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A note on Carleson embedding theorems

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Abstract. In this note, the dyadic Carleson embedding theorem in the multi-parameter setting will be presented. In the case of one parameter, a new simple dyadic proof will be given by using the technique of principal cubes.

1. Introduction

It is well known that Carleson embedding theorems are very useful in harmonic analysis and other fields. First, they can be used to obtain some sharp weighted estimates for Calderón–Zygmund type operators, such as in [HP] and [HPTV]. Secondly, in the multi-parameter theory, they can be used effectively to avoid the complicated geometric structure arguments. For example, in [CXY], the authors presented a rather elementary proof which was not involving the A_∞ condition and Jawerth's methods of redistributing rectangles. Third, they may provide a great convenience when dealing with the connection between the behavior of singular integrals and the geometry of sets and measures (see [MT], [NTV-2]). In this note, we investigate the Carleson embedding theorem associated with rectangles. Our result not only extends the classical result in [HP], [T] to the multi-parameter setting, but also generalizes the weighted version of the Carleson embedding theorem presented in [CXY].

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In order to state our main result, let us first introduce some notations. Throughout this note, $\mathcal{DR}$ and $\mathcal{DQ}$ will be used to denote the following families of all dyadic rectangles and all dyadic cubes in $\mathbb{R}^n$ with sides parallel to the axes:

$$\mathcal{DR}(\mathbb{R}^n) := \left\{2^{-k}(m+[0,1)) : \, k, m \in \mathbb{Z}\right\}^n,$$
$$\mathcal{DQ}(\mathbb{R}^n) := \left\{2^{-k}(m+[0,1)^n) : \, k \in \mathbb{Z}, \, m \in \mathbb{Z}^n\right\}.$$

We denote by P_j the projection on the x_j-axis for $j = 1, \dots, n$. For $R \in \mathcal{DR}(\mathbb{R}^n)$, $I \in \mathcal{DR}(\mathbb{R})$, we define the dyadic rectangle $R_{j,I}$ in the way that

$$R_{j,I} := \left(\prod_{i=1}^{j-1} P_i(R)\right) \times I \times \left(\prod_{i=j+1}^{n} P_i(R)\right), \quad \text{for } j = 1, \dots, n.$$

Our main result in this paper is as follows:

Theorem 1.1. *Let $1 < p \le q < \infty$. Let σ be a weight defined on $\mathbb{R}^n$. Then the following statements are equivalent:*

(a) *The Carleson embedding inequality for dyadic rectangles*

$$\sum_{R \in \mathcal{DR}(\mathbb{R}^n)} \sigma(R)^{q/p} \left(\frac{1}{\sigma(R)} \int_R f \, d\sigma\right)^q \le B_1 \left(\int_{\mathbb{R}^n} f^p \, d\sigma\right)^{q/p}$$

holds for all nonnegative functions $f \in L^p(\sigma)$.

(b) *The testing condition*

$$\sum_{\substack{I \in \mathcal{DQ}(\mathbb{R}) \\ I \subset P_j(R)}} \sigma(R_{j,I})^{q/p} \le B_2 \, \sigma(R)^{q/p}$$

holds for all $R \in \mathcal{DR}(\mathbb{R}^n)$ and $j = 1, \dots, n$.

Moreover, the least possible constants enjoy the property that $B_2 \le B_1 \le c_{n,p,q} B_2^n$.

2. The one-parameter case

Recall that in the one-parameter case, the Carleson embedding theorem for cubes was given in [T] by using the Bellman function method. Later on, by introducing a measure space over the set of dyadic cubes, a simplified proof was given in [HP]. In the special case $p = 2$, the optimal constant problem has been considered in [D] and [NTV-1]. The result in [D] was heavily based on a stopping time argument and the corresponding dyadic maximal inequality. By using the Bellman function, the authors in [NTV-1] gave an alternative proof in one dimension. Unfortunately, the above methods cannot be used to demonstrate the multi-parameter case. Moreover, they are considerably complicated. Now, we prepare to give an elegant and simple proof for the following Carleson embedding theorem in the one-parameter case.

Theorem 2.1. *Let $1 < p \le q < \infty$. Let σ be a weight defined on $\mathbb{R}^n$. Then the following statements are equivalent:*

(i) *The Carleson type embedding inequality for dyadic cubes*

$$\sum_{Q\in\mathcal{DQ}} \sigma(Q)^{q/p} \left(\frac{1}{\sigma(Q)}\int_Q f\,d\sigma\right)^q \le A_1 \left(\int_{\mathbb{R}^n} f^p\,d\sigma\right)^{q/p}$$

holds for all nonnegative functions $f \in L^p(\sigma)$.

(ii) *The following testing condition holds*

$$\sum_{\substack{Q\in\mathcal{DQ}\\ Q\subset Q_0}} \sigma(Q)^{q/p} \le A_2\ \sigma(Q_0)^{q/p}, \quad \forall\, Q_0 \in \mathcal{DQ}.$$

Moreover, the least possible constants satisfy $A_2 \le A_1 \le (2^{1+1/p}p')^q A_2$.

Proof. It suffices to show that (ii) implies (i), since the reverse direction is trivial.

Let Q_0 be a fixed dyadic cube. For the pair (f, σ), the collection of principal cubes $\mathcal{F}$ will be defined by

$$\mathcal{F} := \bigcup_{k=0}^{\infty} \mathcal{F}_k, \qquad \mathcal{F}_0 := \{Q_0\}, \qquad \mathcal{F}_{k+1} := \bigcup_{F\in\mathcal{F}_k} \mathrm{ch}_{\mathcal{F}}(F),$$

where $\mathrm{ch}_{\mathcal{F}}(F)$ is defined to be the set of all maximal dyadic cubes $Q \subset F$ such that

$$\frac{1}{\sigma(Q)}\int_Q f\,d\sigma > \frac{2}{\sigma(F)}\int_F f\,d\sigma.$$

The following observation

$$\begin{aligned}\sum_{F'\in\mathrm{ch}_{\mathcal{F}}(F)} \sigma(F') &\le \left(\frac{2}{\sigma(F)}\int_F f\,d\sigma\right)^{-1} \sum_{F'\in\mathrm{ch}_{\mathcal{F}}(F)} \int_{F'} f\,d\sigma \\ &\le \left(\frac{2}{\sigma(F)}\int_F f\,d\sigma\right)^{-1} \int_F f\,d\sigma = \frac{\sigma(F)}{2}\end{aligned}$$

yields that

$$\sigma(E_{\mathcal{F}}(F)) := \sigma\left(F \setminus \bigcup_{F' \in \mathrm{ch}_{\mathcal{F}}(F)} F'\right) \geq \frac{\sigma(F)}{2}, \tag{2.1}$$

where the sets in the collection $\{E_{\mathcal{F}}(F) : F \in \mathcal{F}\}$ are pairwise disjoint.

For $Q \in \mathcal{DQ}$, we further define the stopping parent $\pi_{\mathcal{F}}(Q)$ by

$$\pi_{\mathcal{F}}(Q) := \min\{F \supset Q : F \in \mathcal{F}\}.$$

Consequently, we can rewrite

$$\begin{aligned}
&\sum_{Q \subset Q_0} \sigma(Q)^{q/p} \left(\frac{1}{\sigma(Q)} \int_Q f\, d\sigma\right)^q \\
&\quad = \sum_{F \in \mathcal{F}} \sum_{Q:\, \pi_{\mathcal{F}}(Q) = F} \sigma(Q)^{q/p} \left(\frac{1}{\sigma(Q)} \int_Q f\, d\sigma\right)^q \\
&\quad \leq \sum_{F \in \mathcal{F}} \left(\frac{2}{\sigma(F)} \int_F f\, d\sigma\right)^q \sum_{Q:\, \pi_{\mathcal{F}}(Q) = F} \sigma(Q)^{q/p} \\
&\quad \leq 2^q A_2 \sum_{F \in \mathcal{F}} \left(\frac{1}{\sigma(F)} \int_F f\, d\sigma\right)^q \sigma(F)^{q/p},
\end{aligned}$$

where we have used the testing condition (ii).

For $0 < p \leq q < \infty$, it holds that $\|\cdot\|_{\ell^p} \geq \|\cdot\|_{\ell^q}$. Then, this fact, together with (2.1), enables us to obtain that

$$\begin{aligned}
&\sum_{Q \subset Q_0} \sigma(Q)^{q/p} \left(\frac{1}{\sigma(Q)} \int_Q f\, d\sigma\right)^q \\
&\leq 2^q A_2 \Bigg\{ \sum_{F \in \mathcal{F}} \left(\frac{1}{\sigma(F)} \int_F f\, d\sigma\right)^p \sigma(F) \Bigg\}^{q/p} \\
&\leq 2^{q+q/p} A_2 \Bigg\{ \sum_{F \in \mathcal{F}} \left(\frac{1}{\sigma(F)} \int_F f\, d\sigma\right)^p \sigma(E_{\mathcal{F}}(F)) \Bigg\}^{q/p} \\
&\leq 2^{q+q/p} A_2 \left(\int_{Q_0} M^{\sigma}_{\mathcal{DQ}}(f 1_{Q_0})^p\, d\sigma\right)^{q/p} \leq (2^{1+1/p} p')^q A_2 \left(\int_{Q_0} f^p\, d\sigma\right)^{q/p}, \quad (2.2)
\end{aligned}$$

where $M^{\sigma}_{\mathcal{DQ}}$ is the following dyadic Hardy–Littlewood maximal operator with respect to the measure $d\sigma$ defined by

$$M^{\sigma}_{\mathcal{DQ}} f(x) \equiv \sup_{x \in Q \in \mathcal{DQ}} \frac{1}{\sigma(Q)} \int_Q |f| d\sigma,$$

and we have used its boundedness: $\|M^{\sigma}_{\mathcal{DQ}}\|_{L^p(\sigma) \to L^p(\sigma)} \leq p'$ in the last inequality of (2.2). This completes the proof of Theorem 2.1. □

3. The multi-parameter case

Let us demonstrate Theorem 1.1. The necessity is obvious, so we only need to show the sufficiency. For simplicity, we shall only prove Theorem 1.1 for $n = 2$, since this case contains all the essential difficulties of the general situation.

From now on, we assume that the weight σ in $\mathbb{R}^2$ satisfies the testing condition (b). For convenience, for any $x_1 \in \mathbb{R}$ and $I_2 \in \mathcal{DR}(\mathbb{R})$, we set

$$\mu_{x_1}(\cdot) := \sigma(x_1, \cdot) \quad \text{and} \quad \mu_{I_2}(\cdot) := \int_{I_2} \sigma(\cdot, x_2)\, dx_2.$$

We begin with giving two observations. First, we will verify that μ_{I_2} satisfies the testing condition (ii) in Theorem 2.1. Indeed, for any $I_1 \in \mathcal{DR}(\mathbb{R})$, setting $R = I_1 \times I_2$, it holds that

$$\sum_{J \subset I_1} \mu_{I_2}(J)^{q/p} = \sum_{J \subset P_1(R)} \sigma(R_{1,J})^{q/p} \le B_2 \sigma(R)^{q/p} = B_2 \mu_{I_2}(I_1)^{q/p}.$$

Secondly, we will verify that $\mu_{x_1}(\cdot)$ also satisfies (ii) for a.e. $x_1 \in \mathbb{R}$. We need to show that the following inequality holds

$$\sum_{J \subset I_2} \mu_{x_1}(J)^{q/p} \le B_2 \mu_{x_1}(I_2)^{q/p}, \quad \forall\, I_2 \in \mathcal{DR}(\mathbb{R}). \tag{3.1}$$

For $I_1 \in \mathcal{DR}(\mathbb{R})$, it follows by setting $R = I_1 \times I_2$ that

$$\sum_{J \subset P_2(R)} \sigma(R_{2,J})^{q/p} \le B_2 \sigma(R)^{q/p}. \tag{3.2}$$

Dividing both sides of (3.2) by $|I_1|^{q/p}$ yields that

$$\sum_{J \subset P_2(R)} \left(\frac{1}{|I_1|} \int_{I_1 \times J} \sigma(t, x_2)\, dx_2\, dt \right)^{q/p} \le B_2 \left(\frac{1}{|I_1|} \int_{I_1 \times I_2} \sigma(t, x_2)\, dx_2\, dt \right)^{q/p}. \tag{3.3}$$

On the both sides of inequality (3.3), considering the Lebesgue point x_1 with respect to the integrals over t, which exists a.e. in $\mathbb{R}$ and shrinking I_1 to x_1, we obtain

$$\sum_{J \subset I_2} \left(\int_J \sigma(x_1, x_2)\, dx_2 \right)^{q/p} \le B_2 \left(\int_{I_2} \sigma(x_1, x_2)\, dx_2 \right)^{q/p},$$

which means that (3.1) is true.

Now, we are in the position to prove Theorem 1.1. Fix a nonnegative function $f \in L^p(\sigma)$. We shall evaluate

$$\mathscr{S} := \sum_{I_1 \in \mathcal{D}(\mathbb{R})} \sum_{I_2 \in \mathcal{D}(\mathbb{R})} \sigma(R)^{q/p} \left(\frac{1}{\sigma(R)} \int_R f \, d\sigma \right)^q,$$

where $R = I_1 \times I_2$. We rewrite

$$\begin{aligned} \mathscr{S} = & \sum_{I_2 \in \mathcal{D}(\mathbb{R})} \sum_{I_1 \in \mathcal{D}(\mathbb{R})} \mu_{I_2}(I_1)^{\frac{q}{p}} \\ & \times \left(\frac{1}{\mu_{I_2}(I_1)} \int_{I_1} \left(\int_{I_2} f(x_1, x_2) \sigma(x_1, x_2) \mu_{I_2}(x_1)^{-1} \, dx_2 \right) \mu_{I_2}(x_1) \, dx_1 \right)^q. \end{aligned}$$

Since μ_{I_2} satisfies the testing condition (ii), it follows from Theorem 2.1 that

$$\begin{aligned} \mathscr{S} &\le A_1 \sum_{I_2 \in \mathcal{D}(\mathbb{R})} \left(\int_{\mathbb{R}} \left(\int_{I_2} f(x_1, x_2) \sigma(x_1, x_2) \mu_{I_2}(x_1)^{-1} \, dx_2 \right)^p \mu_{I_2}(x_1) \, dx_1 \right)^{\frac{q}{p}} \\ &= A_1 \left[\left\{ \sum_{I_2 \in \mathcal{D}(\mathbb{R})} \left(\int_{\mathbb{R}} \left(\int_{I_2} f(x) \sigma(x) \mu_{I_2}(x_1)^{-1} \, dx_2 \right)^p \mu_{I_2}(x_1) \, dx_1 \right)^{\frac{q}{p}} \right\}^{\frac{p}{q}} \right]^{\frac{q}{p}}, \end{aligned}$$

where $B_2 \le A_1 \le (2^{1+1/p} p')^q B_2$. By the integral version of the Minkowski inequality, we obtain

$$\begin{aligned} \mathscr{S} &\le A_1 \left[\int_{\mathbb{R}} \left\{ \sum_{I_2 \in \mathcal{D}(\mathbb{R})} \left(\int_{I_2} f(x) \sigma(x) \mu_{I_2}(x_1)^{-1} \, dx_2 \right)^q \mu_{I_2}(x_1)^{\frac{q}{p}} \right\}^{\frac{p}{q}} dx_1 \right]^{\frac{q}{p}} \\ &= A_1 \left[\int_{\mathbb{R}} \left\{ \sum_{I_2 \in \mathcal{D}(\mathbb{R})} \mu_{x_1}(I_2)^{\frac{q}{p}} \left(\frac{1}{\mu_{x_1}(I_2)} \int_{I_2} f(x) \mu_{x_1}(x_2) \, dx_2 \right)^q \right\}^{\frac{p}{q}} dx_1 \right]^{\frac{q}{p}}. \end{aligned}$$

Here, we have used the fact $\mu_{x_1}(I_2) = \mu_{I_2}(x_1)$. Combining (3.1) with Theorem 2.1, we deduce that

$$\mathscr{S} \le A_1^2 \left(\int_{\mathbb{R}} \int_{\mathbb{R}} f(x_1, x_2)^p \sigma(x_1, x_2) \, dx_2 \, dx_1 \right)^{\frac{q}{p}} = A_1^2 \left(\int_{\mathbb{R}^2} f^p \, d\sigma \right)^{\frac{q}{p}}.$$

The proof of Theorem 1.1 is finished. □

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