let P be a noncyclic Sylow p -subgroup of G , where p is the smallest prime dividing $|G|$. Suppose that there exists a subgroup D of P with $1 < |D| = d < |P|$, and every subgroup of P of order $p^n d$ ($n = 0, 1$) is weakly $\mathcal{H}$ -embedded in G . Then G is p -nilpotent.*

Lemma 2.13 ([20, Theorem 6.3, Appendix C]). *Let P be a normal p -subgroup of G such that $|G/C_G(P)|$ is a power of p . Then $P \leq Z_\infty(G)$.*

Lemma 2.14. *Let P be a normal 2-subgroup of G . Suppose that P has a subgroup D with $1 < |D| = d < |P|$, and every subgroup of P of order $2^n d$ ($n = 0, 1$) is weakly $\mathcal{H}$ -embedded in G . Then $P \leq Z_\infty(G)$.*

PROOF. Let Q be any Sylow subgroup of G with $(2, |Q|) = 1$. Then, by Lemmas 2.11 and 2.12, PQ is 2-nilpotent, and so $PQ = P \times Q$. Hence, by Lemma 2.13, $P \leq Z_\infty(G)$. $\square$

Lemma 2.15 ([1, Lemma 2.19]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let P be a normal p -subgroup of G with $G/P \in \mathfrak{F}$. If $P \leq Z_{\mathfrak{U}}(G)$, then $G \in \mathfrak{F}$.*

Lemma 2.16 ([16, Chapter X]). *Let G be a group. Then:*

- (1) *If $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*
- (2) *$C_G(F^*(G)) \leq F(G)$.*

Lemma 2.17 ([20, Theorem 7.15, Chapter 1]). *Let H be a normal subgroup of G and $H \leq Z_{\mathfrak{U}}(G)$. Then $G/C_G(H)$ is supersolvable.*

3. Proofs

PROOF OF THEOREM 1.1. Suppose that the theorem is false, and let G be a counterexample of minimal order. Then

- (1) $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) \neq 1$. Then, by Lemmas 2.1(3) and 2.2, $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. Hence, by the minimal choice of G , $G/O_{p'}(G)$ is p -nilpotent, and so G is p -nilpotent, a contradiction.

- (2) If K is a proper subgroup of G and $P_1 \leq K$, where P_1 is a Sylow p -subgroup of K such that $P_1 \leq P$ and $|P_1| > d$, then K is p -nilpotent.

By Lemma 2.1(1), every subgroup H of P_1 with $|H| = d$ is weakly $\mathcal{H}$ -embedded in K . Since $N_K(H) \leq N_G(H)$, it follows that $N_K(H)$ is p -nilpotent. Then K satisfies the hypothesis of the theorem. The minimal choice of G implies that K is p -nilpotent.

(3) If $p > 2$, then $O_p(G) \neq 1$.

Suppose that $O_p(G) = 1$. Then $P \leq N_G(Z(J(P))) < G$. By (2), $N_G(Z(J(P)))$ is p -nilpotent. Hence, by Lemma 2.3, G is p -nilpotent, a contradiction.

(4) If $p = 2$ and $O_2(G) = 1$, then $|P| > 2d$.

Suppose that $|P| \leq 2d$. By the hypothesis of the theorem, $|P| \geq 2d$. Then $|P| = 2d$. By (2), $N_G(P)$ is 2-nilpotent. Hence, by Lemma 2.4, G is 2-nilpotent, a contradiction.

(5) If $p = 2$, then $O_2(G) \neq 1$.

Suppose that $O_2(G) = 1$. Let H be a subgroup of P of order d . Then, by the hypothesis of the theorem, H is weakly $\mathcal{H}$ -embedded in G , and so G has a normal subgroup K such that $H^G = HK$ and $H \cap K \in \mathcal{H}(G)$. As $O_2(G) = 1$, we have $K \neq 1$. Suppose that $K \neq G$. If $PK < G$, then, by (2), PK is 2-nilpotent, and so K is 2-nilpotent. By (1), K is a normal 2-group, a contradiction. Hence $G = PK$, and so $P \not\leq K$. As G is not 2-nilpotent, we have $P \cap K \neq 1$. Let P_1 be a maximal subgroup of P such that $P \cap K \leq P_1$. Clearly, $P_1K < G$. By (4), $|P_1| > d$. Then, by (2), P_1K is 2-nilpotent, and so K is 2-nilpotent. Hence, by (1), K is a normal 2-group, a contradiction. Thus $K = G$, so $H^G = K = G$ and $H \in \mathcal{H}(G)$. By the hypothesis of the theorem, $N_G(H)$ is 2-nilpotent, and so $N_G(H)/C_G(H)$ is a 2-group. Hence, by Lemma 2.5, H is a Sylow 2-subgroup of $H^G = G$, a contradiction.

(6) $O_p(G) \neq 1$, where $p \geq 2$.

It follows from (3) and (5).

(7) Let N be a minimal normal subgroup of G such that $N \leq O_p(G)$. Then $|N| < d$, G/N is p -nilpotent and G is p -solvable.

Suppose that $|N| \geq d$. If $|N| = d$, then, by the hypothesis of the theorem, $N_G(N) = G$ is p -nilpotent, a contradiction. If $|N| > d$, then, by the hypothesis of the theorem, every subgroup L of N with $|L| = d$ is weakly $\mathcal{H}$ -embedded in G . Then, by Lemma 2.7, G has a normal subgroup K such that $L^G = LK$ and $L \cap K$ is normal in G . As N is a minimal normal subgroup of G , we have $N = L^G = LK$ and $L \cap K = 1$. Then K is a normal subgroup of G such that $1 < K < N$, a contradiction. Thus $|N| < d$. By Lemma 2.1(2), every subgroup L/N with $|L| = d$ is weakly $\mathcal{H}$ -embedded in G/N . By Lemma 2.2, $N_{G/N}(L/N) = N_G(L)/N$ is p -nilpotent. Then G/N satisfies the hypothesis of the theorem. Hence, by the minimal choice of G , G/N is p -nilpotent, and so G is p -solvable.

(8) G has a unique minimal normal subgroup N and $\Phi(G) = 1$.

See the argument of Part (6) used in the proof of [17, Theorem 3.1].

(9) $N = O_p(G) = C_G(N)$.

See the argument of Part (7) used in the proof of [17, Theorem 3.1].

(10) $|P| > pd$.

Suppose that $|P| \leq pd$. By the hypothesis of the theorem, $|P| \geq pd$. Then $|P| = pd$. By (9), P is not normal in G . By (2), $N_G(P)$ is p -nilpotent. Then, by Lemma 2.4, G is p -nilpotent, a contradiction.

(11) The final contradiction.

By (6), (9) and (7), $G/N = G/O_p(G)$ is p -nilpotent and $|N| < d$. Then G has a normal subgroup M such that $|G/M| = p$. By (10), $|P \cap M| > d$, where $P \cap M$ is a Sylow p -subgroup of M . Then, by (2), M is p -nilpotent, and so G is p -nilpotent, a contradiction. $\square$

Corollary 3.1. *Let p be a prime dividing the order of G , and P a Sylow p -subgroup of G . Suppose that there exists a subgroup D of P with $1 < |D| = d < |P|$ such that every subgroup H of P of order d is c -normal in G and $N_G(H)$ is p -nilpotent. Then G is p -nilpotent.*

Corollary 3.2. *Let p be a prime dividing the order of G and P a Sylow p -subgroup of G . Suppose that there exists a subgroup D of P with $1 < |D| = d < |P|$ such that every subgroup H of P of order d belongs to $\mathcal{H}(G)$ and $N_G(H)$ is p -nilpotent. Then G is p -nilpotent.*

We now prove:

Theorem 3.3. *Let P be a nontrivial normal p -subgroup of G . If every maximal subgroup of P is weakly $\mathcal{H}$ -embedded in G , then $P \leq Z_{\mathcal{U}}(G)$.*

PROOF. Suppose that the theorem is false and consider a counterexample (G, P) for which $|G| + |P|$ is minimal. Then

(1) $\Phi(P) = 1$.

Suppose that $\Phi(P) \neq 1$. Then, by Lemmas 2.1(2), $(G/\Phi(P), P/\Phi(P))$ satisfies the hypothesis of the theorem. Hence, by the minimal choice of (G, P) , the theorem is true for $(G/\Phi(P), P/\Phi(P))$, and so $P/\Phi(P) \leq Z_{\mathcal{U}}(G/\Phi(P))$. Applying Lemma 2.8, $P \leq Z_{\mathcal{U}}(G)$, a contradiction.

(2) Let N be a minimal normal subgroup of G with $N \leq P$. Then $N \neq P$.

Suppose that $N = P$. Let L be a maximal subgroup of $N = P$. Then, by the hypothesis of the theorem, L is weakly $\mathcal{H}$ -embedded in G . Hence, by Lemma 2.7,

G has a normal subgroup K such that $L^G = LK$ and $L \cap K$ is normal in G . Since L is not normal in G , it follows that $K \neq 1$. As $N = P$ is a minimal normal subgroup of G , we have $N = K$. Then $L \cap K = L \cap P = L$ is normal in G , a contradiction.

(3) Let H be a normal subgroup of G contained in P . If $|H| = p$, then $P \leq Z_{\mathfrak{U}}(G)$.

By Lemma 2.1(2), $(G/H, P/H)$ satisfies the hypothesis of the theorem. Then, by the minimal choice of (G, P) , the theorem is true for $(G/H, P/H)$, and so $P/H \leq Z_{\mathfrak{U}}(G/H)$. Hence $P \leq Z_{\mathfrak{U}}(G)$.

(4) Let N be a minimal normal subgroup of G with $N \leq P$. Then there exists a maximal subgroup L of P such that L is not normal in G and $N \not\leq L$.

By (2), $N \neq P$. By (1), $\Phi(P) = 1$. Then there exists a maximal subgroup L of P such that $N \not\leq L$, and so $P = NL$. Suppose that L is normal in G . Then $L \cap N = 1$, and so $|N| = p$. By (3), $P \leq Z_{\mathfrak{U}}(G)$, a contradiction.

(5) The final contradiction.

By (4), P possesses a minimal normal subgroup N of G , and there exists a maximal subgroup L of P such that L is not normal in G and $N \not\leq L$. By the hypothesis of the theorem, L is weakly $\mathcal{H}$ -embedded in G . By Lemma 2.7, G has a normal subgroup K such that $L^G = LK$ and $L \cap K$ is normal in G . As L is not normal in G , we have $L < L^G$, and so $L^G = P$. If $L \cap K = 1$, then $|K| = p$, and so, by (3), $P \leq Z_{\mathfrak{U}}(G)$, a contradiction. Thus $L \cap K \neq 1$. As $N \not\leq L$, we have $N \cap (L \cap K) = 1$. By Lemma 2.1(2), $(G/(L \cap K), P/(L \cap K))$ satisfies the hypothesis of the theorem. Hence, by the minimal choice of (G, P) , the theorem is true for $(G/(L \cap K), P/(L \cap K))$, and so $P/(L \cap K) \leq Z_{\mathfrak{U}}(G/(L \cap K))$. Since $N(L \cap K)/(L \cap K) \leq P/(L \cap K) \leq Z_{\mathfrak{U}}(G/(L \cap K))$, it follows easily that $N(L \cap K)/(L \cap K)$ is of order p , and so $|N| = p$. Then, by (3), $P \leq Z_{\mathfrak{U}}(G)$, a contradiction. $\square$

We need the following lemma:

Lemma 3.4. *Let P be a normal p -subgroup of G of exponent p . If every subgroup of P of order p is weakly $\mathcal{H}$ -embedded in G , then $P \leq Z_{\mathfrak{U}}(G)$.*

PROOF. Suppose that the lemma is false and consider a counterexample (G, P) for which $|G| + |P|$ is minimal. Then $P \not\leq Z_{\mathfrak{U}}(G)$, and so P contains a subgroup H of order p such that $H \not\leq Z_{\mathfrak{U}}(G)$. Then H is not normal in G . By the hypothesis of the lemma, H is weakly $\mathcal{H}$ -embedded in G . Hence, by Lemma 2.7, G has a normal subgroup K such that $H^G = HK$ and $H \cap K = 1$. Clearly, $H^G \leq P$. By Lemma 2.1(1), (G, K) , satisfies the hypothesis of the lemma. Then,

by the minimal choice of (G, P) , the lemma is true for (G, K) , and so $K \leq Z_{\mathfrak{U}}(G)$. Hence $H^G \leq Z_{\mathfrak{U}}(G)$, and so $H \leq Z_{\mathfrak{U}}(G)$, a contradiction. $\square$

We now prove the following two results:

Lemma 3.5. *Let P be a normal p -subgroup of G , where $p > 2$. If every subgroup of P of order p is weakly $\mathcal{H}$ -embedded in G , then $P \leq Z_{\mathfrak{U}}(G)$.*

PROOF. By Lemma 2.9, P possesses a characteristic subgroup D of class at most 2 and of exponent p such that every nontrivial p' -automorphism of P induces a nontrivial automorphism of D . By Lemma 2.1(1), every subgroup of D of order p is weakly $\mathcal{H}$ -embedded in G . Then, by Lemma 3.4, $D \leq Z_{\mathfrak{U}}(G)$. Hence, by Lemma 2.10, $P \leq Z_{\mathfrak{U}}(G)$. $\square$

Theorem 3.6. *Let P be a normal p -subgroup of G , where $p > 2$. Suppose that there exists a subgroup D of P with $1 < |D| = d < |P|$ such that every subgroup of P of order $p^n d$ ($n = 0, 1$) is weakly $\mathcal{H}$ -embedded in G . Then $P \leq Z_{\mathfrak{U}}(G)$.*

PROOF. Suppose that the theorem is false and consider a counterexample (G, P) for which $|G| + |P|$ is minimal. Then

(1) $d > p$.

Suppose that $d = p$. Then, by Lemma 3.5, $P \leq Z_{\mathfrak{U}}(G)$, a contradiction.

(2) $|P| > pd$.

Suppose that $|P| \leq pd$. But, by the hypothesis of the theorem, $|P| \geq pd$. Then $|P| = pd$. Hence, by the hypothesis of the theorem, every maximal subgroup of P is weakly $\mathcal{H}$ -embedded in G . Applying Theorem 3.3, $P \leq Z_{\mathfrak{U}}(G)$, a contradiction.

(3) Let N be a minimal normal subgroup of G contained in P . Then $N \neq P$ and $|N| > p$.

Suppose that $N = P$. By (2), $|P| > pd$. Then P contains a proper subgroup L with $|L| = d$. By the hypothesis of the theorem, L is weakly $\mathcal{H}$ -embedded in G . Then, by Lemma 2.7, G has a normal subgroup K such that $L^G = LK$ and $L \cap K$ is normal in G . Clearly, $L^G \leq P$. Since $N = P$ is a minimal normal subgroup of G , it follows that $L^G = P = N$ and $L \cap K = 1$. Then K is a nontrivial normal subgroup of G with $K < P$, a contradiction. Thus $N \neq P$. Suppose that $|N| = p$. Then, by (1), $d > p = |N|$. By Lemma 2.1(2), $(G/N, P/N)$ satisfies the hypothesis of the theorem. Then, by the minimal choice of (G, P) , the theorem is true for $(G/N, P/N)$, and so $P/N \leq Z_{\mathfrak{U}}(G/N)$. Hence $P \leq Z_{\mathfrak{U}}(G)$, a contradiction. Thus $|N| > p$.

(4) The final contradiction.

Let N be a minimal normal subgroup of G contained in P . Then, by (3), $N < P$ and $|N| > p$. Suppose that $|N| \geq pd$. By Lemma 2.1(1), (G, N) satisfies the hypothesis of the theorem. Then, by the minimal choice of (G, P) , the theorem is true for (G, N) . Hence $N \leq Z_{\mathfrak{U}}(G)$, and so $|N| = p$, a contradiction. Suppose now that $|N| \leq d$. If $|N| < d$, then, by Lemma 2.1(2), $(G/N, P/N)$ satisfies the hypothesis of the theorem, and so, by the minimal choice of (G, P) , the theorem is true for $(G/N, P/N)$, and so $P/N \leq Z_{\mathfrak{U}}(G/N)$. Hence P/N contains a maximal subgroup, say, L/N such that L/N is normal in G/N . By (2), $|L| > d$. Obviously, (G, L) satisfies the hypothesis of the theorem. Then, by the minimal choice of (G, P) , the theorem is true for (G, L) , and so $L \leq Z_{\mathfrak{U}}(G)$. Since $N \leq L \leq Z_{\mathfrak{U}}(G)$, it follows that $|N| = p$, a contradiction. Thus $|N| = d$, and so, by Lemmas 2.1(2) and 3.5, $P/N \leq Z_{\mathfrak{U}}(G/N)$. Let L/N be a minimal normal subgroup of G/N with $L/N \leq P/N$. Since $P/N \leq Z_{\mathfrak{U}}(G/N)$, it follows that $|L/N| = p$, and so $|L| = pd$. As above, $L \leq Z_{\mathfrak{U}}(G)$, and so $|N| = p$, a contradiction. $\square$

PROOF OF THEOREM 1.2. Suppose that the theorem is false and consider a counterexample (G, E) for which $|G| + |E|$ is minimal. If $E = E_p$, then, by Theorem 3.6 and Lemma 2.14, $E_p \leq Z_{\mathfrak{U}}(G)$. Hence, by Lemma 2.15, $G \in \mathfrak{F}$, a contradiction. Thus, $E_p < E$, and so by Lemmas 2.1(1) and 2.12, E possesses a Sylow tower of supersolvable type. Then E_p is normal in G , where p is the largest prime dividing $|E|$. By Lemma 2.1(3), $(G/E_p, E/E_p)$ satisfies the hypothesis of the theorem. Then, by the minimal choice of (G, E) , the theorem is true for $(G/E_p, E/E_p)$, and so $G/E_p \in \mathfrak{F}$. As above, $G \in \mathfrak{F}$, a contradiction. $\square$

PROOF OF THEOREM 1.3. By Lemmas 2.1(1) and 2.12, $F^*(E)$ possesses a Sylow tower of supersolvable type, and so, by Lemma 2.16(1), we get that $F^*(E) = F(E)$. Hence, by Theorem 3.6 and Lemma 2.14, $F(E) \leq Z_{\mathfrak{U}}(G)$, and so, by Lemma 2.17, $G/C_G(F(E)) \in \mathfrak{U}$. Since $\mathfrak{F}$ is a formation containing $\mathfrak{U}$, it follows that $G/C_E(F(E)) = G/(C_G(F(E)) \cap E) \in \mathfrak{F}$. But, by Lemma 2.16(2), $C_E(F(E)) \leq F(E)$. Then $G/F(E) \in \mathfrak{F}$, and hence, by Theorem 1.2, $G \in \mathfrak{F}$. $\square$

References

- [1] M. ASAAD, On c -supplemented subgroups of finite groups, *J. Algebra* **362** (2012), 1–11.
- [2] M. ASAAD, A. A. HELIEL and M. M. AL-SHOMRANI, On weakly $\mathcal{H}$ -subgroups of finite groups, *Comm. Algebra* **40** (2012), 3540–3550.
- [3] M. ASAAD and M. RAMADAN, On weakly $\mathcal{H}$ -embedded subgroups of finite groups, *Comm. Algebra* **44** (2016), 4564–4574.

- [4] M. ASAAD, M. RAMADAN and A. A. HELIEL, Influence of weakly $\mathcal{H}$ -embedded subgroups on the structure of finite groups, *Publ. Math. Debrecen* **91** (2017), 503–513.
- [5] M. BIANCHI, A. G. B. MAURI, M. HERZOG and L. VERARDI, On finite solvable groups in which normality is a transitive relation, *J. Group Theory* **3** (2000), 147–156.
- [6] X. CHEN, W. GUO and A. N. SKIBA, Some conditions under which a finite group belongs to a Baer-local formation, *Comm. Algebra* **42** (2014), 4188–4203.
- [7] P. CSÖRGÖ and M. HERZOG, On supersolvable groups and the nilpotator, *Comm. Algebra* **32** (2004), 609–620.
- [8] K. DOERK and T. HAWKES, Finite Soluble Groups, *Walter de Gruyter, Berlin*, 1992.
- [9] D. GOLDSCHMIDT, Strongly closed 2-subgroups of finite groups, *Ann. of Math. (2)* **102** (1975), 475–489.
- [10] D. GORENSTEIN, Finite Groups, *Harper and Row, New York – London*, 1968.
- [11] X. GUO and X. WEI, The influence of $\mathcal{H}$ -subgroups on the structure of finite groups, *J. Group Theory* **13** (2010), 267–276.
- [12] X. GUO and N. WU, On finite groups with $\mathcal{HC}$ -subgroups, *J. Algebra Appl.* **14** (2015), 1550063, 7 pp, DOI:10.1142/S0219498815500632.
- [13] W. GUO, The Theory of Classes of groups, *Kluwer Academic Publishers Group, Dordrecht; Science Press Beijing, Beijing*, 2000.
- [14] W. GUO, Structure Theory for Canonical Classes of Finite Groups, *Springer, Heidelberg*, 2015.
- [15] W. GUO, F. XIE and B. LI, Some open questions in the theory of generalized permutable subgroups, *Sci. China Ser. A* **52** (2009), 2132–2144.
- [16] B. HUPPERT and N. BLACKBURN, Finite Groups. III, *Springer-Verlag, Berlin – New York*, 1982.
- [17] C. LI and S. QIAO, On weakly $\mathcal{H}$ -subgroups and p -nilpotency of finite groups, *J. Algebra Appl.* **16** (2017), 1750042, 10 pp, DOI: 10.1142/S0219498817500426.
- [18] Y. WANG, C -normality of groups and its properties, *J. Algebra* **180** (1996), 934–965.
- [19] X. WEI and X. GUO, On $\mathcal{HC}$ -subgroups and the structure of finite groups, *Comm. Algebra* **40** (2012), 3245–3256.
- [20] M. WEINSTEIN (ED.), Between Nilpotent and Solvable, *Polygonal Publishing House, Washington, NJ*, 1982.
- [21] X. ZHONG, J. LU and Y. LI, Finite groups all of whose second maximal subgroups are $\mathcal{H}_p$ -groups, *Monatsh. Math.* **172** (2013), 477–486.

MOHAMED ASAAD
 DEPARTMENT OF MATHEMATICS
 FACULTY OF SCIENCE
 CAIRO UNIVERSITY
 GIZA, 12613
 EGYPT

E-mail: moasmo45@hotmail.com

(Received February 10, 2018; revised May 18, 2018)