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Finite groups in which the cores of every non-normal subgroups are trivial

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Abstract. In this paper, we characterize the finite groups G in which $\bigcap_{g\in G} H^g = 1$ or H for every subgroup H in G.

1. Introduction

All groups considered in this paper are finite.

Recall that for a subgroup H of a group G, the subgroup $H_G = \bigcap_{g\in G} H^g$ is called the core of H. A group is called a Dedekind group if its every subgroup is normal. And a subgroup H of a group G is called a TI-subgroup if $H \cap H^x = 1$ or H for all $x \in G$.

If a subgroup H of a group G satisfies $H_G = 1$ or H, then we call it a CT-subgroup for convenience. It is obvious that every subgroup of a Dedekind group is a CT-subgroup.

For a group G, its TI-subgroup is a CT-subgroup. But a CT-subgroup may not be a TI-subgroup.

Example A. Let $G = S_{\{1,2,3,4\}}$ and $\langle(12),(123)\rangle = M \leq G$. Then M is a CT-subgroup. Since $M \cap M^{(14)} = \{(2,3),1\}$, we see M is not a TI-subgroup.

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Example B. Let $G = \mathrm{Aut}(A_6) = P\Gamma L(2,9)$. For any subgroup H of G, H is a CT-subgroup. But there exists a subgroup which is not a TI-subgroup.

In fact, G has six normal subgroups which are isomorphic to $1, A_6, S_6$, M_{10}, $PGL(2,9), G$. $\mathrm{Soc}(G) \cong A_6$ and $G/\mathrm{Soc}(G) \cong C_2 \times C_2$, and then the number of maximal subgroups containing A_6 is 3. So the subgroups containing A_6 are only $S_6, M_{10}, PGL(2,9)$ and G. Therefore, for any subgroup $H \leq G$, if $A_6 \leq H$, then $H \trianglelefteq G$, which implies that $H_G = H$. If $A_6 \nleq H$, then $H_G = 1$. So every subgroup of G is a CT-subgroup. And there exists a subgroup $M \cong S_5$ which is not a TI-subgroup.

A topic of some interest is the classification of groups, in which certain subgroups are assumed to be TI-subgroups. In [7], Walls described the groups all of whose subgroups are TI-subgroups. In [5], Li classified the non-nilpotent groups whose second maximal subgroups are TI-subgroups. In [3] and [6], Guo, Li and Flavell classified the groups whose abelian subgroups are TI-subgroups.

What we are interested in is the structure of groups G in which some subgroups are CT-subgroups. For convenience, we give two notations:

(*) for every subgroup H in G, H is a CT-subgroup.

(**) for every abelian subgroup H in G, H is a CT-subgroup.

In this paper, we have the following results:

Theorem 1. *Let G be a finite group. If for every subgroup H in G, $H_G = \bigcap_{g\in G} H^g = 1$ or H, then one of the following conditions is true:*

(1) *G is a Dedekind group.*

(2) *G is of prime power order, $Z(G)$ is cyclic, and G' is of prime order when $Z(G) \neq 1$.*

(3) *G is a primitive group when $Z(G) = 1$.*

The terminology and notation in this paper are standard. If G is a finite p-group, then $\Omega_1(G) = \langle g \in G | g^p = 1\rangle$.

2. Preliminaries

In this section, we give some basic facts, which are useful for the later use.

Definition 2 ([2]). A finite group G is called primitive if it has a maximal subgroup M such that $M_G = 1$.

Lemma 3 ([1]). *Let G be a finite group. Then G is Dedekindian if and only if G is abelian or $G \cong Q_8 \times A \times B$, where A is an elementary abelian 2-group, B is abelian, and B is of odd order.*

Lemma 4 ([4]). *Assume that π'-group H acts on an abelian π-group G. Then $G = C_G(H) \times [G, H]$.*

Lemma 5. *Let G be a non-Dedekind group. If G satisfies (*), then G has exactly one minimal normal subgroup.*

Proof. For any minimal subgroup N of G, we see G/N is a Dedekind group by the hypothesis. If there are two different minimal normal subgroups N_1 and N_2, then $N_1 \bigcap N_2 = 1$. Hence

$$G \cong G/(N_1 \bigcap N_2) \lesssim (G/N_1 \times G/N_2) \lesssim Q_8 \times Q_8 \times C_2^k \times A = T \times A = L.$$

For any subgroup $H \ntrianglelefteq G$, then $H \ntrianglelefteq L$. Assume that $H = H_1 \times H_2$, where $H_1 \leq T, H_2 \leq A$. It is easy to see that $H_1 \ntrianglelefteq T$. Since $\Omega_1(T) = Z(T)$, there exists an element $x \in H_1 \leq H$ such that $o(x) = 4$. Therefore, $x^2 \in Z(T), \langle x^2 \rangle \trianglelefteq L$. Then $\langle x^2 \rangle \trianglelefteq G$, which implies that $H_G \neq 1$, a contradiction. $\square$

3. Main results

Firstly, we characterize finite groups with non-trivial center which satisfy condition (**).

Lemma 6. *Let G be a non-Dedekind group. If G satisfies (**) and $Z(G) \neq 1$, then G is of prime power order, $Z(G)$ is cyclic, and G' is of prime order.*

Proof. We claim that $Z(G)$ has exactly one minimal subgroup. If not, then there exist two distinct minimal subgroups N, M in $Z(G)$. For any element $a \in G$, we have $\langle a \rangle \cap Z(G) \leq \langle a \rangle_G$. Since G satisfies (**), $\langle a \rangle_G = 1$ or $\langle a \rangle$. If $\langle a \rangle \cap Z(G) > 1$, then $\langle a \rangle_G = \langle a \rangle$, and so $\langle a \rangle \trianglelefteq G$. If $\langle a \rangle \cap Z(G) = 1$, then $\langle a \rangle N \trianglelefteq G$ by $N \leq (\langle a \rangle N)_G$. Similarly, we see $\langle a \rangle M \trianglelefteq G$. Then $\langle a \rangle = \langle a \rangle N \cap \langle a \rangle M \trianglelefteq G$. Thus G is a Dedekind group, a contradiction. Hence $Z(G)$ has exactly one minimal subgroup, and then $Z(G)$ is cyclic.

Assume that N is the unique minimal subgroup of $Z(G)$ and $|N| = p$. For any cyclic subgroup H in G, we see $1 \neq N \leq (HN)_G$, and so $HN \trianglelefteq G$. Thus G/N is a Dedekind group.

If G/N is non-abelian, then

$$G/N = \bar{A} \times \langle \bar{a}, \bar{b} | \bar{a}^4 = 1, \bar{a}^2 = \bar{b}^2, [\bar{a}, \bar{b}] = \bar{a}^2 \rangle,$$

and $G = NA\langle a,b\rangle$, where $\bar{A}$ is an abelian group. If $p = 2$ and $a^4 = 1$, then $[a^2, b] = [a, b]^2 = a^4 = 1$ and $[a^2, x] = [a, x]^2 = 1$, for any element $x \in A$. Hence $a^2 \in Z(G)$, and then $\langle a^2\rangle = N$ by the above paragraph, a contradiction. If $p = 2$ and $a^4 \neq 1$, then, assuming that $a^2 = b^2 n$ $(n \in N)$, $a^4 = [a^2, b] = [b^2 n, b] = 1$, a contradiction. When $p > 2$, it is easy to see that $[a^{2p}, b] = a^{4p} = 1$ and $[a^{2p}, g] = [a, g]^{2p} = 1$, for any element $g \in A$. Then $a^{2p} \in Z(G)$, hence $Z(G)$ is not of prime power order, a contradiction.

So G/N is abelian. Assume that $G/N = \times_{i=1}^{k}\langle\bar{y_i}\rangle$, where $|\langle\bar{y_i}\rangle| = q_i^{m_i}$, q_i is a prime and m_i is an integer, for all $i \in \{1, 2, \ldots, k\}$. We see $[y_i^p, y_j] = 1$, for any $i, j \in \{1, 2, \ldots, k\}$ by $|N| = p$. Noting $G = \langle N, y_1, y_2, \ldots, y_k\rangle$, $y_i^p \in Z(G)$ for any $i \in \{1, 2, \ldots, k\}$. If $y_i^p \neq 1$, then $N \leq \langle y_i^p\rangle$, and so $p|q_i^{m_i}$. Thus $p = q_i$, for any $i \in \{1, 2, \ldots, k\}$. So G/N is a p-group.

Since G is a non-Dedekind group, $G' \neq 1$, and then $G' = N$ is of order p. The proof is complete. □

Lemma 7. *Let G be a p-group. If $Z(G)$ is cyclic and $|G'| = p$, then G satisfies condition (*).*

Proof. Let H be a subgroup of G. If $H_G \neq 1$, then $H_G \cap Z(G) \neq 1$, and therefore $G' \leq H_G \cap Z(G)$. Then $G' \leq H_G \leq H$ and $H \trianglelefteq G$, which implies $H_G = H$. Hence G satisfies condition (*). □

Now we obtained the following theorem:

Theorem 8. *Let G be a non-Dedekind group, and $Z(G) \neq 1$. Then the following conclusions are equivalent:*

(1) *G satisfies condition (*);*

(2) *G satisfies condition (**);*

(3) *G is of prime power order, $Z(G)$ is cyclic, and G' is of prime order.*

In fact, (*) is not equivalent to (**) when $Z(G) = 1$.

Example C. The Dihedral group $G = \langle a, b | a^2 = 1, b^9 = 1, b^a = b^{-1}\rangle$. Then G does not satisfy (*), but G satisfies (**).

It is easy to see that $Z(G) = 1$, $\mathrm{Syl}_3(G) = \langle b\rangle \trianglelefteq G$, and then G has exactly one subgroup of order 3, which is $\langle b^3\rangle$. The subgroup of order 2 is in $\{\langle ab^i\rangle | i = 0, 1, 2, 3, 4, 5, 6, 7, 8\}$. For any subgroup H of order 6, we see that $\langle b^3\rangle \leq H$, and there exists $i \in \{0, \ldots, 8\}$ such that $\langle ab^i\rangle \leq H$. Since $[ab^i, b^3] \neq 1$, H is non-abelian.

For any abelian subgroup $K \leq G$, then $|K| = 1, 2, 3$ or 9. If $|K| = 1, 3, 9$, then $K \trianglelefteq G$, $K_G = K$. If $|K| = 2$, then $K_G = 1$. So G satisfies (**).

Considering subgroup $H = \langle a, b^3\rangle$, $H \ntrianglelefteq G$, by $[b,a] = b^2$, and $H_G = \langle b^3\rangle$. Hence G does not satisfies (*).

Secondly, we study finite groups G with trivial center which satisfy condition (*).

Theorem 9. *Let G be a finite non-Dedekind group, and $Z(G) = 1$. Then G satisfies condition (*) if and only if G is a primitive group,* $\mathrm{Soc}(G)$ *is a minimal normal subgroup, and $G/\mathrm{Soc}(G)$ is Dedekindian.*

Proof. Suppose that G satisfies condition (*). By Lemma 5, we see that $\mathrm{Soc}(G)$ is the unique minimal normal subgroup, and $G/\mathrm{Soc}(G)$ is Dedekindian by condition (*).

If $\mathrm{Soc}(G)$ is abelian, then we may assume that the unique normal subgroup $\mathrm{Soc}(G) = N \cong C_p^k$. Then there exists a subgroup $P \in \mathrm{Syl}_p(G)$ such that $N \leq P$. Hence $P \trianglelefteq G$ by (*). Let $G = P \rtimes H$. If $H \trianglelefteq G$, then $G = P \times H$ and $1 \neq Z(P) \leq Z(G)$, which contradicts to $Z(G) = 1$. So $H \ntrianglelefteq G$. Since $G/N = (P/N) \times (HN/N)$, $[P,H] \leq N$. By $Z(P)\,\mathrm{char}\,P \trianglelefteq G$, we see $N \leq Z(P)$. It follows that $C_N(H) = 1$ from $Z(G) = 1$. Using Lemma 4, $N = C_N(H)[N,H] = [N,H]$. Thus $[P,H] = N$. Using Lemma 4 again, we see that $P = C_P(H)[P,H] = C_P(H)N$. Since N is the unique minimal normal subgroup, we see that $N \leq Z(P)$ and $[C_P(H), N] = 1$. Hence $C_P(H) \trianglelefteq P$. Noting that $G = PH$, we have $C_P(H) \trianglelefteq G$. Therefore, $N \leq C_P(H)$ or $C_P(H) = 1$. If $N \leq C_P(H)$, then $N \leq Z(G)$, a contradiction. So $C_P(H) = 1$. Then $P = C_P(H)[P,H] = [P,H] = N$. So $N \nleq \Phi(G)$, $\Phi(G) = 1$, and therefore there exists a maximal subgroup M of G such that $M_G = 1$ by (*). Thus G is a primitive group. If $\mathrm{Soc}(G)$ is not abelian, then $\Phi(G) = 1$, because $\Phi(G)$ is nilpotent. So G is a primitive group.

Conversely, if $H \leq G$ and $H_G \neq 1$, then $\mathrm{Soc}(G) \leq H$. Since $G/\mathrm{Soc}(G)$ is Dedekindian, we see that $H \trianglelefteq G$, $H_G = H$. So G satisfies condition (*). The proof is complete. □

Since a soluble primitive group has exactly one minimal normal subgroup, we get the following result.

Corollary 10. *Let G be a finite non-Dedekind soluble group, and $Z(G) = 1$. Then G satisfies condition (*) if and only if G is a primitive group and $G/\mathrm{Soc}(G)$ is Dedekindian.*

However, for a non-soluble group G, the above conclusion does not hold.

Example D. $G = D \times D$, where D is a non-abelian simple group.

It is easy to see that G is primitive and $G/\mathrm{Soc}(G) = 1$. But G does not satisfy condition (*).

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