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Orlicz spaces on hypergroups

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Abstract. For a locally compact hypergroup K and a Young function φ, we study the Orlicz space $L^{\varphi}(K)$ and provide a sufficient condition for $L^{\varphi}(K)$ to be an algebra under convolution of functions. We show that a closed subspace of $L^{\varphi}(K)$ is a left ideal if and only if it is left translation invariant. We apply the basic theory developed here to characterize the space of multipliers of the Morse–Transue space $M^{\varphi}(K)$. We also investigate the multipliers of $L^{\varphi}(\mathcal{S}, \pi_K)$, where S is the support of the Plancherel measure π_K associated to a commutative hypergroup K.

1. Introduction

Hypergroups are generalization of groups. Here we deal with hypergroups which are analogous to locally compact groups. It is needless to say that L^p-spaces on locally compact groups are central objects in harmonic analysis and have plenty of applications in mathematics and otherwise. Orlicz spaces are natural generalizations of L^p-spaces. In fact, the index p is replaced by a continuous function φ with certain properties. Orlicz spaces on locally compact groups have been studied extensively by a large number of authors. In this note, a study of Orlicz spaces on hypergroups is attempted.

A hypergroup is a locally compact space with a convolution product which maps each pair of points to a probability measure with compact support. The notion of hypergroups is a probabilistic generalization of locally compact groups

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wherein the convolution of two points corresponds to the point mass measure at their product. Hypergroups are independently created by Dunkl [7], Jewett [10] and Spector [17] with the purpose of doing standard harmonic analysis. We follow Jewett [10] for basic notation and terminology of hypergroups. For details of hypergroups, one can refer to ([3], [7], [10]–[11], [17]).

Let K be a hypergroup with the Haar measure m, and let $L^1(K)$ be its hypergroup algebra. The study of $L^1(K)$ has been extensively carried out by many researchers. The Banach space $L^p(K)$ for $1 < p < \infty$ is a Banach algebra if and only if K is a compact hypergroup.

To find more on Orlicz spaces, one can refer to [1]–[2], [9], [12] and [14]–[15]. M. M. Rao commented in [14, p. 3613] that a study of Orlicz spaces for hypergroups would be interesting. This motivates us to work on this topic.

Section 2 contains basic definitions and results related to Orlicz spaces on hypergroups in the form we need. In Section 3, we give a sufficient condition for $L^\varphi(K)$ to become a Banach algebra, and show existence of a bounded approximate identity for $L^\varphi(K)$ in L_1-norm which is used further to characterize closed left ideals of the Orlicz algebra $L^\varphi(K)$. In Section 4, as an application of the theory developed in Section 3, we study the multiplier space $C_{V_\varphi}(K)$ of $M^\varphi(K)$, the norm closure of $C_c(K)$ in $L^\varphi(K)$, and prove that it can be identified with the dual of nicely described space denoted by $\breve{A}_\varphi(K)$. In the last section, we give a characterization of multipliers of $L^\varphi(\mathcal{S}, \pi_K)$, where S is the support of the Plancherel measure π_K when the hypergroup K is commutative.

2. Preliminaries

Let K be a locally compact hypergroup with a left Haar measure m. Denote the set of all complex valued m-measurable functions on K by $L^0(K)$. A nonzero convex function $\varphi : \mathbb{R} \to [0, \infty]$ is called a *Young function* if it is even, left continuous with $\varphi(0) = 0$ and is not identically infinity. Here we note that every Young function is an integral of a non-decreasing left continuous function [13, Theorem 1]. Thus φ' is non-decreasing, and hence φ is increasing for $x \geq 0$. A Young function φ is called a *nice Young function* or *N-function* if it satisfies the following conditions:

$$\lim_{x\to 0} \frac{\varphi(x)}{x} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty.$$

For any Young function φ, and $y \in \mathbb{R}$, define,

$$\psi(y) = \sup\{x|y| - \varphi(x) : x \geq 0\}.$$

It can easily be seen that ψ is also a Young function. This is called the *complimentary Young function* to φ. Further, it turns out that φ is the complimentary Young function to ψ. The pair (φ, ψ) is called a *complimentary pair* of Young functions.

A Young function φ is Δ_2*-regular* if there exists a constant $C > 0$ and $x_0 > 0$ such that $\varphi(2x) \leq C\varphi(x)$ for all $x \geq x_0$ when K is compact, and $\varphi(2x) \leq C\varphi(x)$ for all $x \geq 0$ when K is noncompact. We write $\varphi \in \Delta_2$ if φ satisfies the Δ_2-regularity condition. Given a Young function φ, the *modular function* $\rho_\varphi : L^0(K) \to \mathbb{R}$ is defined by $\rho_\varphi(f) := \int_K \varphi(|f|)\, dm$. For a given Young function φ, the *Orlicz space* is defined by

$$L^\varphi(K) := \left\{f \in L^0(K) : \rho_\varphi(af) < \infty, \text{ for some } a > 0\right\}.$$

Then the Orlicz space is a Banach space with respect to the *Orlicz norm* $\|\cdot\|_\varphi$ defined for $f \in L^\varphi(K)$ by

$$\|f\|_\varphi = \sup\left\{\int_K |fg|\, dm : \int_K \psi(|g|) \leq 1\right\}$$

where ψ is the complimentary Young function to φ. Another norm $\|\cdot\|_\varphi^0$ on $L^\varphi(K)$ called *Luxemburg norm* is defined as

$$\|f\|_\varphi^0 := \inf\left\{r > 0 : \int_K \varphi\left(\frac{|f|}{r}\right)\, dm \leq 1\right\}.$$

It is known that these two norms are equivalent. In fact, $\|\cdot\|_\varphi^0 \leq \|\cdot\|_\varphi \leq 2\,\|\cdot\|_\varphi^0$ and $\|f\|_\varphi^0 \leq 1$ if and only if $\rho_\varphi(f) \leq 1$. If (φ, ψ) is a complementary pair of Δ_2-functions, then it is a complementary pair of N-Young functions. If (φ, ψ) is a complementary pair of N-Young functions and $\varphi \in \Delta_2$, then the dual space of $(L^\varphi(K), \|\cdot\|_\varphi)$ is $(L^\psi(K), \|\cdot\|_\psi^0)$. In fact, the duality is given by $\langle f, g\rangle = \int_K f(x)\, g(x)\, dm(x)$. Let $C_c(K)$ denote the space of continuous functions with compact support on K. The closure of $C_c(K)$ inside $L^\varphi(K)$ is denoted by $M^\varphi(K)$. If $\varphi \in \Delta_2$, then $L^\varphi(K) = M^\varphi(K)$ so that $C_c(K)$ is dense in $L^\varphi(K)$. If $f \in L^\varphi(K)$ and $g \in L^\psi(K)$ where ψ is complimentary Young function to φ, then $fg \in L^1(K)$ and the following Hölder's inequality [13, Remark 1, p. 62] holds:

$$\int_K |f(t)g(t)|\, dm(t) \leq \|f\|_\varphi^0\, \|g\|_\psi. \tag{1}$$

For more details on Orlicz spaces, see [13].

Let K be a locally compact hypergroup, and let $M(K)$ denote the corresponding (associative) measure algebra of all complex regular Borel measures on K. We call a locally compact hypergroup simply a hypergroup if no confusion arises. The involution of an element $s \in K$ is denoted by $\check{s}$. Let p_s be the unit point mass at s. If f is a Borel function on K and $x, y \in K$, the right translate f^y (also denoted by $L^y(f)$) is defined by

$$f^y(x) = L^y(f)(x) = \int_K f \, d(p_x * p_y),$$

whenever the integral exists. We shall also denote this by $f(x * y)$, although $x * y$ may not represent a point in K.

If $\mu \in M(K)$, the convolutions $\mu * f$ and $f * \mu$ are defined by

$$\mu * f(x) = \int_K f(\check{y} * x) \, d\mu(y) \quad \text{and} \quad f * \mu(x) = \int_K f(x * \check{y}) \, d\mu(y).$$

If f and g are Borel functions, their convolution $f * g$ is defined by

$$f * g(x) = \int_K f(x * y) g(\check{y}) \, dm(y),$$

whenever it makes sense. Throughout this article, K denotes a hypergroup with a fixed Haar measure m.

3. Orlicz algebra on hypergroups

In this section, we develop some basic results related to Orlicz spaces for hypergroups. We provide a sufficient condition for the Orlicz space $L^\varphi(K)$ to become a Banach algebra and characterize the closed left ideals of the Banach algebra $L^\varphi(K)$.

Lemma 3.1. *The Orlicz space $L^\varphi(K)$ is translation invariant, i.e., $f \in L^\varphi(K)$ implies $f_s \in L^\varphi(K)$ for every $s \in K$.*

Proof. If $f \in L^0(G)$, then it is clear from [10, Lemma 3.1D] that $f_s \in L^0(G)$. Now, let $f \in L^\varphi(K)$, i.e., $\rho_\varphi(af) = \int_K \varphi(a|f|) \, dm < \infty$ for some $a > 0$. Then

$$\begin{aligned}\rho_\varphi(af_s) &= \int_K \varphi\left(\left|\int_K af(z) \, d(p_s * p_t)(z)\right|\right) dm(t) \\ &\leq \int_K \varphi\left(\int_K |af(z)| \, d(p_s * p_t)(z)\right) dm(t).\end{aligned}$$

By using Jensen's inequality [13, Proposition 5, p. 62], we get

$$\begin{aligned}\rho_\varphi(af_s) &\leq \int_K \left(\int_K \varphi\left(a|f|\right)(z)\; d(p_s * p_t)(z) \right) dm(t) = \int_K \varphi\left(a|f|\right)_s(t)\, dm(t) \\ &= \int_K \varphi\left(a|f|\right)(t)\, dm(t) = \rho_\varphi(af) < \infty,\end{aligned}$$

where the penultimate equality follows from [10, Lemma 3.3 F]. □

Corollary 3.2. *Let K be a hypergroup, and let φ be a Young function such that $\varphi \in \Delta_2$. Then, for any $s \in K$ and $f \in L^\varphi(K)$, we have $\|f_s\|_\varphi \leq \|f\|_\varphi$.*

For $\varphi(x) = \frac{|x|^p}{p}$ for $1 \leq p < \infty$, Corollary 3.2 turns into [10, Lemma 3.3 B].

Lemma 3.3. *Let K be a hypergroup, and let φ be a Young function. Then $L^\varphi(K) \subset L^1(K)$ if and only if there exists $d > 0$ such that $\|f\|_1 \leq d\,\|f\|_\varphi$ for all $f \in L^\varphi(K)$.*

Proof. The "if" part is apparent. For the converse, assume that $L^\varphi(K) \subset L^1(K)$. Note that $L^\varphi(K)$ is also a Banach space with the norm $\|\cdot\| := \|\cdot\|_\varphi + \|\cdot\|_1$. The identity map $I : (L^\varphi(K), \|\cdot\|) \to (L^\varphi(K), \|\cdot\|_\varphi)$ is a continuous bijection. Therefore, by the open mapping theorem, there exists $d > 0$ such that $\|f\| \leq d\,\|f\|_\varphi$. Thus, we have $\|f\|_1 \leq d\,\|f\|_\varphi$ for all $f \in L^\varphi(K)$. □

Lemma 3.4. *Let φ be a finite Young function. If (K, m) is a finite measure space or the right derivative $\varphi'(0) > 0$, then $L^\varphi(K) \subset L^1(K)$. In particular, the conclusion holds if K is a compact hypergroup.*

Proof. Suppose that the right derivative $\varphi'(0) > 0$. Then we have $|u|\varphi'(0) \leq \varphi(|u|)$ for all $u \in \mathbb{R}$. Indeed, $|u|\varphi'(0) = \int_0^{|u|} \varphi'(0)dx \leq \int_0^{|u|} \varphi'(x)dx = \varphi(|u|)$, where the penultimate inequality holds as φ' is increasing. Thus, for $f \in L^\varphi(K)$, we get $\|f\|_1 \leq \frac{1}{\varphi'(0)}\|f\|_\varphi$ so that $L^\varphi(K) \subset L^1(K)$.

Next, assume that $m(K) < \infty$. Since φ is convex, there exist $c > 0$ and $u_0 > 0$ such that $\varphi(u) \geq c\,u$ for all $u \geq u_0$. If $f \in L^\varphi(K)$, then $\rho_\varphi(af) < \infty$ for some $a > 0$. Set $N := \{s \in K : a|f(s)| < u_0\}$. Then

$$\begin{aligned}\int_K |f(s)|\, dm(s) &= \frac{1}{a}\left(\int_N |af(s)|\, dm(s) + \int_{K\setminus N} |af(s)|\, dm(s) \right) \\ &\leq \frac{1}{a}\left(u_0\, m(K) + \frac{1}{c}\rho_\varphi(f) \right) < \infty.\end{aligned}$$

Thus $f \in L^\varphi(K)$ implies that $f \in L^1(K)$. □

The following Theorem provides a sufficient condition on the Banach space $L^\varphi(K)$ to become a Banach algebra.

Theorem 3.5. *Let K be a hypergroup, and let φ be a Young function. If $L^\varphi(K) \subset L^1(K)$, then the Orlicz space $L^\varphi(K)$ is a Banach algebra under convolution of functions. If K is commutative, then algebra $L^\varphi(K)$ is commutative.*

PROOF. Suppose that $L^\varphi(K) \subset L^1(K)$ holds. Then by Lemma 3.3, there exists $d > 0$ such that

$$\|f\|_1 \leq d\,\|f\|_\varphi \tag{2}$$

for all $f \in L^\varphi(K)$. In fact, we can choose $d = 1$ by replacing $\|\cdot\|_\varphi$ by an equivalent norm, denoted by $\|\cdot\|_\varphi$ again. Let $f, g \in L^\varphi(K)$. By Fubini's Theorem we have

$$\begin{aligned}
\|f * g\|_\varphi &= \sup\left\{\int_K |(f*g)h| : \rho_\psi(h) \leq 1\right\} \\
&\leq \sup\left\{\int_K |f(s)| \int_K |g(\check{s} * t)h(t)|dm(t)\,dm(s) : \rho_\psi(h) \leq 1\right\} \\
&\leq \|f\|_1 \|g_{\check{s}}\|_\varphi \leq \|f\|_\varphi \|g\|_\varphi,
\end{aligned}$$

where the last inequality follows from (2) and Corollary 3.2. Therefore $L^\varphi(K)$ is a Banach algebra. □

The converse of Lemma 3.4 (that $L^\varphi(K)$ is a Banach algebra if K is compact), is the well-known L^p-conjecture when $\varphi(x) = \frac{|x|^p}{p}$. This was established by SAEKI [16] in 1990 for a locally compact group. TABATABAIE and HAGHIGHIFAR claimed that L^p-conjecture is true for the locally compact hypergroups [18].

Lemma 3.6. *Let K be a hypergroup, and let φ be a finite Young function. Then $L^\varphi(K)$ is a left Banach $M(K)$-module. In particular, $L^\varphi(K)$ is a left Banach $L^1(K)$-module.*

PROOF. Let μ be a bounded positive measure such that $\mu(K) < \infty$, and let $f \in L^\varphi(K)$ be a positive function. For the complimentary function ψ of φ, if $h \in L^\psi(K)$, then

$$\langle \mu * f, h\rangle = \int_K \int_K f(\check{s} * t)h(t)d\mu(s)dm(t) = \int_K \int_K f(t)h_s(t)\,dm(t)d\mu(s).$$

Thus, by Hölder's inequality (1), we get

$$\langle \mu * f, h\rangle \leq \int_K \|f\|_\varphi^0 \|h_s\|_\varphi d\mu(s).$$

By Corollary 3.2, $\langle \mu * f, h\rangle \leq \|f\|_\varphi^0 \|h\|_\varphi \|\mu\|$, which is finite. Hence, the proposition follows from [13, Proposition IV(1)]. □

Theorem 3.7. *Let K be a hypergroup, and let φ be a Young function such that $\varphi \in \Delta_2$. Then the map $s \mapsto f_s$ from K to $L^\varphi(K)$ is continuous.*

Proof. Let $f \in C_c(K)$, and let $S = \operatorname{supp}(f)$, the support of f, which is the closure of the set of points of K which are not mapped to zero under f. Suppose $s_0 \in K$ and V_{s_0} is a compact neighbourhood s_0. Note that $\operatorname{supp}(f_s) \subset s * S$ for all $s \in V_{s_0}$. Set $W = V_{s_0} \cup \{s\} * S \cup V_{s_0} * S$. Because W is a compact set, from Lemma 3.4, we have $L^\psi(W) \subset L^1(W)$ where ψ is a complementary Young function to φ. By Lemma 3.3, there exists $d > 0$ such that

$$\|g\chi_W\|_1 \leq d\|g\chi_W\|_\psi \leq 2d\,\|g\chi_W\|_\psi^0 \leq 2d\|g\|_\psi^0 \leq 2d, \tag{3}$$

for every $g \in L^\psi(K)$ satisfying $\rho_\psi(g) \leq 1$. By (3), for all $s \in V_{s_0}$,

$$\begin{aligned}
\|f_s - f_{s_0}\|_\varphi &= \sup\left\{\int_K |(f_s - f_{s_0})g|\,dm\ : \rho_\psi(g) \leq 1\right\}\\
&\leq \|f_s - f_{s_0}\|_\infty \sup\left\{\int_W |g|dm : \rho_\psi(g) \leq 1\right\}\ (\text{since supp}\,(f_s - f_{s_0}) \subset W)\\
&\leq 2d\|f_s - f_{s_0}\|_\infty.
\end{aligned}$$

Then by [3, Lemma 1.2.28], $\|f_s - f_{s_0}\|_\infty < \frac{\epsilon}{2d}$ for a neighbourhood $U_{s_0} \subset V_{s_0}$ of s_0. Therefore, it follows that for $f \in C_c(K)$, the map $s \mapsto f_s$ is continuous.

Now, suppose that $f \in L^\varphi(K)$ is an arbitrary function. Since $\varphi \in \Delta_2$, $C_c(K)$ is dense in $L^\varphi(K)$, the continuity of $s \mapsto f_s$, for $f \in L^\varphi(K)$, can be shown by using a density argument. Indeed, for any $\epsilon > 0$, there exists $h \in C_c(K)$ such that $\|f - h\|_\varphi < \frac{\epsilon}{4}$. Also from the first part of the proof, there exists a neighbourhood U_{s_0} of s_0 such that $\|h_s - h_{s_0}\|_\varphi < \frac{\epsilon}{2}$ for all $s \in U_{s_0}$. Therefore, for all $s \in U_{s_0}$,

$$\|f_s - f_{s_0}\|_\varphi \leq \|f_s - h_s\|_\varphi + \|h_s - h_{s_0}\|_\varphi + \|h_{s_0} - f_{s_0}\|_\varphi.$$

Thus by Corollary 3.2, $\|f_s - f_{s_0}\|_\varphi \leq 2\|f - h\|_\varphi + \|h_s - h_{s_0}\|_\varphi < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence, $s \mapsto f_s$ is continuous. □

It is apparent that the above results hold true if each left translation is replaced by the corresponding right translation. Symbolically, for any $f \in L^\varphi(K)$, we have, $f^s \in L^\varphi(K)$, $\|f^s\|_\varphi \leq \|f\|_\varphi$ and the map $s \mapsto f^s$ is continuous.

Theorem 3.8. *Let K be a hypergroup, and let φ be a Δ_2-regular Young function. If $L^\varphi(K)$ is an algebra, then it has a left bounded approximate identity which is bounded in L^1-norm.*

Proof. Since $C_c(K)$ is contained in $L^\varphi(K)$, for every symmetric compact neighbourhood V of identity $e \in K$, there exists $e_V(\geq 0) \in L^\varphi(K)$ with $\mathrm{supp}(e_V) \subset V$ such that $\|e_V\|_1 = 1$. Let $\mathcal{V}$ denote the collection of all neighbourhoods of identity. Then $\mathcal{V}$ is a directed set with respect to inclusion of sets, and $(e_V)_{V\in\mathcal{V}}$ is a left approximate identity of $L^\varphi(K)$.

Indeed, for any $\epsilon > 0$, by Theorem 3.7, there exists a symmetric relatively compact neighbourhood $W \in \mathcal{V}$ such that $\|f_{\check{t}} - f\|_\varphi < \epsilon$ for all $t \in W$. Then for all $V \geq W$ with $V \in \mathcal{V}$, we have

$$\begin{aligned}
&\int_K |(e_V * f - f)h|dm \\
&\quad = \int_K \left| \left(\int_K e_V(t) f(\check{t} * s)\, dm(t) - f(s) \int_K e_V(t)\, dm(t) \right) h(s) \right| dm(s) \\
&\quad \leq \int_K e_V(t) \int_K |f_{\check{t}}(s) - f(s)||h(s)|\, dm(s)\, dm(t) \\
&\quad \leq \int_K e_V(t) \|h\|_\psi \|f_{\check{t}} - f\|_\varphi^0\, dm(t) \qquad \text{(by Hölder's inequality (1))},
\end{aligned}$$

for all $h \in L^\psi(K)$ where ψ is the complementary Young function of φ.

Now, by taking the supremum over $h \in L^\psi(K)$ with $\rho_\psi(h) \leq 1$, we get

$$\|e_V * f - f\|_\varphi < \epsilon.$$

Therefore, $(e_V)_{V\in\mathcal{V}}$ is a left approximate identity. □

Theorem 3.9. *Let φ be a Young function such that $\varphi \in \Delta_2$, and let the Orlicz space $L^\varphi(K)$ be an algebra. Then a closed subset I of $L^\varphi(K)$ is a left ideal if and only if it is a left translation invariant subspace of $L^\varphi(K)$.*

Proof. Let I be a closed left ideal. Let $f \in I$ and $s \in K$. Using Theorem 3.8, we pick a bounded approximate identity (e_α) of $L^\varphi(K)$. Then, for any $\epsilon > 0$, there exists α_0 such that $\|e_\alpha * f - f\|_\varphi < \epsilon$ for every $\alpha \geq \alpha_0$. By Lemma 3.1, it follows that $(e_\alpha)_s * f \in I$. Hence, by Corollary 3.2,

$$\|(e_\alpha)_s * f - f_s\| \leq \|e_\alpha * f - f\|_\varphi < \epsilon.$$

Since I is closed, we get $f_s \in I$.

Conversely, suppose that I is a closed left translation invariant subspace. Suppose that I is not an ideal, that is, there exist $f \in L^\varphi(K)$ and $g \in I$ such that $f * g \notin I$. Hence, by the Hahn–Banach Theorem, there exists a bounded linear functional Λ on $L^\varphi(K)$ such that $\Lambda|_I = 0$ and $\Lambda(f * g) \neq 0$. Since $\varphi \in \Delta_2$, the dual of $L^\varphi(K)$ is $L^\psi(K)$ where ψ is the complimentary Young function to φ. Hence, the continuous linear functional $\Lambda \in (L^\varphi(K))^*$ is uniquely determine by $g_1 \in L^\psi(K)$, such that

$$\Lambda(h) = \int_K hg_1\, dm \qquad (\forall\, h \in L^\varphi(K)).$$

Thus,

$$\begin{aligned}\Lambda(f * g) &= \int_K \left(\int_K f(t)g(\check{t} * s)\, dm(t)\right) g_1(s)\, dm(s)\\ &= \int_K \left(\int_K g_1(s)g_{\check{t}}(s)\, dm(s)\right) f(t)dm(t) = \int_K f(t)\Lambda(g_{\check{t}})\, dm(t) = 0,\end{aligned}$$

as I is left translation invariant and $\Lambda|_I = 0$, which is a contradiction. Hence I is a left ideal. $\square$

By considering $\varphi(x) = \frac{|x|^p}{p}$ for $1 \le p < \infty$ in Theorem 3.9, we get the following new result for Lebesgue algebra $L^p(K)$. The case $p = 1$ was established by Chilana and Ross [4] for commutative hypergroups, and by Litvinov [11] for general hypergroups.

Theorem 3.10. *If $L^p(K)$ is a Banach algebra for $1 < p < \infty$, then a closed subspace I of $L^p(K)$ is a left ideal if and only if it is left translation invariant.*

Proposition 3.11. *Let φ be a Young function such that the right derivative $\varphi'(0) > 0$. Then $L^\varphi(K)$ is a left ideal of $L^1(K)$.*

4. The space of multipliers of $M^\varphi(K)$

Suppose φ is a Young function. The set of all measurable functions f such that $\rho_\varphi(af) < \infty$ for all $a > 0$ is denoted by $M^\varphi(K)$. It is well-known that $M^\varphi(K) \neq 0$ if and only if φ is finite. Moreover, if φ is finite, then $M^\varphi(K)$ is a subspace of $L^\varphi(K)$, and in fact, $M^\varphi(K)$ is the norm closure of $C_c(K)$ in $L^\varphi(K)$. In this case, $M^\varphi(K)$ is called the *subspace of finite elements* or *Morse–Transue space*. For more details, see [9] and [13]. Throughout this section, we will deal with only finite Young functions.

A bounded linear operator T on $M^\varphi(K)$ is called a *convolutor* or a *multiplier* if $T(f * g) = T(f) * g$ for all $f, g \in C_c(K)$. The space of multipliers $C_{V_\varphi}(K)$ is a closed subspace of $\mathcal{B}(M^\varphi(K))$.

Suppose $\mathfrak{C}(K)$ is the set of all compact neighbourhoods of $e \in K$. For $P \in \mathfrak{C}(K)$, define

$$\check{A}_{\varphi,P}(K) := \Bigg\{ u \in C_c(K) : u = \sum_{n=1}^{\infty} g_n * \check{f}_n, (f_n) \subset M^{\varphi}(P), (g_n) \subset M^{\psi}(P), \text{ with } \sum_{n=1}^{\infty} \|f_n\|_{\varphi}^0 \|g_n\|_{\psi}^0 < \infty \Bigg\}.$$

The norm of $u \in \check{A}_{\varphi,P}(K)$ is given by

$$\|u\|_{\check{A}_{\varphi,P}(K)} = \inf \left\{ \sum_{n=1}^{\infty} \|f_n\|_{\varphi}^0 \|g_n\|_{\psi}^0 : u = \sum_{n=1}^{\infty} g_n * \check{f}_n \right\}.$$

Set $\check{A}_{\varphi}(K) = \bigcup_{P \in \mathfrak{C}(K)} \check{A}_{\varphi,P}(K)$. Then $\check{A}_{\varphi}(K)$ is a subspace of $C_c(K)$, and for $u \in \check{A}_{\varphi}(K)$,

$$\|u\|_{\check{A}_{\varphi}(K)} = \inf\{\|u\|_{\check{A}_{\varphi,P}(K)} : u \in \check{A}_{\varphi,P}(K), P \in \mathfrak{C}(K)\}.$$

It can be proved as in the group case, with $\varphi(x) = |x|^p/p$ (see [5]), that $\check{A}_{\varphi}(K)$ is a normed space with the above norm. This space need not be an algebra under pointwise multiplication. For instance, for $\varphi(x) = \frac{|x|^2}{2}$, Vrem [19, Example 4.12] showed that for a hypergroup [10, 9.1C] with three elements, $A_2(K)$ is not a normed algebra.

Lemma 4.1. *Let K be a hypergroup, and let φ be an N-function. If $T \in C_{V_{\varphi}}$, then there exists a net $(e_\alpha) \in C_c(K)$ with $\|e_\alpha\|_1 = 1$ such that if we set $T_\alpha(f) = T(e_\alpha * f)$ for every $f \in M^{\varphi}(K)$, then*

(i) *for each α, $\|T_\alpha\| \le \|T\|$; and*

(ii) $\lim_{\alpha} \|T_\alpha(f) - T(f)\|_{\varphi}^0 = 0$ *for each $f \in C_c(K)$.*

Proof. By Theorem 3.8, we can choose a net $(e_\alpha) \subset C_c(K)$ with $\|e_\alpha\|_1 = 1$ such that $\lim_{\alpha}\|e_\alpha * f - f\|_{\varphi}^0 = 0$ for any $f \in M^{\varphi}(K)$. Suppose T_α is as in the statement of the Lemma. Since T_α is a multiplier and $\|T_\alpha(f)\|_{\varphi}^0 \le \|T\| \, \|f\|_{\varphi}^0$ for any $f \in M^{\varphi}(K)$, we get $\|T_\alpha\| \le \|T\|$. Further,

$$\lim_{\alpha} \|T_\alpha(f) - T(f)\|_{\varphi}^0 = \lim_{\alpha} \|T(e_\alpha * f - f)\|_{\varphi}^0 = \|T\| \lim_{\alpha} \|e_\alpha * f - f\|_{\varphi}^0 = 0. \quad \square$$

The next result is an obvious analogue of a theorem of Cowling [5, Theorem 2] in the context of Orlicz spaces on hypergroups. One can find a generalization of Cowling's result in [1] for Orlicz spaces on locally compact groups.

Theorem 4.2. *Let K be a hypergroup, and let (φ, ψ) be a complementary pair of N-functions. Suppose φ is Δ_2-regular. Then the dual of $\check{A}_\varphi(K)$ can be identified with $C_{V_\varphi}(K)$ as Banach spaces.*

Proof. Let $T \in C_{V_\varphi}(K)$, $h \in \check{A}_\varphi(K)$. Then for some $P \in \mathfrak{C}(K)$, $h = \sum_{n=1}^{\infty} g_n * \check{f}_n$ with $f_n, g_n \in C_c(K)$ and $\operatorname{supp}(f_n)$, $\operatorname{supp}(g_n) \subset P$. Define $\Phi_T : \check{A}_\varphi(K) \to \mathbb{C}$ by

$$\Phi_T(h) = \sum_{n=1}^{\infty} \langle Tf_n, g_n \rangle.$$

Then, Φ_T is linear and

$$|\Phi_T(h)| \le 2 \sum_{n=1}^{\infty} \|Tf_n\|_\varphi^0 \|g_n\|_\varphi^0 \le 2\|T\| \sum_{n=1}^{\infty} \|f_n\|_\varphi^0 \|g_n\|_\varphi^0 < \infty. \tag{4}$$

To see that $\Phi_T(h)$ is independent of the representation of h, it is enough to show that $\Phi_T(h) = 0$ whenever $h = 0$.

Suppose that $P \in \mathfrak{C}(K)$ and $h \in \check{A}_\varphi(K)$ with $h = \sum_{n=1}^{\infty} g_n * \check{f}_n = 0$. By Lemma 4.1, there exists a net $(e_\alpha) \subset C_c(K)$ such that for $f \in M^\varphi(K)$,

$$T(e_\alpha * f) = (Te_\alpha) * f = T_\alpha f.$$

For each α we have,

$$\begin{aligned}\left| \sum_{n=1}^{\infty} \langle T_\alpha f_n, g_n \rangle \right| &\le 2 \sum_{n=1}^{\infty} \|T_\alpha f_n\|_\varphi^0 \|g_n\|_\varphi^0 \le 2\|T_\alpha\| \sum_{n=1}^{\infty} \|f_n\|_\varphi^0 \|g_n\|_\varphi^0 \\ &\le 2\|T\| \sum_{n=1}^{\infty} \|f_n\|_\varphi^0 \|g_n\|_\varphi^0 < \infty.\end{aligned}$$

Thus, the series $\sum_{n=1}^{\infty} \langle T_\alpha f_n, g_n \rangle$ converges uniformly in α, and we get $\|\Phi_T\| \le 2\|T\|$. Therefore,

$$\lim_\alpha \sum_{n=1}^{\infty} \langle T_\alpha f_n, g_n \rangle = \sum_{n=1}^{\infty} \lim_\alpha \langle T_\alpha f_n, g_n \rangle = \sum_{n=1}^{\infty} \langle Tf_n, g_n \rangle.$$

On the other hand,

$$\begin{aligned}\sum_{n=1}^{\infty} \langle T_\alpha f_n, g_n \rangle &= \sum_{n=1}^{\infty} \langle (Te_\alpha) * f_n, g_n \rangle = \sum_{n=1}^{\infty} \langle \chi_{P_1} \cdot Te_\alpha, g_n * \check{f}_n \rangle \quad (\text{with } P_1 = P * \check{P}) \\ &= \left\langle \chi_{P_1} \cdot Te_\alpha, \sum_{n=1}^{\infty} g_n * \check{f}_n \right\rangle = 0.\end{aligned}$$

Therefore, $\Phi_T(h) = 0$. Hence, Φ_T is a well-defined linear functional on $\check{A}_\varphi(K)$. Moreover,

$$\begin{aligned}\|T\| &= \sup\{\|Tf\|_\varphi^0 : f \in C_c(K),\ \|f\|_\varphi^0 \leq 1\}\\ &\leq \sup\{|\langle Tf, g\rangle| : f, g \in C_c(K),\ \|f\|_\varphi^0 \leq 1, \|g\|_\psi \leq 1\}\\ &\leq \sup\{|\Phi_T(h)| : h = g * \check{f},\ \|h\|_{\check{A}_\varphi} \leq 1\} \leq \|\Phi_T\|.\end{aligned}$$

Now, it remains to show that $\Phi : T \mapsto \Phi_T$ is surjective. For $F \in (\check{A}_\varphi)^*$ and $f \in C_c(K)$, define F_f as $F_f(g) = F(g * \check{f})$ for $g \in C_c(K)$. Then,

$$|F_f(g)| = |F(g * \check{f})| \leq \|F\|\|f\|_\varphi^0\|g\|_\psi < \infty.$$

Thus F_f is a linear functional on a dense subspace of $M^\psi(K)$. Since both φ, ψ are N-functions and φ is Δ_2-regular, we have $(M^\psi(K))^* = L^\varphi(K) = M^\varphi(K)$. Therefore, there exists a unique function $T(f) \in M^\varphi(K)$ such that

$$F_f(g) = F(g * \check{f}) = \langle Tf, g\rangle \quad \text{for each } g \in C_c(K).$$

Thus $\|Tf\|_\varphi^0 = \|F_f\|_\varphi^0 \leq \|F\|\|f\|_\varphi^0$ so that $\|T\| \leq \|F\|$. Since $\varphi \in \Delta_2$, $C_c(K)$ is dense in $M^\varphi(K)$, and hence, T can be extended to $M^\varphi(K)$ with $\|T\| \leq \|F\|$. Now, for $f, g \in C_c(K)$ and $\forall\, h \in L^\psi(K)$,

$$\langle (Tf) * g, h\rangle = \langle Tf, h * \check{g}\rangle = F_f(h * \check{g}) = F(h * \check{g} * \check{f}) = \langle T(f * g), h\rangle.$$

Hence, $T \in C_{V_\varphi}(K)$. □

5. The multiplier space of $L^\varphi(\mathcal{S}, \pi)$

In this section, we characterize the multiplier space of $L^\varphi(\mathcal{S}, \pi)$ where S is the support of the Plancherel measure π_K on the dual of a commutative hypergroup K, and obtain a relation between the multiplier spaces of $L^\varphi(\mathcal{S}, \pi)$ and $L^\psi(\mathcal{S}, \pi)$ where (φ, ψ) is a pair of complementary Young functions. We begin this section with a few basic definitions to make it self-contained.

Let K be a commutative hypergroup with the Haar measure m. Denote the space of complex valued continuous bounded function defined on K by $C^b(K)$. The dual of K is defined by $\widehat{K} = \{\chi \in C^b(K) : \chi(x * y) = \chi(x)\chi(y), \chi(\check{x}) = \overline{\chi(x)}$ and $\chi(e) = 1\ \forall\, x, y \in K\}$. Equip $\widehat{K}$ with the compact-open topology so that $\widehat{K}$ is a locally compact Hausdorff space. In general, $\widehat{K}$ may not have a naturally defined hypergroup structure. The Fourier transform of $f \in L^1(K)$ is defined by

$$\widehat{f}(\chi) = \int_K f(x)\overline{\chi(x)}\, dm(x), \qquad \forall\, \chi \in \widehat{K}.$$

There exists a unique positive Borel measure π_K on $\widehat{K}$ called the *Plancherel measure* such that

$$\int_K |f(x)|^2 \, dx = \int_{\widehat{K}} |\widehat{f}(\chi)|^2 \, d\pi_K(\chi), \qquad \forall\, f \in L^2(K) \cap L^1(K).$$

Note that the support $\mathcal{S}$ of π_K, unlike the group case, need not be the whole of $\widehat{K}$ [3, Example 2.2.49]. The extension of the Fourier transform from $L^1(K) \cap L^2(K)$ to $L^2(K)$ is called the *Plancherel transform.* We denote the Plancherel transform of $f \in L^2(K)$ by $\mathfrak{p}(f)$. The Plancherel transform is an isometric isomorphism from $L^2(K,m)$ onto $L^2(\mathcal{S},\pi_K)$ [3, p. 91]. Thus, for $f,g \in L^2(K,m)$,

$$\int_K f(s)\,\overline{g(s)}\, dm(s) = \int_{\mathcal{S}} \mathfrak{p}(f)(\chi)\,\overline{\mathfrak{p}(g)(\chi)}\, d\pi_K(\chi),$$

and hence

$$\int_K f(s)\, g(s)\, dm(s) = \int_{\mathcal{S}} \mathfrak{p}(f)(\chi)\,\mathfrak{p}(g)(\chi)\, d\pi_K(\chi),$$

which is called the *Plancherel identity.*

If $\widehat{K}$ is not a hypergroup, we can't talk of translation map, and therefore, the usual definition of multiplier does not work. The notion of multiplier is generalised to $L^\varphi(\mathcal{S},\pi_K)$ by means of the Plancherel transform.

A bounded linear operator T on $L^\varphi(\mathcal{S},\pi_K)$ is called a *multiplier* if there exists a function $h \in L^\infty(K,m)$ such that $T(g) = \mathfrak{p}(h\mathfrak{p}^{-1}g)$ for every $g \in L^\varphi(\mathcal{S},\pi_K) \cap L^2(\mathcal{S},\pi_K)$. The uniqueness of the Fourier transform ensures that T is well-defined. We denote the set of all multipliers of $L^\varphi(\mathcal{S},\pi_K)$ by $\mathcal{M}(L^\varphi(\mathcal{S},\pi_K))$, and the set of corresponding functions by $\mathfrak{M}(L^\varphi(\mathcal{S},\pi_K))$.

Theorem 5.1. *Let (φ,ψ) be a pair of complementary Young functions which are Δ_2-regular. The multiplier spaces of $L^\varphi(\mathcal{S},\pi_K))$ and $L^\psi(\mathcal{S},\pi_K))$ are isometrically isomorphic.*

Proof. Let $T \in \mathcal{M}(L^\varphi(\mathcal{S},\pi_K))$. Then, by definition, there exists a bounded function $h \in L^\infty(K,m)$ such that $T(g) = \mathfrak{p}(h\mathfrak{p}^{-1}g)$ for all $g \in L^\varphi(\mathcal{S},\pi_K) \cap L^2(\mathcal{S},\pi_K)$. For $f \in C_c(\mathcal{S}) \cap L^\psi(\mathcal{S},\pi_K)$, define

$$F_f(g) = \int_{\mathcal{S}} Tf(\chi)g(\chi)\, d\pi_K(\chi), \quad \text{for all } g \in C_c(\mathcal{S}).$$

Note that, by the Plancherel identity, it follows that

$$\begin{aligned}F_f(g) &= \int_{\mathcal{S}} \mathfrak{p}(h\mathfrak{p}^{-1}f)(\chi)\, g(\chi)\, d\pi_K(\chi) = \int_K (h\mathfrak{p}^{-1}f)(s)\, (\mathfrak{p}^{-1}g)(s)\, dm(s)\\ &= \int_{\mathcal{S}} \mathfrak{p}(h\mathfrak{p}^{-1}g)(\chi)\, f(\chi)\, d\pi_K(\chi) = \int_{\mathcal{S}} Tg(\chi) f(\chi)\, d\pi_K(\chi).\end{aligned}$$

Therefore, by Hölder's inequality,

$$|F_f(g)| = \left| \int_{\mathcal{S}} Tg(\chi) f(\chi)\, d\pi_K(\chi) \right| \le \|Tg\|_\varphi\, \|f\|^0_\psi \le \|T\|_\varphi \|g\|_\varphi\, \|f\|^0_\psi.$$

Since φ is Δ_2-regular, $C_c(\mathcal{S})$ is dense in $L^\varphi(\mathcal{S}, \pi_K)$. Therefore, F_f can be extended to $L^\varphi(\mathcal{S}, \pi_K)$ without changing the norm. Again, as $(\varphi,\, \psi)$ is Δ_2-regular (hence N-functions), by duality of $L^\varphi(\mathcal{S}, \pi_K)$ and $L^\psi(\mathcal{S}, \pi_K)$, $Tf \in L^\psi(\mathcal{S}, \pi_K)$, and moreover,

$$\|Tf\|_\psi = \|F_f\| \le \|T\|_\varphi\, \|f\|^0_\psi.$$

Thus, T restricted to $C_c(\mathcal{S})$ defines a bounded linear transformation of $C_c(\mathcal{S})$ into $L^\psi(\mathcal{S}, \pi_K)$. Since $C_c(\mathcal{S})$ is dense in $L^\psi(\mathcal{S}, \pi_K)$, T restricted to $C_c(\mathcal{S})$ can be uniquely extended to a bounded linear operator of $L^\psi(\mathcal{S}, \pi_K)$ without changing the norm. Denote this operator on $L^\psi(\mathcal{S}, \pi_K)$ by $\widetilde{T}$.

Suppose, $f \in L^\psi(\mathcal{S}, \pi_K) \cap L^2(\mathcal{S}, \pi_K)$, and let $(f_n) \subset C_c(\mathcal{S})$ be a sequence such that $\lim\limits_n \|f_n - f\|_\psi = 0$. Since $\widetilde{T}$ is continuous, $\lim\limits_n \|Tf_n - \widetilde{T}f\|_\psi = \lim\limits_n \|\widetilde{T}f_n - \widetilde{T}f\|_\psi = 0$. By the Plancherel identity, we have, for all $g \in C_c(\mathcal{S})$,

$$\begin{aligned}&\int_{\mathcal{S}} \widetilde{T}f(\chi)\, g(\chi)\, d\pi_K(\chi)\\ &= \lim_{n\to\infty} \int_{\mathcal{S}} Tf_n(\chi)\, g(\chi)\, d\pi_K(\chi) = \lim_{n\to\infty} \int_{\mathcal{S}} \mathfrak{p}(h\mathfrak{p}^{-1}f_n)(\chi)\, g(\chi)\, d\pi_K(\chi)\\ &= \lim_{n\to\infty} \int_K h(s)(\mathfrak{p}^{-1}f_n)(s)\mathfrak{p}^{-1}(g)(s)\, dm(s) = \lim_{n\to\infty} \int_{\mathcal{S}} f_n(\chi)\, \mathfrak{p}(h\mathfrak{p}^{-1}g)(\chi)\, d\pi_K(\chi)\\ &= \int_{\mathcal{S}} f(\chi)\, \mathfrak{p}(h\mathfrak{p}^{-1}g)(\chi)\, d\pi_K(\chi) = \int_K h(s)\, (\mathfrak{p}^{-1}f)(s)(\mathfrak{p}^{-1}g)(s)\, dm(s)\\ &= \int_{\mathcal{S}} \mathfrak{p}(h\mathfrak{p}^{-1}f)(\chi)\, g(\chi)\, d\pi_K(\chi).\end{aligned}$$

Since $C_c(\mathcal{S})$ is dense in $L^\varphi(\mathcal{S}, \pi_K)$, we conclude that $\widetilde{T}(f) = \mathfrak{p}(h\mathfrak{p}^{-1}(f))$ for all $f \in L^\psi(\mathcal{S}, \pi) \cap L^2(\mathcal{S}, \pi_K)$. Therefore, $\widetilde{T} \in \mathcal{M}(L^\psi(\mathcal{S}, \pi_K))$ and $\|\widetilde{T}\|_\psi \le \|T\|_\varphi$. The inverse of $T \mapsto \widetilde{T}$ is obtained by interchanging the role of φ and ψ, which shows that $T \mapsto \widetilde{T}$ is an isometric isomorphism. □

Degenfeld-Schonburg [6] proved the above result for $\varphi(x) = \frac{|x|^p}{p}$ and $\psi(x) = \frac{|x|^q}{q}$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 1. By the proof of Theorem 5.1, it follows that if T is a bounded linear operator on $L^\varphi(\mathcal{S}, \pi_K)$, the condition that $T(f) = \mathfrak{p}(h\mathfrak{p}^{-1}f)$ for all $f \in C_c(\mathcal{S})$ and for some $h \in L^\infty(K, m)$ is sufficient for T to be a multiplier.

Our next result generalizes a result of Hahn on a locally compact abelian group with $\varphi(x) = \frac{|x|^p}{p}$, $1 < p < \infty$ (see [8]). For a hypergroup, the case $\varphi(x) = \frac{|x|^p}{p}$, $1 < p < \infty$, was estabished by Degenfeld-Schonburg [6, Proposition 4.3.7].

Theorem 5.2. *Let (φ, ψ) be a pair of Δ_2-regular complementary Young functions, and let h be a bounded measurable function on K. Then the following statements are equivalent:*

(i) $h \in M(L^\varphi(\mathcal{S}, \pi_K))$.

(ii) *There exists a constant C such that*

$$\left| \int_K h(\mathfrak{p}^{-1}f)(\mathfrak{p}^{-1}g)\, dm \right| \le C\|f\|_\varphi^0 \, \|g\|_\psi^0$$

for all f, $g \in C_c(\mathcal{S})$.

Proof. Let $h \in \mathfrak{M}(L^\varphi(\mathcal{S}, \pi_K))$ so that $\mathfrak{p}(h\mathfrak{p}^{-1}f) \in L^\varphi(\mathcal{S}, \pi)$ for all $f \in L^\varphi(\mathcal{S}, \pi) \cap L^2(\mathcal{S}, \pi)$. We define a bounded linear operator T from $C_c(\mathcal{S})$ to $L^\varphi(\mathcal{S}, \pi_K)$ by $T(f) = \mathfrak{p}(h\mathfrak{p}^{-1}f)$. Then,

$$\|T\| = \sup\{\|\mathfrak{p}(h\mathfrak{p}^{-1}f)\|_\varphi : f \in C_c(\mathcal{S}), \|f\|_\varphi \le 1\}.$$

Since (φ, ψ) is a pair of complementary Δ_2-regular functions (hence N-functions), $L^\psi(\mathcal{S}, \pi_K)$ is the dual of $L^\varphi(\mathcal{S}, \pi_K)$ and $C_c(\mathcal{S})$ is dense in $L^\psi(\mathcal{S}, \pi_K)$. Hence we have

$$\|T\| = \sup\left\{ \left| \int_S \mathfrak{p}(h\mathfrak{p}^{-1}f) g \, d\pi_K \right| : f, g \in C_c(\mathcal{S}), \|f\|_\varphi^0 \le 1, \ \|g\|_\psi^0 \le 1 \right\}.$$

By the Plancherel identity,

$$\|T\| = \sup\left\{ \left| \int_K h(\mathfrak{p}^{-1}f)\mathfrak{p}^{-1}g \, dm \right| : f, g \in C_c(\mathcal{S}), \|f\|_\varphi^0 \le 1, \ \|g\|_\psi^0 \le 1 \right\},$$

and thus (ii) holds with $C = \|T\|$.

Conversely, assume that (ii) holds for a bounded measurable function defined on K. If we reverse the above argument, we see that $T : C_c(S) \to L^\varphi(\mathcal{S}, \pi_K)$ defined by $Tf := \mathfrak{p}(h\mathfrak{p}^{-1}f)$ has norm bounded by C. Therefore, T can be extended to $L^\varphi(\mathcal{S}, \pi_K)$ with the same norm. By Remark 1, we have $\|h\|_\infty \le \|T\| = C$ so that $h \in \mathfrak{M}(L^\varphi(\mathcal{S}, \pi_K))$. $\square$

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