

Finite groups with two relative subgroup commutativity degrees

By MIHAI-SILVIU LAZOREC (Iași) and MARIUS TĂRNĂUCEANU (Iași)

Abstract. In this paper, we show that there is an infinite number of finite groups with two relative subgroup commutativity degrees. Also, we indicate a sufficient condition such that a finite group has at least three relative subgroup commutativity degrees, and we prove that D_6 is the only finite dihedral group with two relative subgroup commutativity degrees. Finally, we study the density of the set containing all relative subgroup commutativity degrees of finite groups.

1. Introduction

The large amount of results (see [2]–[3], [6]–[10], [12], [14]) obtained during the last decades using the commutativity degree of a finite group G determined the introduction and study of some new probabilistic aspects of finite group theory. In this paper, we will focus on two such concepts, which were introduced in [19] and [20]. More exactly, we will prove some results related to the relative subgroup commutativity degree of a subgroup H of G defined by

$$sd(H, G) = \frac{1}{|L(H)||L(G)|} |\{(H_1, G_1) \in L(H) \times L(G) \mid H_1 G_1 = G_1 H_1\}|,$$

which is a generalization of the subgroup commutativity degree of G that is given by

$$sd(G) = \frac{1}{|L(G)|^2} |\{(H, K) \in L(G)^2 \mid HK = KH\}|,$$

where $L(G)$ denotes the subgroup lattice of G .

Mathematics Subject Classification: Primary: 20D60, 20P05; Secondary: 20D30, 20E45.

Key words and phrases: subgroup commutativity degree, relative subgroup commutativity degree, subgroup lattice.

We recall that if $G = \bigtimes_{i=1}^k G_i$, where $(G_i)_{i=1,k}$ is a family of finite groups having coprime orders, the subgroup lattice of G is decomposable. Hence, every subgroup H of G can be written as $H = \bigtimes_{i=1}^k H_i$, where $H_i \in L(G_i)$, $\forall i = 1, 2, \dots, k$. Moreover, we have

$$sd(G) = \prod_{i=1}^k sd(G_i) \quad \text{and} \quad sd(H, G) = \prod_{i=1}^k sd(H_i, G_i).$$

A relevant property of the function

$$f : L(G) \longrightarrow [0, 1] \quad \text{given by} \quad f(H) = sd(H, G),$$

is that it is constant on each conjugacy class of subgroups of G . It is easy to see that $|\text{Im } f| = 1$ if and only if G is an Iwasawa group (i.e., a nilpotent modular group). Having in mind these results, our aim is to study the following class of groups:

$$\mathcal{C} = \{G = \text{finite group} \mid |\text{Im } f| = 2\}.$$

We note that a similar problem involving the number of non-zero values taken by a generalization of the commutativity degree of a finite group is illustrated in [13].

The paper is organized as follows. Section 2 describes all finite groups with two relative commutativity degrees under the additional assumption that they have one or two conjugacy classes of non-normal subgroups. A consequence of this study is the fact that $\mathcal{C}$ contains an infinite number of groups. We prove a criterion that indicates which groups do not belong to $\mathcal{C}$ in Section 3. Using this result, we detect some well-known classes of groups which are not included in $\mathcal{C}$. Also, we show that $S_3 \cong D_6$ is the unique finite dihedral group contained in $\mathcal{C}$. We end our paper by studying the density of the set $\{sd(H, G) \mid G = \text{finite group}, H \in L(G)\}$ in $[0, 1]$.

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [5], [16]. For subgroup lattice concepts, we refer the reader to [15], [17]–[18].

2. The connection between $\mathcal{C}$ and the conjugacy classes of non-normal subgroups of a finite group

For a finite group G , we denote the number of its conjugacy classes of non-normal subgroups and its normal subgroup lattice by $\gamma(G)$ and $N(G)$, respectively. We recall that any normal subgroup of G is permutable, i.e., it permutes

with all other subgroups of G . It is obvious that $sd(\{1\}, G) = 1$. Also, in general, once that $\gamma(G)$ increases, the probability that G has more than 2 relative subgroup commutativity degrees also increases. Therefore, it is natural to think that a group G may belong to $\mathcal{C}$ if $\gamma(G) = 1$ or $\gamma(G) = 2$. This additional hypothesis will help us in our study, because all finite groups having one or two conjugacy classes of non-normal subgroups were described in [1] and [11], respectively. More exactly, the following two theorems were proved.

Theorem 2.1. *Let G be a finite group. Then $\gamma(G) = 1$ if and only if G is isomorphic to one of the following groups:*

- (1) $\mathbb{Z}_p \rtimes \mathbb{Z}_{q^n}$, where $[\mathbb{Z}_p, \Phi(\mathbb{Z}_{q^n})] = 1$, p, q are primes such that $q|p-1$ and n is a positive integer;
- (2) $M(p^n) = \langle x, y \mid x^{p^{n-1}} = y^p = 1, x^y = x^{1+p^{n-2}} \rangle$, where p is a prime and $n \geq 3$ if $p \geq 3$ or $n \geq 4$ if $p = 2$.

Theorem 2.2. *Let G be a finite group. Then $\gamma(G) = 2$ if and only if G is isomorphic to one of the following groups:*

- (1) A_4 ;
- (2) $\langle x, y \mid x^q = y^{p^n} = 1, x^y = x^k \rangle$, where p, q are prime numbers such that $p^2|q-1$, $n > 1$ and $k^{p^2} \equiv 1 \pmod{q}$ with $k \neq 1$;
- (3) $\langle x, y, z \mid x^r = y^{p^n} = z^q = [x, z] = [y, z] = 1, x^y = x^k \rangle$, where p, q, r are prime numbers such that $p \neq q$, $q \neq r$, $p|r-1$ and $k^p \equiv 1 \pmod{r}$ with $k \neq 1$;¹
- (4) $\langle x, y \mid x^{q^2} = y^{p^n} = 1, x^y = x^k \rangle$, where p, q are primes such that $p|q-1$ and $k^p \equiv 1 \pmod{q^2}$ with $k \neq 1$;
- (5) $M(p^n) \times \mathbb{Z}_q$, where p, q are primes such that $p \neq q$ and $n \geq 3$ if $p \geq 3$ or $n \geq 4$ if $p = 2$;
- (6) $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$;
- (7) Q_{16} ;
- (8) $\langle x, y \mid x^4 = y^{2^n} = 1, y^x = y^{1+2^{n-1}} \rangle$, where $n \geq 3$;
- (9) D_8 .

Our aim is to pass through the two lists of groups and find the ones that are contained in $\mathcal{C}$. Some answers are provided by our next two results.

¹Note that the condition $q \neq r$ must be added to the group (3) in [11, Theorem 1], as shows the example $G = S_3 \times \mathbb{Z}_3$; in this case, we have $q = r = 3$, but $\gamma(G) = 3$.

Theorem 2.3. *Let G be a finite group such that $\gamma(G) = 1$. Then $G \in \mathcal{C}$ if and only if $G \cong S_3$.*

PROOF. We saw that there are only two classes of groups having the property $\gamma(G) = 1$. The modular p -groups $M(p^n)$ are Iwasawa, so $|\text{Im } f| = 1$. Consequently, they are not contained in $\mathcal{C}$. We assume that $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{q^n}$, where p, q are primes such that $q|p-1$. There are p subgroups isomorphic to $\mathbb{Z}_{q^n}$, and they form the conjugacy class of non-normal subgroups of G . Two different conjugates cannot commute, since G does not contain a subgroup of order q^{n+1} . Therefore,

$$sd(\mathbb{Z}_{q^n}, G) = \frac{|L(\mathbb{Z}_{q^{n-1}})||L(G)| + |N(G)| + 1}{|L(\mathbb{Z}_{q^n})||L(G)|} = \frac{n(2n+p+1) + 2(n+1)}{(n+1)(2n+p+1)},$$

while

$$sd(G) = \frac{|N(G)||L(G)| + p(|N(G)| + 1)}{|L(G)|^2} = \frac{(2n+1)(2n+p+1) + 2p(n+1)}{(2n+p+1)^2}.$$

Then

$$G \in \mathcal{C} \iff sd(\mathbb{Z}_{q^n}, G) = sd(G) \iff p = 2 + \frac{1}{n} \text{ or } p = 1.$$

But p, q are primes such that $q|p-1$, so $(p, q, n) = (3, 2, 1)$. Consequently, $G \in \mathcal{C}$ if and only if $G \cong S_3$. For any other triplet (p, q, n) , we have $|\text{Im } f| = 3$. $\square$

Theorem 2.4. *Let G be a finite group with $\gamma(G) = 2$. Then $G \in \mathcal{C}$ if and only if $G \cong S_3 \times \mathbb{Z}_q$, where $q \geq 5$ is a prime number.*

PROOF. We are moving our attention to the list of groups provided by Theorem 2.2. First, we will inspect the nilpotent groups that have two non-normal conjugacy classes of subgroups. It is easy to see that if $G \cong Q_{16}$ or $G \cong D_8$, we obtain that $|\text{Im } f| = 3$. Also, if $G \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_4$, we get $|\text{Im } f| = 4$. Going further, for a triplet (p, q, n) chosen as it is indicated in Theorem 2.2, since $M(p^n)$ and $\mathbb{Z}_q$ are Iwasawa groups having coprime orders, the group $G \cong M(p^n) \times \mathbb{Z}_q$ is also Iwasawa, so $|\text{Im } f| = 1$. If G is a group of type (8), then it is a non-hamiltonian 2-group that contains an abelian normal subgroup $N \cong \mathbb{Z}_{2^n}$. According to [17, Theorem 9], since $\frac{G}{N}$ is cyclic and there exist $x \in G$ and an integer $m = n-1 \geq 2$ such that $G = \langle N, x \rangle$ and $g^x = g^{1+2^m}$, $\forall g \in N$, we deduce that G is modular. Consequently, G is an Iwasawa group and we have $|\text{Im } f| = 1$.

The groups that are not covered yet are non-nilpotent. It is easy to see that $|\text{Im } f| = 5$ for $G \cong A_4$. If G is a group of type (2), then $G \cong \mathbb{Z}_q \rtimes \mathbb{Z}_{p^n}$, where p, q are primes such that $p^2|q-1$ and $n > 1$. There are q subgroups isomorphic to $\mathbb{Z}_{p^n}$ and q subgroups isomorphic to $\mathbb{Z}_{p^{n-1}}$, and they form the two

non-normal conjugacy classes of subgroups of G . We denote them by C_1 and C_2 , respectively. Since $\mathbb{Z}_{p^n}$ is a cyclic p -group, it contains one subgroup isomorphic to $\mathbb{Z}_{p^{n-1}}$. Therefore, two different conjugates contained in C_2 cannot commute. Also, the same thing can be said about two different conjugates taken from C_1 . Then, we have

$$\begin{aligned} sd(\mathbb{Z}_{p^{n-1}}, G) &= \frac{|L(\mathbb{Z}_{p^{n-2}})||L(G)| + |N(G)| + 2}{|L(\mathbb{Z}_{p^{n-1}})||L(G)|} = \frac{(n-1)(n+q) + n + 1}{n(n+q)}, \\ sd(\mathbb{Z}_{p^n}, G) &= \frac{|L(\mathbb{Z}_{p^{n-2}})||L(G)| + 2(|N(G)| + 2)}{|L(\mathbb{Z}_{p^n})||L(G)|} = \frac{(n-1)(n+q) + 2(n+1)}{(n+1)(n+q)}, \\ sd(\mathbb{Z}_q \rtimes \mathbb{Z}_{p^{n-1}}, G) &= \frac{(2n-1)|L(G)| + q(|N(G)| + 2)}{|L(\mathbb{Z}_q \rtimes \mathbb{Z}_{p^{n-1}})||L(G)|} = \frac{(2n-1)(n+q) + q(n+1)}{(2n+q-1)(n+q)}, \\ sd(G) &= \frac{|N(G)||L(G)| + 2q(|N(G)| + 2)}{|L(G)|^2} = \frac{n(n+q) + q(n+1)}{(n+q)^2}. \end{aligned}$$

Now, to find $|\text{Im } f|$, we must study when two of the above (relative) subgroup commutativity degrees are equal, under the assumptions that p, q are primes such that $p^2|q-1$ and $n > 1$. After making some computation, we deduce that $|\text{Im } f| = 4$ if and only if $(p, q, n) = (2, 5, 4)$, and $|\text{Im } f| = 5$ if and only if $(p, q, n) \neq (2, 5, 4)$.

A group G of type (4) is isomorphic to $\mathbb{Z}_{q^2} \rtimes \mathbb{Z}_{p^n}$, where p, q are primes such that $p|q-1$ and n is a positive integer. There are q subgroups isomorphic to $\mathbb{Z}_q \rtimes \mathbb{Z}_{p^n}$ which form a non-normal conjugacy class of subgroups. Each of them has q subgroups isomorphic to $\mathbb{Z}_{p^n}$, so the second non-normal conjugacy class contains q^2 subgroups. The reader can check if two non-normal subgroups commute using similar techniques with the ones applied previously. We obtain the following results:

$$\begin{aligned} sd(\mathbb{Z}_{p^n}, G) &= \frac{|L(\mathbb{Z}_{p^{n-1}})||L(G)| + |N(G)| + 2}{|L(\mathbb{Z}_{p^n})||L(G)|} = \frac{n(3n+q^2+q+1) + 3(n+1)}{(n+1)(3n+q^2+q+1)}, \\ sd(\mathbb{Z}_q \rtimes \mathbb{Z}_{p^n}, G) &= \frac{|L(\mathbb{Z}_{p^{n-1}q})||L(G)| + q(|N(G)| + 2) + |N(G)| + q + 1}{|L(\mathbb{Z}_q \rtimes \mathbb{Z}_{p^n})||L(G)|} \\ &= \frac{2n(3n+q^2+q+1) + 3q(n+1) + 3n+q+2}{(2n+q+1)(3n+q^2+q+1)}, \\ sd(G) &= \frac{|N(G)||L(G)| + q^2(|N(G)| + 2) + q(|N(G)| + q + 1)}{|L(G)|^2} \\ &= \frac{(3n+1)(3n+q^2+q+1) + 3q^2(n+1) + q(3n+q+2)}{(3n+q^2+q+1)^2}. \end{aligned}$$

The properties of the triplet (p, q, n) lead to stating that the above 3 (relative) subgroup commutativity degrees are different. Consequently, $|\text{Im } f| = 4$ for a group of type (4).

Finally, if G is a group of type (3), then $G \cong (\mathbb{Z}_r \rtimes \mathbb{Z}_{p^n}) \times \mathbb{Z}_q$, where p, q, r are prime numbers such that $p \neq q$, $q \neq r$, $p|r-1$, and n is a positive integer. We remark that the group $\mathbb{Z}_r \rtimes \mathbb{Z}_{p^n}$ is exactly the non-nilpotent group having one conjugacy class of non-normal subgroups described in Theorem 2.1. Also, it is obvious that $(rp, q) = 1$. Consequently, the first conjugacy class of non-normal subgroups contains r subgroups isomorphic to $\mathbb{Z}_{p^n}$, while the second class is formed by r subgroups isomorphic to $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$. Moreover, we have

$$\begin{aligned} sd(\mathbb{Z}_{p^n}, G) &= sd(\mathbb{Z}_{p^n}, \mathbb{Z}_r \rtimes \mathbb{Z}_{p^n})sd(\{1\}, \mathbb{Z}_q) = \frac{n(2n+r+1) + 2(n+1)}{(n+1)(2n+r+1)}, \\ sd(\mathbb{Z}_{p^n} \times \mathbb{Z}_q, G) &= sd(\mathbb{Z}_{p^n}, \mathbb{Z}_r \rtimes \mathbb{Z}_{p^n})sd(\mathbb{Z}_q) = \frac{n(2n+r+1) + 2(n+1)}{(n+1)(2n+r+1)}, \\ sd(\mathbb{Z}_r \rtimes \mathbb{Z}_{p^n}, G) &= sd(\mathbb{Z}_r \rtimes \mathbb{Z}_{p^n})sd(\{1\}, \mathbb{Z}_q) = \frac{(2n+1)(2n+r+1) + 2r(n+1)}{(2n+r+1)^2}, \\ sd(G) &= sd(\mathbb{Z}_r \rtimes \mathbb{Z}_{p^n})sd(\mathbb{Z}_q) = \frac{(2n+1)(2n+r+1) + 2r(n+1)}{(2n+r+1)^2}. \end{aligned}$$

Then

$$G \in \mathcal{C} \iff sd(G) = sd(\mathbb{Z}_{p^n}, G) \iff r = 2 + \frac{1}{n} \text{ or } r = 1.$$

Following the properties of p, q, r and n , we have $G \in \mathcal{C}$ if and only if $G \cong S_3 \times \mathbb{Z}_q$, where q is a prime number such that $q \geq 5$. We finish our proof by stating that for a group of type (3), we get $|\text{Im } f| = 3$ for all other possible choices of the quadruplet (p, q, r, n) . $\square$

Looking to our last proof, we deduce that even if $\gamma(G) = n$, where $n \geq 3$ is an integer, we always find a group G that belongs to $\mathcal{C}$. An example would be $G \cong S_3 \times \mathbb{Z}_{5^{n-1}}$. Also, our previous proof indicates that we can generalize our last result by replacing $\mathbb{Z}_q$, where $q \geq 5$ is a prime number, with any Iwasawa group G such that 6 and $|G|$ are relatively prime.

Corollary 2.5. *Let G be a finite Iwasawa group such that $(6, |G|) = 1$. Then $S_3 \times G$ is contained in $\mathcal{C}$.*

PROOF. The subgroup lattice of $S_3 \times G$ is decomposable since $(6, |G|) = 1$. Hence, any subgroup of $S_3 \times G$ is of type $H_1 \times H_2$, where H_1 and H_2 are subgroups of S_3 and G , respectively. Since G is an Iwasawa group and the only conjugacy class of non-normal subgroups of S_3 contains 3 subgroups isomorphic

to $\mathbb{Z}_2$, the subgroups of $S_3 \times G$ that are not permutable are of type $\mathbb{Z}_2 \times H$, where $H \in L(G)$. Consequently, in this case, the function f takes only 2 values: 1 and $\frac{5}{6}$. Therefore, $|\text{Im } f| = 2$, so $S_3 \times G \in \mathcal{C}$. $\square$

We end this section by pointing out that we did not fully describe the set $\mathcal{C}$, and another remark is that we did not find any nilpotent group that belongs to this set. Hence, it is natural to indicate the following two open problems:

Problem 2.6. *Show that*

$$\mathcal{C} = \{S_3 \times G \mid G = \text{finite Iwasawa group such that } (6, |G|) = 1\}.$$

Problem 2.7. *Does $\mathcal{C}$ contain nilpotent groups?*

3. How can we find finite groups that do not belong to $\mathcal{C}$?

In the previous section, we used the number of conjugacy classes of non-normal subgroups of a finite group G to establish that $\mathcal{C}$ contains an infinite number of groups. Our next aim is to find a condition which guarantees that a finite group G is not a part of $\mathcal{C}$. For a subgroup H of G , we will consider the following set

$$C(H) = \{K \in L(G) \mid HK = KH\}.$$

Proposition 3.1. *Let G be a finite group. If $sd(G) < \frac{1}{2} + \frac{|N(G)|+1}{2|L(G)|}$, then $|\text{Im } f| > 2$.*

PROOF. Assume that $|\text{Im } f| \leq 2$. Then $|\text{Im } f| = 2$, since $|\text{Im } f| = 1$ would imply that G is Iwasawa and this is contradicting our hypothesis. Let H be a minimal subgroup of G such that $sd(H, G) \neq 1$. Then $sd(H, G) = sd(G)$ as a consequence of the fact that $|\text{Im } f| = 2$. On the other hand, we have $sd(K, G) = 1$, for any proper subgroup K of H . Hence,

$$\begin{aligned} sd(G) &= sd(H, G) = \frac{1}{|L(H)||L(G)|} \sum_{K \in L(H)} |C(K)| \\ &= \frac{1}{|L(H)||L(G)|} [(|L(H)| - 1)|L(G)| + |C(H)|] \\ &\geq \frac{1}{|L(H)||L(G)|} [(|L(H)| - 1)|L(G)| + |N(G)| + 1] \\ &= 1 - \frac{|L(G)| - |N(G)| - 1}{|L(H)||L(G)|} \geq 1 - \frac{|L(G)| - |N(G)| - 1}{2|L(G)|} = \frac{1}{2} + \frac{|N(G)| + 1}{2|L(G)|}, \end{aligned}$$

which again contradicts our hypothesis. This argument completes our proof. $\square$

We remark that the upper bound $\frac{1}{2} + \frac{|N(G)|+1}{2|L(G)|}$ is the best possible one, since if $G \cong S_3$, we have $sd(G) = \frac{1}{2} + \frac{|N(G)|+1}{2|L(G)|}$, but $|\text{Im } f| = 2$ as we already saw. Also, our previous result mainly states that a group G may be contained in $\mathcal{C}$ if its subgroup commutativity degree is “large”. Consequently, we expect that a class of groups is not included in $\mathcal{C}$ if its subgroup commutativity degree vanishes asymptotically. Some examples of well-known classes of groups having this property were indicated in [19]. They are

- the dihedral groups

$$D_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, yxy^{-1} = x^{2^{n-1}-1} \rangle, \quad n \geq 3,$$

- the generalized quaternion groups

$$Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^4 = 1, yxy^{-1} = x^{2^{n-1}-1} \rangle, \quad n \geq 3,$$

- the quasi-dihedral groups

$$S_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{2^{n-2}-1} \rangle, \quad n \geq 4.$$

Corollary 3.2. *The sets $\{D_{2^n}\}_{n \geq 3}$, $\{Q_{2^n}\}_{n \geq 3}$ and $\{S_{2^n}\}_{n \geq 4}$ are not included in $\mathcal{C}$.*

PROOF. It is easy to see that

$$|\text{Im } f| = \begin{cases} 1, & \text{if } G \cong Q_8, \\ 3, & \text{if } G \cong D_8, G \cong Q_{16}, G \cong S_{16}, \\ 4, & \text{if } G \cong D_{16}, G \cong Q_{32}. \end{cases}$$

Remark that, since $|\text{Im } f| \neq 2$, none of the above-listed 6 groups are contained in $\mathcal{C}$. We will check if the inequality

$$sd(G) < \frac{1}{2} + \frac{|N(G)|+1}{2|L(G)|} \tag{1}$$

holds for a finite group G isomorphic to a group contained in any of the sets $\{D_{2^n}\}_{n \geq 5}$, $\{Q_{2^n}\}_{n \geq 6}$ and $\{S_{2^n}\}_{n \geq 5}$. The following explicit formulas were indicated in [19] and [21]:

$$sd(D_{2^n}) = \frac{(n-2)2^{n+2} + n2^{n+1} + (n-1)^2 + 8}{(n-1+2^n)^2},$$

$$\begin{aligned}
sd(Q_{2^n}) &= \frac{(n-3)2^{n+1} + n2^n + (n-1)^2 + 8}{(n-1+2^{n-1})^2}, \\
sd(S_{2^n}) &= \frac{(n-3)2^{n+1} + n2^n + (3n-2)2^{n-1} + (n-1)^2 + 8}{(n-1+3 \cdot 2^{n-2})^2}, \\
|N(D_{2^n})| &= |N(Q_{2^n})| = |N(S_{2^n})| = n+3.
\end{aligned}$$

Using Wolfram|Alpha [22], we deduce that inequality (1) is satisfied for a finite group G isomorphic to a group that is an element of any of the above 3 infinite families. Hence, for such a group we have $|\text{Im } f| > 2$ as Proposition 3.1 indicates. Consequently, G does not belong to $\mathcal{C}$. $\square$

Theorem 3.3. $S_3 \cong D_6$ is the unique finite dihedral group contained in $\mathcal{C}$.

PROOF. Assume that the dihedral group

$$D_{2n} = \langle x, y \mid x^n = y^2 = 1, yxy = x^{-1} \rangle$$

belongs to $\mathcal{C}$. Then $sd(H, D_{2n}) = sd(K, D_{2n})$, for every two non-Iwasawa subgroups H and K of D_{2n} . The subgroup structure of D_{2n} is well-known: given a divisor r of n , D_{2n} possesses a subgroup isomorphic to $\mathbb{Z}_r$, namely $H_0^r = \langle x^{\frac{n}{r}} \rangle$, and $\frac{n}{r}$ subgroups isomorphic to D_{2r} , namely $H_i^r = \langle x^{\frac{n}{r}}, x^{i-1}y \rangle$, $i = 1, 2, \dots, \frac{n}{r}$. Then $|L(D_{2n})| = \tau(n) + \sigma(n)$, where $\tau(n)$ and $\sigma(n)$ are the number and the sum of all divisors of n , respectively. Also, by [19], we have $|C(H_i^r)| = \tau(n) + x_i^r$, where x_i^r is the number of solutions of

$$\frac{n}{[r, s]} \mid 2(i-j) \quad \text{with} \quad s \mid n \quad \text{and} \quad j = 1, 2, \dots, \frac{n}{s}. \quad (2)$$

Since n cannot be of type 2^m (see Corollary 3.2), we distinguish the following two cases.

Case 1. n is odd, say $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, with each p_i an odd prime.

By taking $H = H_1^1$ and $K = H_1^{p_i}$, one obtains

$$|C(H)| = 2\tau(n) \quad \text{and} \quad |C(K)| = 2\tau(n) + (p_i - 1)\tau\left(\frac{n}{p_i^{\alpha_i}}\right),$$

implying that

$$sd(H, D_{2n}) = \frac{3\tau(n) + \sigma(n)}{2(\tau(n) + \sigma(n))}$$

and

$$sd(K, D_{2n}) = \frac{(2p_i + 4)\tau(n) + 2\sigma(n) + (p_i - 1)\tau\left(\frac{n}{p_i^{\alpha_i}}\right)}{(p_i + 3)(\tau(n) + \sigma(n))}.$$

These equalities lead to

$$\sigma(n) - \tau(n) = 2\tau\left(\frac{n}{p_i^{\alpha_i}}\right). \quad (3)$$

Since i is arbitrary, we infer that $\alpha_1 = \alpha_2 = \dots = \alpha_k = \alpha$, and (3) becomes

$$\prod_{i=1}^k \frac{p_i^{\alpha+1} - 1}{p_i - 1} = (\alpha + 1)^{k-1}(\alpha + 3).$$

It is easy to see that

$$\frac{p_i^{\alpha+1} - 1}{p_i - 1} \geq (\alpha + 1)^2, \quad \forall i = 1, 2, \dots, k,$$

which implies that

$$(\alpha + 1)^{k-1}(\alpha + 3) \geq (\alpha + 1)^{2k}, \quad \text{i.e., } (\alpha + 3) \geq (\alpha + 1)^{k+1}.$$

From this inequality it follows that $k = \alpha = 1$ and $p_1 = 3$. Thus $n = 3$, as desired.

Case 2. $n = 2^m n'$, where $m \geq 1$ and $n' = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ is odd.

Similarly, by taking $H = H_1^1$ and $K = H_1^{p_i}$, one obtains

$$|C(H)| = \tau(n) + (2m + 1)\tau(n')$$

and

$$|C(K)| = \tau(n) + (2m + 1) \left[\tau(n') + (p_i - 1)\tau\left(\frac{n'}{p_i^{\alpha_i}}\right) \right],$$

implying that

$$sd(H, D_{2n}) = \frac{2\tau(n) + \sigma(n) + (2m + 1)\tau(n')}{2(\tau(n) + \sigma(n))}$$

and

$$sd(K, D_{2n}) = \frac{(p_i + 3)\tau(n) + 2\sigma(n) + (2m + 1)(p_i + 1)\tau(n') + (2m + 1)(p_i - 1)\tau\left(\frac{n'}{p_i^{\alpha_i}}\right)}{(p_i + 3)(\tau(n) + \sigma(n))},$$

respectively. Then

$$\sigma(n) - (2m + 1)\tau(n') = 2(2m + 1)\tau\left(\frac{n'}{p_i^{\alpha_i}}\right), \quad (4)$$

and again we have $\alpha_1 = \alpha_2 = \dots = \alpha_k = \alpha$. Thus (4) becomes

$$(2^{m+1} - 1)\sigma(n') = (2m + 1)(\alpha + 1)^{k-1}(\alpha + 3). \quad (5)$$

But we have $2^{m+1} - 1 \geq 2m + 1$ with equality if and only if $m = 1$, and $\sigma(n') \geq (\alpha + 1)^{k-1}(\alpha + 3)$ with equality if and only if $k = \alpha = 1$ and $p_1 = 3$. So, (5) leads to $n = 6$. In this case, we observe that

$$sd(H, D_{12}) = sd(K, D_{12}) = \frac{13}{16} \neq \frac{101}{128} = sd(D_{12}, D_{12}) = sd(D_{12})$$

(i.e., the function f has at least three distinct values 1, $\frac{13}{16}$, and $\frac{101}{128}$), a contradiction.

Hence $n = 3$, completing the proof. $\square$

4. Density result for the relative subgroup commutativity degree

Let $\alpha \in [0, 1]$. We will consider the following two sets:

$$A = \{sd(G) \mid G = \text{finite group}\}$$

and

$$B = \{sd(H, G) \mid G = \text{finite group}, H \in L(G)\}.$$

An interesting question that appeared after the subgroup commutativity degree of a finite group G was introduced was: if there exists a sequence of groups $(G_n)_{n \in \mathbb{N}}$, such that $\lim_{n \rightarrow \infty} sd(G_n) = \alpha$. In other words, is the set A dense in $[0, 1]$? This question remains open, but a positive answer is given if we work with the set B .

Theorem 4.1. *The set B is dense in $[0, 1]$.*

PROOF. Let $\alpha \in [0, 1]$. It is obvious that $\alpha = 1$ is a limit point. If $\alpha = 0$, then we choose the sequence $(D_{2^n}, D_{2^n})_{n \geq 3}$, and we have $\lim_{n \rightarrow \infty} sd(D_{2^n}, D_{2^n}) = \lim_{n \rightarrow \infty} sd(D_{2^n}) = 0$, as it was proved in [19]. Further, we consider $\alpha = \frac{a}{b} \in \mathbb{Q} \cap (0, 1)$, where a, b are positive integers with $a < b$. The proof of Theorem 2.3 provides the explicit formula that allows us to compute $sd(\mathbb{Z}_{q^n}, \mathbb{Z}_p \rtimes \mathbb{Z}_{q^n})$, where p, q are primes such that $q|p - 1$ and n is a positive integer. We remark that

$$\lim_{p \rightarrow \infty} sd(\mathbb{Z}_{q^n}, \mathbb{Z}_p \rtimes \mathbb{Z}_{q^n}) = \lim_{p \rightarrow \infty} \frac{n(2n + p + 1) + 2(n + 1)}{(n + 1)(2n + p + 1)} = \frac{n}{n + 1}.$$

Further, we consider the sequence $(q_i)_{i \in \mathbb{N}}$, where each q_i is a prime number of the form $4k + 3$, with $k \in \mathbb{N}$. Since $(4q_i, 1) = 1$, there is a sequence of primes $(p_i)_{i \in \mathbb{N}}$ of the form $4hq_i + 1$. In this way, for each prime q_i , we find a prime p_i such that $q_i | p_i - 1$. Moreover, the sequences $(p_i)_{i \in \mathbb{N}}, (q_i)_{i \in \mathbb{N}}$ are strictly increasing and $p_i \neq q_j, \forall i, j \in \mathbb{N}$. Now, let $(k_n^1), (k_n^2), \dots, (k_n^{b-a})$ be some strictly increasing and disjoint subsequences of $\mathbb{N}$. From our previous discussion about prime numbers, we can choose the sequences $(H_j^n, G_j^n)_{n \in \mathbb{N}} = (\mathbb{Z}_{q_{k_n^j}^{a+j-1}}, \mathbb{Z}_{q_{k_n^j}^{a+j-1}} \rtimes \mathbb{Z}_{p_{k_n^j}})_{n \in \mathbb{N}}$, where $j = 1, 2, \dots, b-a$. Then

$$\lim_{n \rightarrow \infty} sd(H_j^n, G_j^n) = \frac{a+j-1}{a+j}, \quad \forall j = 1, 2, \dots, b-a.$$

Moreover, we can build the sequence $\left(\bigtimes_{j=1}^{b-a} H_j^n, \bigtimes_{j=1}^{b-a} G_j^n \right)_{n \in \mathbb{N}}$, and use the fact

that $L \left(\bigtimes_{j=1}^{b-a} G_j^n \right)$ is decomposable for each positive integer n , which, again, is a consequence of the above discussion about prime numbers. Finally, we get that

$$\lim_{n \rightarrow \infty} sd \left(\bigtimes_{j=1}^{b-a} H_j^n, \bigtimes_{j=1}^{b-a} G_j^n \right) = \prod_{j=1}^{b-a} \lim_{n \rightarrow \infty} sd(H_j^n, G_j^n) = \prod_{j=1}^{b-a} \frac{a+j-1}{a+j} = \frac{a}{b} = \alpha.$$

This implies that $(0, 1) \cap \mathbb{Q} \subseteq \overline{B}$. Since 0 and 1 were other limit points and $B \subseteq [0, 1]$, we have $[0, 1] \cap \mathbb{Q} \subseteq \overline{B} \subseteq [0, 1]$. Finally, we deduce that B is dense in $[0, 1]$, as a consequence of the fact that the closure of $[0, 1] \cap \mathbb{Q}$ is $[0, 1]$. $\square$

ACKNOWLEDGEMENTS. The authors are grateful to the reviewers for their remarks, which improved the previous version of the paper.

References

- [1] R. BRANDL, Groups with few non-normal subgroups, *Comm. Algebra* **23** (1995), 2091–2098.
- [2] A. K. DAS, R. K. NATH AND M. R. POURNAKI, A survey on the estimation of commutativity in finite groups, *Southeast Asian Bull. Math.* **37** (2013), 161–180.
- [3] R. M. GURALNICK AND G. R. ROBINSON, On the commuting probability in finite groups, *J. Algebra* **300** (2006), 509–528.
- [4] W. H. GUSTAFSON, What is the probability that two group elements commute?, *Amer. Math. Monthly* **80** (1973), 1031–1034.

- [5] B. HUPPERT, Endliche Gruppen. I, *Springer Verlag, Berlin – New York*, 1967, 1968.
- [6] P. LESCOT, Sur certains groupes finis, *Rev. Math. Spéciales* **8** (1987), 276–277.
- [7] P. LESCOT, Degré de commutativité et structure d'un groupe fini (1), *Rev. Math. Spéciales* **8** (1988), 276–279.
- [8] P. LESCOT, Degré de commutativité et structure d'un groupe fini (2), *Rev. Math. Spéciales* **4** (1989), 200–202.
- [9] P. LESCOT, Isoclinism classes and commutativity degrees of finite groups, *J. Algebra* **177** (1995), 847–869.
- [10] P. LESCOT, Central extensions and commutativity degree, *Comm. Algebra* **29** (2001), 4451–4460.
- [11] H. MOUSAVI, On finite groups with few non-normal subgroups, *Comm. Algebra* **27** (1999), 3143–3151.
- [12] R. K. NATH, Commutativity degree of a class of finite groups and consequences, *Bull. Aust. Math. Soc.* **88** (2013), 448–452.
- [13] R. K. NATH and M. K. YADAV, On the probability distribution associated to commutator word map in finite groups, *Internat. J. Algebra Comput.* **25** (2015), 1107–1124.
- [14] D. J. RUSIN, What is the probability that two elements of a finite group commute?, *Pacific J. Math.* **82** (1979), 237–247.
- [15] R. SCHMIDT, Subgroup Lattices of Groups, *Walter de Gruyter, Berlin*, 1994.
- [16] M. SUZUKI, Group Theory. I, II, *Springer Verlag, Berlin – New York*, 1982, 1986.
- [17] M. TĂRNĂUCEANU, Groups determined by posets of subgroups, *Matrix Rom, București*, 2006.
- [18] M. TĂRNĂUCEANU, Contributions to the study of subgroup lattices, *Ed. Matrix Rom, București*, 2016.
- [19] M. TĂRNĂUCEANU, Subgroup commutativity degrees of finite groups, *J. Algebra* **321** (2009), 2508–2520.
- [20] M. TĂRNĂUCEANU, Addendum to “Subgroup commutativity degrees of finite groups”, *J. Algebra* **337** (2011), 363–368.
- [21] M. TĂRNĂUCEANU, Normality degrees of finite groups, *Carpathian J. Math.* **33** (2017), 115–126.
- [22] WOLFRAM|ALPHA, Wolfram Alpha LLC, 2009, <http://www.wolframalpha.com>.

MIHAI-SILVIU LAZOREC
FACULTY OF MATHEMATICS
“AL.I.CUZA” UNIVERSITY
BD. CAROL I, NO. 11
IAȘI - 700506
ROMANIA

E-mail: mihai.lazorec@student.uaic.ro

MARIUS TĂRNĂUCEANU
FACULTY OF MATHEMATICS
“AL.I.CUZA” UNIVERSITY
BD. CAROL I, NO. 11
IAȘI - 700506
ROMANIA

E-mail: tarnauc@uaic.ro

URL: <http://www.math.uaic.ro/~martar>

(Received March 12, 2018; revised July 27, 2018)