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Pseudo-random subsets constructed by using Fermat quotients

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Abstract. Let p be a prime, and let n be an arbitrary integer with $(n,p)=1$. The Fermat quotient $q_p(n)$ is defined as the unique integer with

$$q_p(n) \equiv \frac{n^{p-1}-1}{p} \pmod{p}, \qquad 0 \le q_p(n) \le p-1.$$

We also define $q_p(kp)=0$ for $k \in \mathbb{Z}$. In this paper, we study the pseudo-randomness of subsets constructed by Fermat quotients, by using the estimates for exponential sums and character sums with Fermat quotients.

1. Introduction

For a prime p and an integer n with $(n,p)=1$, the Fermat quotient $q_p(n)$ is defined as

$$q_p(n) \equiv \frac{n^{p-1}-1}{p} \pmod{p}, \qquad 0 \le q_p(n) \le p-1.$$

We also define $q_p(kp)=0$, $k \in \mathbb{Z}$. It is easy to show that $q_p(u) = q_p(p^2 \pm u)$ for $p > 2$, and

$$q_p(a+kp) \equiv q_p(a) - ka^{-1} \pmod{p} \tag{1.1}$$

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for any $a, k \in \mathbb{Z}$ with $(a, p) = 1$. Fermat quotients arose from the study of the first case of Fermat's Last Theorem, which claimed that the equation

$$x^n + y^n + z^n = 0, \qquad 2 \nmid n, \qquad (n, xyz) = 1, \qquad n > 1 \qquad (1)_n$$

has no solutions in integers x, y and z.

In 1823, Sophie Germain proved that if n and $2n+1$ are both prime, then $(1)_n$ has no solutions in integers. In 1909, Wieferich showed that if $(1)_p$ has solutions in integers x, y and z, then $q_p(2) = 0$. Shortly later, Mirimanoff extended the result to $q_p(3) = 0$. In 1917, Pollaczek proved that if prime p is sufficiently large and $(1)_p$ has solutions in integers x, y and z, then $q_p(u) = 0$ for each prime $u \le 31$. In 1988, Granville and Monagan showed that if $(1)_p$ has solutions in integers x, y and z, then $q_p(u) = 0$ for all primes $u \le 89$. See [13] for an excellent survey of the first case of Fermat's Last Theorem.

Fermat quotients have numerous applications in computational and algebraic number theory (see [1]–[2], [4], [6], [12], [16]–[19]). For example, we define the sets

$$\begin{aligned}
I_p(N) &= |\{q_p(u) : 1 \le u \le N, (u, p) = 1\}|, \\
M(p) &= |\{q_p(u) : 1 \le u \le p-1\}|, \\
F(p) &= |\{u : 0 \le u \le p-1, q_p(u) = u\}|, \\
R(p, a) &= |\{u : 0 \le u \le p-1, q_p(u) = a\}|, \\
\vartheta(p) &= \frac{1}{p^2} \cdot |\{(u, v) : 0 \le u, v \le p-1, q_p(u) = q_p(v)\}|, \\
U(p; k, h) &= |\{u : 0 \le u \le p-1, q_p(u) \equiv z \pmod p \text{ for } z \in [k+1, \dots, k+h]\}|.
\end{aligned}$$

Vandiver [20] showed that $\sqrt{p} - 1 \le M(p) \le p - \sqrt{\dfrac{p-1}{2}}$. Fouché [11] proved $R(p, a) \le p^{1/2+o(1)}$. Ostafe and Shparlinski [16] obtained the following estimations:

$$\begin{aligned}
M(p) &\ge (1 + o(1)) \frac{p}{(\log p)^2}, & F(p) &\ll p^{11/12+o(1)}, \\
\vartheta(p) &\le p^{-3/4+o(1)}, & U(p; k, h) &\le h^{1/2} p^{1/2+o(1)}.
\end{aligned}$$

Chen and Winterhof [6] gave certain generalization and improvement on Ostafe and Shparlinski's result.

Proposition 1.1. *Let $P(x) \in \mathbb{F}_p[x]$ be of degree $1 \le d < p$. Then we have*

$$\#\{u : 1 \le u \le p-1, q_p(u) = P(u)\} \ll \begin{cases} d^{1/4} p^{5/6}, & 1 \le d \le p^{1/3}, \\ d^{1/8} p^{7/8}, & d > p^{1/3}, \end{cases}$$

as $p \to \infty$.

SHPARLINSKI [17] proved some lower bounds on the image size $I_p(N)$.

Proposition 1.2. *For every p and $N < p$, we have*

$$I_p(N) \geq (1 + o(1)) \frac{N^2}{p(\log N)^2}.$$

For every p and arbitrary fixed $\epsilon > 0$, there exists some $\delta > 0$ such that, for $p^\epsilon < N < p$, we have $I_p(N) \geq p^\delta$.

Shparlinski [17] also studied the primitive roots and power non-residues among the values of the Fermat quotients.

Proposition 1.3. *For every p, there exists $n \leq p^{3/4+o(1)}$ such that $q_p(n)$ is a primitive root modulo p. For every p and positive integer $d \mid p-1$, there exists $n \leq p^{3/4+o(1)}$ such that $q_p(n)$ is a d-th power non-residue modulo p.*

In this paper, we further study the pseudo-randomness of subsets constructed by using Fermat quotients. Let $\mathcal{R} \subset \{1, 2, \ldots, N\}$, and define the sequence

$$E_N = E_N(\mathcal{R}) = \{e_1, e_2, \ldots, e_N\} \in \left\{1 - \frac{|\mathcal{R}|}{N},\ -\frac{|\mathcal{R}|}{N}\right\}^N$$

by

$$e_n = \begin{cases} 1 - \frac{|\mathcal{R}|}{N}, & \text{for } n \in \mathcal{R}, \\ -\frac{|\mathcal{R}|}{N}, & \text{for } n \notin \mathcal{R}. \end{cases}$$

DARTYGE and SÁRKÖZY [8] introduced the measures of pseudo-randomness: The *well-distribution measure* of the subset $\mathcal{R}$ is defined by

$$W(\mathcal{R}, N) = \max_{a,b,t} \left| \sum_{j=0}^{t-1} e_{a+jb} \right|,$$

where the maximum is taken over all a, b, $t \in \mathbb{N}$ with $1 \leq a \leq a + (t-1)b \leq N$. The *correlation measure of order k* of the subset $\mathcal{R}$ is defined by

$$C_k(\mathcal{R}, N) = \max_{M,D} \left| \sum_{n=1}^{M} e_{n+d_1} \cdots e_{n+d_k} \right|,$$

where the maximum is taken over all $D = (d_1, \ldots, d_k)$ and M with $0 \leq d_1 < \cdots < d_k \leq N - M$.

The subset $\mathcal{R}$ is considered as a pseudo-random subset if both $W(\mathcal{R}, N)$ and $C_k(\mathcal{R}, N)$ (at least for small k) are "small" in terms of N. Later many pseudo-random subsets were given and studied. For example, DARTYGE, MOSAKI and SÁRKÖZY [7] proved the following results.

Proposition 1.4. *Assume that p is an odd prime number, $f(x) \in \mathbb{F}_p[x]$ is of degree $d \geq 2$. Let $r \in \mathbb{Z}, s \in \mathbb{N}, s < p/2$. Define $\mathcal{R} \subset \{1, \ldots, p\}$ by*

$$\mathcal{R} = \{n : 1 \leq n \leq p,\ \exists h \in \{r, r+1, \ldots, r+s-1\} \text{ with } f(n) \equiv h \pmod{p}\}.$$

Then we have

$$W(\mathcal{R}, p) < 2d\sqrt{p}(\log p)^2.$$

And for $2 \leq k \leq d-1$, we have

$$C_k(\mathcal{R}, p) < 2d\sqrt{p}(1 + \log p)^{k+1}.$$

Dartyge and Sárközy [9] presented large families of pseudo-random subsets formed by power residues.

Proposition 1.5. *Let $p \geq 2$ be a prime number, $d \mid p-1$ and $f \in \mathbb{F}_p[x]$ of degree k. Define*

$$\mathcal{R} = \left\{x \in \mathbb{F}_p : \exists y \in \mathbb{F}_p^* \text{ with } f(x) \equiv y^d \pmod{p}\right\}.$$

Write $f = f_1^{\alpha_1} \cdots f_r^{\alpha_r}$ for the factorization into irreducible factors in $\overline{\mathbb{F}}_p$, and suppose that $(d, \alpha_1, \ldots, \alpha_r) = 1$. Then

$$W(\mathcal{R}, p) \leq 20k\sqrt{p}\log p.$$

Let f be a polynomial of degree k with no multiple roots in $\overline{\mathbb{F}}_p$, and let $k, l \in \mathbb{N}$ such that one of the following conditions is satisfied:

(i) *l=2;*

(ii) *d is a prime divisor of $p-1$, and $(4l)^k < p$;*

(iii) *the polynomial $x^{p-1} + \cdots + x + 1$ is irreducible in $\mathbb{F}_d[x]$ and $\max(k, l) < p$.*

Then we have

$$C_l(\mathcal{R}, p) \leq lk\left(1 + \frac{20k\log p}{\sqrt{p}}\right)^l + 9\left(\frac{d-1}{d}\right)^l\left(1 + \frac{10k\log p}{\sqrt{p}}\right)^l lk\sqrt{p}\log p.$$

Dartyge, Sárközy and Szalay [10] studied the pseudo-randomness of certain subsets related to primitive roots.

Proposition 1.6. *Let p be an odd prime, $k, r \in \mathbb{N}$, $f(x) \in \mathbb{F}_p[x]$ of degree D. Let*

$$\mathcal{G}_p = \{g_1, g_2, \ldots, g_{\phi(p-1)}\}, \qquad (0 < g_1 < g_2 < \cdots < g_{\phi(p-1)} < p)$$

be the set of the primitive roots modulo p. Define the subset $\mathcal{R} \subset \mathbb{F}_p$ by

$$\mathcal{R} = \{g^k : g \in \mathcal{G}_p, \exists x \in \mathbb{F}_p^* \text{ with } f(g^k) = x^r\}.$$

If $f(x)$ is irreducible over $\mathbb{F}_p$, then we have

$$W(\mathcal{R}, p) \ll D2^{\omega(\frac{p-1}{k})}p^{1/2}\log p.$$

Moreover, if we also assume that $D \geq 2$ or $D = r = 1$, we also have

$$C_K(\mathcal{R}, p) \ll KD\left((1 + O(Dp^{-1/2}))2^{\omega(\frac{p-1}{k})}\right)^K p^{1/2}\log p.$$

Chen [3] constructed a family of pseudo-random subsets modulo pq, where p, q are two distinct primes satisfying "RSA type" with $2 < p < q < 2p$, and studied the well-distribution and correlation measures. Liu and Song [15] further studied the correlation measures of the subsets.

In this paper, we study the pseudo-random subsets constructed by using Fermat quotients. Our results are the following.

Theorem 1.1. *Let p be an odd prime number, and let $r, s \in \mathbb{N}$ with $1 \leq r < r + s - 1 \leq p - 1$. Define $\mathcal{R} \subset \{1, \ldots, p^2\}$ by*

$$\mathcal{R} = \left\{n : 1 \leq n \leq p^2,\ \exists h \in \{r, r+1, \ldots, r+s-1\} \text{ with } q_p(n) \equiv h \pmod{p}\right\}.$$

Then we have

$$W\left(\mathcal{R}, p^2\right) \ll p(\log p)^2,$$

and

$$C_2\left(\mathcal{R}, p^2\right) \ll p(\log p)^3.$$

Furthermore, if $s = o(p)$, then

$$C_3\left(\mathcal{R}, p^2\right) \ll s^2 + p^{\frac{3}{2}}\left(\log p\right)^3.$$

While $s = \left[\frac{p}{3}\right]$, we have

$$C_3\left(\mathcal{R}, p^2\right) \geq \frac{1}{54}p^2 + O\left(p\left(\log p\right)^4\right).$$

Theorem 1.2. *Let p be an odd prime number, and $d \mid p-1$. Define $\mathcal{R} \subset \{1, \dots, p^2\}$ by*

$$\mathcal{R} = \left\{n : 1 \le n \le p^2,\ \exists\ y \text{ such that } 1 \le y \le p-1 \text{ and } q_p(n) \equiv y^d \pmod p\right\}.$$

Then we have

$$W\left(\mathcal{R}, p^2\right) \ll p^{\frac{3}{2}} \log p,$$

and

$$C_k\left(\mathcal{R}, p^2\right) \ll k p^{\frac{5}{3}},$$

where the implied constant depends on d.

Theorem 1.3. *Let p be an odd prime number, and let $\mathcal{G}_p$ be the set of the primitive roots modulo p. Define $\mathcal{R} \subset \{1, \dots, p^2\}$ by*

$$\mathcal{R} = \left\{n : 1 \le n \le p^2, q_p(n) \in \mathcal{G}_p\right\}.$$

Then we have

$$W\left(\mathcal{R}, p^2\right) \ll 2^{\omega(p-1)} p^{\frac{3}{2}} \log p.$$

and

$$C_k\left(\mathcal{R}, p^2\right) \ll k 2^{k\omega(p-1)} p^{\frac{5}{3}}.$$

Remark 1.1. Our results show that the subsets in Theorems 1.2 and 1.3 enjoy good pseudo-random properties. However, the subset in Theorem 1.1 is not pseudo-random.

2. Exponential sums and character sums with Fermat quotients

The exponential sums with Fermat quotients have been studied by many authors. Write $e(y) = \mathrm{e}^{2\pi i y}$. Heath-Brown [14] proved the following estimation by using the Pólya–Vinogradov bound and the Burgess bound.

Proposition 2.1. *For any integer a coprime to p, we have*

$$\sum_{\substack{M < n \le M+N \\ p \nmid n}} e\left(\frac{a q_p(n)}{p}\right) \ll N^{1/2} p^{3/8},$$

uniformly for $M, N \ge 1$. In particular,

$$\sum_{n=1}^{p-1} e\left(\frac{a q_p(n)}{p}\right) \ll p^{7/8},$$

uniformly for $p \nmid a$.

Ostafe and Shparlinski [16] obtained a non-trivial bound of exponential sums with linear combinations of Fermat quotients.

Proposition 2.2. *For any integer $s \geq 1$, we have*

$$\max_{(a_0,\ldots,a_s,p)=1} \left| \sum_{u=M+1}^{M+N} e\left(\frac{\sum_{j=0}^{s-1} a_j q_p(u+j)}{p} \right) \right| \ll sp \log p,$$

uniformly over M and $p^2 > N \geq 1$.

Using Proposition 2.2, they studied the distribution of the points

$$\left(\frac{q_p(u)}{p}, \ldots, \frac{q_p(u+s-1)}{p} \right), \qquad u = M+1, \ldots, M+N,$$

in the s-dimensional cube. Moreover, Chen, Ostafe and Winterhof [5] gave the following generalization.

Proposition 2.3. *Let $d_0, d_1, \ldots, d_{s-1}$ be integers with $0 \leq d_0 < d_1 < \cdots < d_{s-1} < p^2$. Suppose that no triple (d_l, d_h, d_t) satisfies $d_l \equiv d_h \equiv d_t \pmod p$ for $0 \leq l < h < t < s$. We have*

$$\max_{(a_0,\ldots,a_s,p)=1} \left| \sum_{u=1}^{N} e\left(\frac{\sum_{j=0}^{s-1} a_j q_p(u+d_j)}{p} \right) \right| \ll s \max\left\{ p \log p, N p^{-\frac{1}{2}} \right\}$$

for $1 \leq N \leq p^2$. If $s = 2$ or $d_{s-1} < p$, the stronger bound $sp \log p$ holds.

Now, we give the following identities on certain exponential sums of Fermat quotients.

Theorem 2.1. *Let $p > 2$ be a prime, and let u be an integer with $(u, p) = 1$. For any integer z, we have*

$$\left| \sum_{n=1}^{p^2} e\left(\frac{u q_p(n)}{p} \right) e\left(\frac{zn}{p^2} \right) \right| = p.$$

PROOF. Write $n = a + bp$, we get

$$\sum_{n=1}^{p^2} e\left(\frac{uq_p(n)}{p}\right) e\left(\frac{zn}{p^2}\right) = \sum_{a=1}^{p}\sum_{b=0}^{p-1} e\left(\frac{uq_p(a+bp)}{p}\right) e\left(\frac{z(a+bp)}{p^2}\right)$$
$$= \sum_{a=1}^{p-1}\sum_{b=0}^{p-1} e\left(\frac{uq_p(a+bp)}{p}\right) e\left(\frac{z(a+bp)}{p^2}\right) + \sum_{b=0}^{p-1} e\left(\frac{z(1+b)}{p}\right).$$

Assume that $p \nmid z$. By (1.1), we have

$$\sum_{n=1}^{p^2} e\left(\frac{uq_p(n)}{p}\right) e\left(\frac{zn}{p^2}\right) = \sum_{a=1}^{p-1}\sum_{b=0}^{p-1} e\left(\frac{uq_p(a+bp)}{p}\right) e\left(\frac{z(a+bp)}{p^2}\right)$$
$$= \sum_{a=1}^{p-1}\sum_{b=0}^{p-1} e\left(\frac{u\left(q_p(a) - ba^{-1}\right)}{p}\right) e\left(\frac{z(a+bp)}{p^2}\right)$$
$$= \sum_{a=1}^{p-1} e\left(\frac{uq_p(a)}{p}\right) e\left(\frac{za}{p^2}\right) \sum_{b=0}^{p-1} e\left(\frac{b(z-ua^{-1})}{p}\right)$$
$$= p \sum_{\substack{a=1 \\ z\equiv ua^{-1} \pmod p}}^{p-1} e\left(\frac{uq_p(a)}{p}\right) e\left(\frac{za}{p^2}\right)$$

Hence,

$$\left|\sum_{n=1}^{p^2} e\left(\frac{uq_p(n)}{p}\right) e\left(\frac{zn}{p^2}\right)\right| = p.$$

Suppose that $p \mid z$. By (1.1), we get

$$\sum_{n=1}^{p^2} e\left(\frac{uq_p(n)}{p}\right) e\left(\frac{zn}{p^2}\right) = \sum_{a=1}^{p-1}\sum_{b=0}^{p-1} e\left(\frac{uq_p(a+bp)}{p}\right) e\left(\frac{z(a+bp)}{p^2}\right) + p$$
$$= \sum_{a=1}^{p-1}\sum_{b=0}^{p-1} e\left(\frac{u\left(q_p(a) - ba^{-1}\right)}{p}\right) e\left(\frac{z(a+bp)}{p^2}\right) + p$$
$$= \sum_{a=1}^{p-1} e\left(\frac{uq_p(a)}{p}\right) e\left(\frac{za}{p^2}\right) \sum_{b=0}^{p-1} e\left(-\frac{bua^{-1}}{p}\right) + p = p.$$

This proves Theorem 2.1. □

Theorem 2.2. *Let $p > 2$ be a prime, and let u_1, u_2, u_3 be integers with $(u_1u_2u_3, p) = 1$. Let d_1, k_2, k_3 and M be integers such that $0 \le d_1 < d_1 + k_2p < d_1 + k_3p \le p^2 - M$. Then we have*

$$\sum_{n=1}^{M} e\left(\frac{u_1q_p(n+d_1) + u_2q_p(n+d_1+k_2p) + u_3q_p(n+d_1+k_3p)}{p}\right)$$
$$= \begin{cases} M, & \text{if } u_1+u_2+u_3 \equiv 0 \pmod p \text{ and } u_2k_2+u_3k_3 \equiv 0 \pmod p, \\ O(p), & \text{if } u_1+u_2+u_3 \equiv 0 \pmod p \text{ and } u_2k_2+u_3k_3 \not\equiv 0 \pmod p, \\ O(p\log p), & \text{if } u_1+u_2+u_3 \not\equiv 0 \pmod p. \end{cases}$$

Proof. We may suppose that $M \ge p$, since otherwise the claim is trivial. By (1.1), we get

$$\sum_{n=1}^{M} e\left(\frac{u_1q_p(n+d_1) + u_2q_p(n+d_1+k_2p) + u_3q_p(n+d_1+k_3p)}{p}\right)$$
$$= \sum_{\substack{n=1\\(n+d_1,p)=1}}^{M} e\left(\frac{u_1q_p(n+d_1)+u_2q_p(n+d_1+k_2p)+u_3q_p(n+d_1+k_3p)}{p}\right) + \sum_{\substack{n=1\\(n+d_1,p)>1}}^{M} 1$$
$$= \sum_{\substack{n=1\\(n+d_1,p)=1}}^{M} e\left(\frac{(u_1+u_2+u_3)q_p(n+d_1)-(u_2k_2+u_3k_3)(n+d_1)^{-1})}{p}\right) + \sum_{\substack{n=1\\(n+d_1,p)>1}}^{M} 1.$$

Case 1. $u_1 + u_2 + u_3 \equiv 0 \pmod p$ and $u_2k_2 + u_3k_3 \equiv 0 \pmod p$. We have

$$\sum_{n=1}^{M} e\left(\frac{u_1q_p(n+d_1) + u_2q_p(n+d_1+k_2p) + u_3q_p(n+d_1+k_3p)}{p}\right) = M.$$

Case 2. $u_1 + u_2 + u_3 \equiv 0 \pmod p$ and $u_2k_2 + u_3k_3 \not\equiv 0 \pmod p$. It is not hard to show that

$$\sum_{n=1}^{M} e\left(\frac{u_1q_p(n+d_1) + u_2q_p(n+d_1+k_2p) + u_3q_p(n+d_1+k_3p)}{p}\right)$$
$$= \sum_{\substack{n=1\\(n+d_1,p)=1}}^{M} e\left(-\frac{(u_2k_2+u_3k_3)(n+d_1)^{-1})}{p}\right) + \sum_{\substack{n=1\\(n+d_1,p)>1}}^{M} 1 =$$

$$= \sum_{0\le k\le\left[\frac{M}{p}\right]-1} \sum_{\substack{v=1\\(v+d_1,p)=1}}^{p} e\left(-\frac{(u_2k_2+u_3k_3)(kp+v+d_1)^{-1})}{p}\right) + O(p)$$

$$= \sum_{0\le k\le\left[\frac{M}{p}\right]-1} \sum_{\substack{v=1\\(v+d_1,p)=1}}^{p} e\left(-\frac{(u_2k_2+u_3k_3)(v+d_1)^{-1})}{p}\right) + O(p) \ll p.$$

Case 3. $u_1+u_2+u_3 \not\equiv 0 \pmod p$. By (1.1) and the properties of residue systems, we have

$$\sum_{n=1}^{M} e\left(\frac{u_1q_p(n+d_1)+u_2q_p(n+d_1+k_2p)+u_3q_p(n+d_1+k_3p)}{p}\right)$$

$$= \sum_{\substack{n=1\\(n+d_1,p)=1}}^{M} e\left(\frac{(u_1+u_2+u_3)q_p(n+d_1)-(u_2k_2+u_3k_3)(n+d_1)^{-1})}{p}\right) + \sum_{\substack{n=1\\(n+d_1,p)>1}}^{M} 1$$

$$= \sum_{0\le k\le\left[\frac{M}{p}\right]-1} \sum_{\substack{v=1\\(v+d_1,p)=1}}^{p} e\left(\frac{(u_1+u_2+u_3)q_p(kp+v+d_1)}{p}\right.$$

$$\left. -\frac{(u_2k_2+u_3k_3)(kp+v+d_1)^{-1})}{p}\right) + O(p)$$

$$= \sum_{\substack{v=1\\(v+d_1,p)=1}}^{p} e\left(\frac{(u_1+u_2+u_3)q_p(v+d_1)-(u_2k_2+u_3k_3)(v+d_1)^{-1})}{p}\right)$$

$$\times \sum_{0\le k\le\left[\frac{M}{p}\right]-1} e\left(-\frac{(u_1+u_2+u_3)(v+d_1)^{-1}k}{p}\right) + O(p)$$

$$\ll \sum_{\substack{v=1\\(v+d_1,p)=1}}^{p} \left|\sum_{0\le k\le\left[\frac{M}{p}\right]-1} e\left(-\frac{(u_1+u_2+u_3)(v+d_1)^{-1}k}{p}\right)\right| + p$$

$$\ll \sum_{\substack{v=1\\(v+d_1,p)=1}}^{p} \frac{1}{\left\langle -\frac{(u_1+u_2+u_3)(v+d_1)^{-1}}{p}\right\rangle} + p \ll \sum_{0<|v|\le\frac{p-1}{2}} \frac{1}{\left\langle\frac{v}{p}\right\rangle} + p \ll p\log p,$$

where $\langle\alpha\rangle$ denotes the distance of the real number α to the nearest integer.

This completes the proof of Theorem 2.2. □

On the other hand, character sums with Fermat quotients have also been studied by many authors. For example, from [12] we know that

$$\sum_{u=0}^{N-1}\chi\left(q_p(au+b)\right) \ll N^{1-\frac{1}{v}}p^{\frac{5v+1}{4v^2}}(\log p)^{\frac{1}{v}}, \tag{2.1}$$

for $1 \le N \le p^2$ and every fixed integer $v \ge 1$, where χ is a non-trivial multiplicative character modulo p, and a, b are integers with $(a, p^2) \ne p^2$. Gomez and Winterhof [12] further studied the following character sums.

Proposition 2.4. *Let $\chi_1, \chi_2, \dots, \chi_l$ be nontrivial multiplicative characters modulo p. Then we have*

$$\sum_{u=0}^{N-1}\chi_1\left(q_p\left(u+d_1\right)\right)\cdots\chi_l\left(q_p\left(u+d_l\right)\right) \ll \max\left\{\frac{lN}{p^{\frac{1}{3}}},\ lp^{\frac{3}{2}}\log p\right\},$$

for any integers $0 \le d_1 < \cdots < d_l \le p^2-1$ and $1 \le N \le p^2$.

3. Proof of Theorem 1.1

For integer h with $(h,p)=1$, by (1.1), we get

$$\sum_{\substack{n=1\\ q_p(n)\equiv h\,(\mathrm{mod}\ p)}}^{p^2} 1 = \sum_{\substack{n=1\\ (n,p)=1\\ q_p(n)\equiv h\,(\mathrm{mod}\ p)}}^{p^2} 1 = \sum_{a=1}^{p-1}\sum_{\substack{k=0\\ q_p(a+kp)\equiv h\,(\mathrm{mod}\ p)}}^{p-1} 1$$

$$= \sum_{\substack{a=1\\ q_p(a)-ka^{-1}\equiv h\,(\mathrm{mod}\ p)}}^{p-1}\sum_{k=0}^{p-1} 1 = \sum_{a=1}^{p-1}\sum_{\substack{k=0\\ ka^{-1}\equiv q_p(a)-h\,(\mathrm{mod}\ p)}}^{p-1} 1 = p-1.$$

Suppose that $\mathcal{A} \subset \{1, 2, \dots, p-1\}$. Then

$$\sum_{\substack{n=1\\ q_p(n)\in\mathcal{A}}}^{p^2} 1 = |\mathcal{A}|\cdot(p-1). \tag{3.1}$$

Noting that $1 \le r < r+s-1 \le p-1$, we have

$$|\mathcal{R}| = s(p-1).$$

It is easy to show that, for $1 \le n \le p^2$,

$$\frac{1}{p}\sum_{h=r}^{r+s-1}\sum_{|u|\le\frac{p-1}{2}} e\left(\frac{u(q_p(n)-h)}{p}\right) = \begin{cases}1, & n \in \mathcal{R},\\ 0, & n \notin \mathcal{R}.\end{cases}$$

Hence,

$$\begin{aligned} e_n &= \frac{1}{p}\sum_{h=r}^{r+s-1}\sum_{|u|\le\frac{p-1}{2}} e\left(\frac{u(q_p(n)-h)}{p}\right) - \frac{|\mathcal{R}|}{p^2}\\ &= \frac{1}{p}\sum_{1\le|u|\le\frac{p-1}{2}}\left(\sum_{h=r}^{r+s-1} e\left(-\frac{uh}{p}\right)\right) e\left(\frac{uq_p(n)}{p}\right) + O\left(\frac{1}{p}\right), \end{aligned} \tag{3.2}$$

where the term $\frac{|\mathcal{R}|}{p^2}$ is compensated by the contribution of $u = 0$.

For a, b, $t \in \mathbb{N}$ with $1 \le a \le a + (t-1)b \le p^2$, from (3.2) and Theorem 2.1, we get

$$\begin{aligned} \sum_{j=0}^{t-1} e_{a+jb} &= \frac{1}{p}\sum_{1\le|u|\le\frac{p-1}{2}}\left(\sum_{h=r}^{r+s-1} e\left(-\frac{uh}{p}\right)\right)\sum_{j=0}^{t-1} e\left(\frac{uq_p(a+jb)}{p}\right) + O(p)\\ &= \frac{1}{p^3}\sum_{1\le|u|\le\frac{p-1}{2}}\left(\sum_{h=r}^{r+s-1} e\left(-\frac{uh}{p}\right)\right)\sum_{n=1}^{p^2} e\left(\frac{uq_p(n)}{p}\right)\sum_{j=0}^{t-1}\\ &\times\sum_{z=1}^{p^2} e\left(\frac{z(n-(a+jb))}{p^2}\right) + O(p)\\ &= \frac{1}{p^3}\sum_{1\le|u|\le\frac{p-1}{2}}\left(\sum_{h=r}^{r+s-1} e\left(-\frac{uh}{p}\right)\right)\sum_{z=1}^{p^2}\left(\sum_{j=0}^{t-1} e\left(-\frac{z(a+jb)}{p^2}\right)\right)\\ &\times\sum_{n=1}^{p^2} e\left(\frac{uq_p(n)}{p}\right) e\left(\frac{zn}{p^2}\right) + O(p) \ll p(\log p)^2. \end{aligned}$$

Hence,

$$W\left(\mathcal{R}, p^2\right) \ll p(\log p)^2.$$

For integers d_1, d_2 and M with $0 \le d_1 < d_2 \le p^2 - M$, from (3.2) and Proposition 2.3, we get

$$\sum_{n=1}^{M} e_{n+d_1}e_{n+d_2} = \sum_{n=1}^{M}\left(\frac{1}{p}\sum_{1\le|u_1|\le\frac{p-1}{2}}\left(\sum_{h_1=r}^{r+s-1} e\Big(-\frac{u_1h_1}{p}\Big)\right)e\left(\frac{u_1q_p(n+d_1)}{p}\right)+O\left(\frac{1}{p}\right)\right)$$
$$\times\left(\frac{1}{p}\sum_{1\le|u_2|\le\frac{p-1}{2}}\left(\sum_{h_2=r}^{r+s-1} e\Big(-\frac{u_2h_2}{p}\Big)\right)e\left(\frac{u_2q_p(n+d_2)}{p}\right)+O\left(\frac{1}{p}\right)\right)$$
$$=\frac{1}{p^2}\sum_{1\le|u_1|\le\frac{p-1}{2}}\left(\sum_{h_1=r}^{r+s-1} e\Big(-\frac{u_1h_1}{p}\Big)\right)\sum_{1\le|u_2|\le\frac{p-1}{2}}\left(\sum_{h_2=r}^{r+s-1} e\Big(-\frac{u_2h_2}{p}\Big)\right)$$
$$\times\sum_{n=1}^{M} e\left(\frac{u_1q_p(n+d_1)+u_2q_p(n+d_2)}{p}\right)+O\left(p\log p\right)\ll p(\log p)^3.$$

Therefore,

$$C_2\left(\mathcal{R},p^2\right)\ll p(\log p)^3.$$

For integers d_1,d_2,d_3 and M with $0\le d_1<d_2<d_3\le p^2-M$, by (3.2), we have

$$\sum_{n=1}^{M} e_{n+d_1}e_{n+d_2}e_{n+d_3}$$
$$=\sum_{n=1}^{M}\left(\frac{1}{p}\sum_{1\le|u_1|\le\frac{p-1}{2}}\left(\sum_{h_1=r}^{r+s-1} e\left(-\frac{u_1h_1}{p}\right)\right)e\left(\frac{u_1q_p(n+d_1)}{p}\right)+O\left(\frac{1}{p}\right)\right)$$
$$\times\left(\frac{1}{p}\sum_{1\le|u_2|\le\frac{p-1}{2}}\left(\sum_{h_2=r}^{r+s-1} e\left(-\frac{u_2h_2}{p}\right)\right)e\left(\frac{u_2q_p(n+d_2)}{p}\right)+O\left(\frac{1}{p}\right)\right)$$
$$\times\left(\frac{1}{p}\sum_{1\le|u_3|\le\frac{p-1}{2}}\left(\sum_{h_3=r}^{r+s-1} e\left(-\frac{u_3h_3}{p}\right)\right)e\left(\frac{u_3q_p(n+d_3)}{p}\right)+O\left(\frac{1}{p}\right)\right)$$
$$=\frac{1}{p^3}\sum_{1\le|u_1|\le\frac{p-1}{2}}\left(\sum_{h_1=r}^{r+s-1} e\left(-\frac{u_1h_1}{p}\right)\right)\sum_{1\le|u_2|\le\frac{p-1}{2}}\left(\sum_{h_2=r}^{r+s-1} e\left(-\frac{u_2h_2}{p}\right)\right)$$
$$\times\sum_{1\le|u_3|\le\frac{p-1}{2}}\left(\sum_{h_3=r}^{r+s-1} e\left(-\frac{u_3h_3}{p}\right)\right)$$
$$\times\sum_{n=1}^{M} e\left(\frac{u_1q_p(n+d_1)+u_2q_p(n+d_2)+u_3q_p(n+d_3)}{p}\right)+O\left(p\left(\log p\right)^2\right).$$

We consider d_1, d_2, d_3 in two cases.

Case 1. $d_1 \equiv d_2 \equiv d_3 \pmod p$ does not hold. Then from Proposition 2.3, we get

$$\sum_{n=1}^{M} e_{n+d_1} e_{n+d_2} e_{n+d_3} \ll p^{\frac{3}{2}} (\log p)^3.$$

Case 2. $d_1 \equiv d_2 \equiv d_3 \pmod p$. Write $d_2 = d_1 + k_2 p$, $d_3 = d_1 + k_3 p$, where $0 < k_2 < k_3 < p$. Then from Theorem 2.2, we have

$$\begin{aligned}\sum_{n=1}^{M} e_{n+d_1} e_{n+d_2} e_{n+d_3} &= \frac{M}{p^3} \sum_{1 \le |u_1| \le \frac{p-1}{2}} \sum_{1 \le |u_2| \le \frac{p-1}{2}} \sum_{1 \le |u_3| \le \frac{p-1}{2}} \left(\sum_{h_1=r}^{r+s-1} e\left(-\frac{u_1 h_1}{p}\right)\right) \\ &\qquad \begin{matrix} u_1+u_2+u_3 \equiv 0 \pmod p \\ u_2 k_2 + u_3 k_3 \equiv 0 \pmod p \end{matrix} \\ &\quad \times \left(\sum_{h_2=r}^{r+s-1} e\left(-\frac{u_2 h_2}{p}\right)\right) \left(\sum_{h_3=r}^{r+s-1} e\left(-\frac{u_3 h_3}{p}\right)\right) + O\left(p (\log p)^4\right) \\ &= \frac{M}{p^3} \sum_{h_1=r}^{r+s-1} \sum_{h_2=r}^{r+s-1} \sum_{h_3=r}^{r+s-1} \sum_{1 \le |u| \le \frac{p-1}{2}} \\ &\quad \times e\left(\frac{u\left((1-k_3 k_2^{-1}) h_1 + k_3 k_2^{-1} h_2 - h_3\right)}{p}\right) + O\left(p (\log p)^4\right). \end{aligned} \tag{3.3}$$

Assume that $s = o(p)$. Then

$$\begin{aligned}\sum_{n=1}^{M} e_{n+d_1} e_{n+d_2} e_{n+d_3} &= \frac{M}{p^2} \sum_{\substack{h_1=r}}^{r+s-1} \sum_{h_2=r}^{r+s-1} \sum_{h_3=r}^{r+s-1} 1 - \frac{M s^3}{p^3} + O\left(p (\log p)^4\right) \\ &\qquad (1-k_3 k_2^{-1}) h_1 + k_3 k_2^{-1} h_2 - h_3 \equiv 0 \pmod p \\ &\ll \frac{M s^2}{p^2} + p (\log p)^4. \end{aligned}$$

Therefore,

$$C_3\left(\mathcal{R}, p^2\right) \ll s^2 + p^{\frac{3}{2}} (\log p)^3.$$

Now taking $M = p^2 - 2p$, $d_1 = 0$, $d_2 = p$ and $d_3 = 2p$ in (3.3), we have

$$\begin{aligned}&\sum_{n=1}^{p^2-2p} e_{n+d_1} e_{n+d_2} e_{n+d_3} \\ &= \frac{p^2-2p}{p^3} \sum_{h_1=r}^{r+s-1} \sum_{h_2=r}^{r+s-1} \sum_{h_3=r}^{r+s-1} \sum_{1 \le |u| \le \frac{p-1}{2}} e\left(\frac{u (2h_2 - h_1 - h_3)}{p}\right) + O\left(p (\log p)^4\right) = \end{aligned}$$

$$= \frac{p^2-2p}{p^3} \sum_{h_1=0}^{s-1} \sum_{h_2=0}^{s-1} \sum_{h_3=0}^{s-1} \sum_{1 \le |u| \le \frac{p-1}{2}} e\left(\frac{u\,(2h_2-h_1-h_3)}{p}\right) + O\left(p\,(\log p)^4\right)$$

$$= \frac{p^2-2p}{p^2} \sum_{\substack{h_1=0 \\ h_1+h_3=2h_2}}^{s-1} \sum_{h_2=0}^{s-1} \sum_{h_3=0}^{s-1} 1 - \frac{p^2-2p}{p^3} s^3 + O\left(p\,(\log p)^4\right).$$

Noting that

$$\sum_{\substack{h_1=0 \\ h_1+h_3=2h_2}}^{s-1} \sum_{h_2=0}^{s-1} \sum_{h_3=0}^{s-1} 1 = \sum_{h_2=0}^{s-1} \sum_{\substack{h_1=0 \\ h_1+h_3=2h_2}}^{s-1} \sum_{h_3=0}^{s-1} 1$$

$$= \sum_{0 \le 2h_2 \le s-1} (2h_2+1) + \sum_{s \le 2h_2 \le 2s-2} (2s-1-2h_2)$$

$$= \sum_{\substack{1 \le t \le s \\ 2 \nmid t}} t + \sum_{\substack{1 \le t \le s-1 \\ 2 \nmid t}} t = \begin{cases} \dfrac{s^2+1}{2}, & \text{if } 2 \nmid s, \\ \dfrac{s^2}{2}, & \text{if } 2 \mid s. \end{cases}$$

Thus we have

$$\sum_{n=1}^{p^2-2p} e_{n+d_1} e_{n+d_2} e_{n+d_3} = \frac{s^2}{2} - \frac{s^3}{p} + O\left(p\,(\log p)^4\right).$$

Assume that $s = \left[\frac{p}{3}\right]$. We immediately get

$$\sum_{n=1}^{p^2-2p} e_{n+d_1} e_{n+d_2} e_{n+d_3} = \frac{1}{54} p^2 + O\left(p\,(\log p)^4\right).$$

Therefore,

$$C_3\left(\mathcal{R}, p^2\right) \ge \frac{1}{54} p^2 + O\left(p\,(\log p)^4\right).$$

This proves Theorem 1.1.

4. Proof of Theorem 1.2

From (3.1), we get

$$|\mathcal{R}| = \frac{(p-1)^2}{d}.$$

On the other hand, we denote by χ_0 the principal character modulo p. Assume that $q_p(n) \neq 0$. By the orthogonality of characters of order d, we have

$$\frac{1}{d}\sum_{\substack{\chi^d=\chi_0\\ \chi \bmod p}} \chi\left(q_p(n)\right) = \begin{cases} 1, & \text{if } \exists\ y \text{ such that } 1 \le y \le p-1 \text{ and } q_p(n) \equiv y^d (\bmod\ p),\\ 0, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} 1, & n \in \mathcal{R},\\ 0, & n \notin \mathcal{R}. \end{cases}$$

Hence,

$$e_n = \frac{1}{d}\sum_{\substack{\chi^d=\chi_0\\ \chi \bmod p}} \chi\left(q_p(n)\right) - \frac{|\mathcal{R}|}{p^2} = \frac{1}{d}\sum_{\substack{\chi^d=\chi_0\\ \chi\neq\chi_0\\ \chi \bmod p}} \chi\left(q_p(n)\right) + O\left(\frac{1}{p}\right). \tag{4.1}$$

For a, b, $t \in \mathbb{N}$ with $1 \le a \le a + (t-1)b \le p^2$, from (4.1) and (2.1), we get

$$\sum_{j=0}^{t-1} e_{a+jb} = \sum_{j=0}^{t-1}\left(\frac{1}{d}\sum_{\substack{\chi^d=\chi_0\\ \chi\neq\chi_0\\ \chi \bmod p}} \chi\left(q_p(a+jb)\right) + O\left(\frac{1}{p}\right)\right)$$

$$= \frac{1}{d}\sum_{\substack{\chi^d=\chi_0\\ \chi\neq\chi_0\\ \chi \bmod p}} \sum_{j=0}^{t-1} \chi\left(q_p(a+jb)\right) + O\left(p\right) \ll p^{\frac{3}{2}}\log p.$$

Therefore,

$$W\left(\mathcal{R}, p^2\right) \ll p^{\frac{3}{2}}\log p.$$

For integers $d_1, d_2, \ldots, d_k$ and M with $0 \le d_1 < d_2 < \cdots < d_k \le p^2 - M$, from (4.1) and Proposition 2.4, we have

$$\sum_{n=1}^{M} e_{n+d_1}\cdots e_{n+d_k}$$

$$= \sum_{n=1}^{M}\left(\frac{1}{d}\sum_{\substack{\chi_1^d=\chi_0\\ \chi_1\neq\chi_0\\ \chi_1 \bmod p}} \chi_1\left(q_p(n+d_1)\right) + O\left(\frac{1}{p}\right)\right)\times\cdots$$

$$\times \left(\frac{1}{d} \sum_{\substack{\chi_k^d = \chi_0 \\ \chi_k \neq \chi_0 \\ \chi_k \bmod p}} \chi_k\left(q_p(n+d_k)\right) + O\left(\frac{1}{p}\right) \right)$$

$$= \frac{1}{d^k} \sum_{\substack{\chi_1^d = \chi_0 \\ \chi_1 \neq \chi_0 \\ \chi_1 \bmod p}} \cdots \sum_{\substack{\chi_k^d = \chi_0 \\ \chi_k \neq \chi_0 \\ \chi_k \bmod p}} \sum_{n=1}^{M} \chi_1\left(q_p(n+d_1)\right) \cdots \chi_k\left(q_p(n+d_k)\right) + O\left(kp\right) \ll kp^{\frac{5}{3}}.$$

Hence,

$$C_k\left(\mathcal{R}, p^2\right) \ll kp^{\frac{5}{3}}.$$

This completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3

By (3.1), we have

$$|\mathcal{R}| = (p-1)\phi(p-1).$$

Suppose that $q_p(n) \neq 0$. Noting that

$$\frac{\phi(p-1)}{p-1} \sum_{s|p-1} \frac{\mu(s)}{\phi(s)} \sum_{\substack{\text{ord } \chi = s \\ \chi \bmod p}} \chi(q_p(n)) = \begin{cases} 1, & \text{if } q_p(n) \text{ is a primitive root modulo } p, \\ 0, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} ll1, & n \in \mathcal{R}, \\ 0, & n \notin \mathcal{R}. \end{cases}$$

Thus, we have

$$e_n = \frac{\phi(p-1)}{p-1} \sum_{s|p-1} \frac{\mu(s)}{\phi(s)} \sum_{\substack{\text{ord } \chi = s \\ \chi \bmod p}} \chi\left(q_p(n)\right) - \frac{|\mathcal{R}|}{p^2}$$

$$= \frac{\phi(p-1)}{p-1} \sum_{\substack{s|p-1 \\ s>1}} \frac{\mu(s)}{\phi(s)} \sum_{\substack{\text{ord } \chi = s \\ \chi \bmod p}} \chi\left(q_p(n)\right) + O\left(\frac{1}{p}\right). \tag{5.1}$$

For a, b, $t \in \mathbb{N}$ with $1 \le a \le a + (t-1)b \le p^2$, from (5.1) and (2.1), we get

$$\begin{aligned}\sum_{j=0}^{t-1} e_{a+jb} &= \sum_{j=0}^{t-1}\left(\frac{\phi(p-1)}{p-1}\sum_{\substack{s|p-1\\ s>1}}\frac{\mu(s)}{\phi(s)}\sum_{\substack{\operatorname{ord}\ \chi=s\\ \chi \bmod p}}\chi\left(q_p(a+jb)\right)+O\left(\frac{1}{p}\right)\right)\\ &= \frac{\phi(p-1)}{p-1}\sum_{\substack{s|p-1\\ s>1}}\frac{\mu(s)}{\phi(s)}\sum_{\substack{\operatorname{ord}\ \chi=s\\ \chi \bmod p}}\sum_{j=0}^{t-1}\chi\left(q_p(a+jb)\right)+O\left(p\right)\\ &\ll 2^{\omega(p-1)}p^{\frac{3}{2}}\log p.\end{aligned}$$

Therefore,

$$W\left(\mathcal{R}, p^2\right) \ll 2^{\omega(p-1)}p^{\frac{3}{2}}\log p.$$

For integers $d_1, d_2, \cdots, d_k$ and M with $0 \le d_1 < d_2 < \cdots < d_k \le p^2 - M$, from (5.1) and Proposition 2.4 we have

$$\begin{aligned}&\sum_{n=1}^{M} e_{n+d_1}\cdots e_{n+d_k}\\ &= \sum_{n=1}^{M}\left(\frac{\phi(p-1)}{p-1}\sum_{\substack{s_1|p-1\\ s_1>1}}\frac{\mu(s_1)}{\phi(s_1)}\sum_{\substack{\operatorname{ord}\ \chi_1=s_1\\ \chi_1\neq\chi_0\\ \chi_1 \bmod p}}\chi_1\left(q_p(n+d_1)\right)+O\left(\frac{1}{p}\right)\right)\times\cdots\\ &\quad\times\left(\frac{\phi(p-1)}{p-1}\sum_{\substack{s_k|p-1\\ s_k>1}}\frac{\mu(s_k)}{\phi(s_k)}\sum_{\substack{\operatorname{ord}\ \chi_k=s_k\\ \chi_k\neq\chi_0\\ \chi_k \bmod p}}\chi_k\left(q_p(n+d_k)\right)+O\left(\frac{1}{p}\right)\right)\\ &= \frac{\phi^k(p-1)}{(p-1)^k}\sum_{\substack{s_1|p-1\\ s_1>1}}\frac{\mu(s_1)}{\phi(s_1)}\sum_{\substack{\operatorname{ord}\ \chi_1=s_1\\ \chi_1\neq\chi_0\\ \chi_1 \bmod p}}\cdots\sum_{\substack{s_k|p-1\\ s_k>1}}\frac{\mu(s_k)}{\phi(s_k)}\sum_{\substack{\operatorname{ord}\ \chi_k=s_k\\ \chi_k\neq\chi_0\\ \chi_k \bmod p}}\\ &\quad\times\sum_{n=1}^{M}\chi_1\left(q_p(n+d_1)\right)\cdots\chi_k\left(q_p(n+d_k)\right)+O\left(k2^{k\omega(p-1)}p\right)\ll k2^{k\omega(p-1)}p^{\frac{5}{3}}.\end{aligned}$$

Hence,

$$C_k\left(\mathcal{R}, p^2\right) \ll k2^{k\omega(p-1)}p^{\frac{5}{3}}.$$

This completes the proof of Theorem 1.3.

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