

A generalization of the Gelfand–Kolmogoroff theorem

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Abstract. We introduce the notion of a ring isomorphism in norm. For such maps, we obtain an extension of the Gelfand–Kolmogoroff theorem by showing that a ring isomorphism in norm between spaces of continuous functions on compact spaces is a weighted composition operator.

1. Introduction

Let X be a compact Hausdorff space, and let $C(X)$ denote the Banach algebra of all continuous real-valued functions on X equipped with the supremum norm. The classical Banach–Stone theorem asserts that the linear metric structure of $C(X)$ determines the topology of X (see [1], [7]). More precisely, every surjective isometry T between $C(X)$ and $C(Y)$ must be of the form $Tf = h \cdot (f \circ \tau)$ (that is, it is a *weighted composition operator*), where $\tau : Y \rightarrow X$ is a homeomorphism and $h \in C(Y)$ satisfies $|h| = 1$. In 1966, HOLSZTYŃSKI [5] showed that any non-surjective isometry between spaces of continuous functions is also a weighted composition operator.

The Banach–Stone theorem has found a large number of extensions, generalizations and variants in many different contexts (see, for example, [2]). Among

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them, one line of research which started with the Banach–Stone theorem established a link between the algebraic properties. In this direction, the following well-known result was obtained by GELFAND and KOLMOGOROFF [3] in 1939.

Theorem 1.1 (Gelfand–Kolmogoroff). *Suppose that X and Y are compact Hausdorff spaces. If $C(X)$ and $C(Y)$ are isomorphic as rings, then X and Y are homeomorphic and every ring isomorphism $T : C(X) \rightarrow C(Y)$ is of the form $Tf = f \circ \tau$, where $\tau : Y \rightarrow X$ is a homeomorphism.*

Indeed, this result can be obtained by the Banach–Stone theorem, since ring isomorphism implies isometry (see, for example, GILLMAN and JERISON [4, 1J.6]). The main purpose of this short note is to generalize this theorem by considering some class of more general maps.

2. Main results

Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras. If $T : \mathcal{A} \rightarrow \mathcal{B}$ is a bijection so that

$$\|T(f \cdot g)\| = \|Tf \cdot Tg\| \quad (1)$$

and

$$\|T(f + g)\| = \|Tf + Tg\| \quad (2)$$

for all $f, g \in \mathcal{A}$, then we call T a *ring isomorphism in norm*. We say that T is unital if T preserves the identity element. A natural question is whether every unital ring isomorphism in norm from $\mathcal{A}$ onto $\mathcal{B}$ is indeed a ring isomorphism.

The next theorem provides a positive answer for the class of $C(X)$ -spaces and strengthens the Gelfand–Kolmogoroff theorem.

Theorem 2.1. *Suppose that X and Y are compact Hausdorff spaces. If there exists a ring isomorphism in norm $T : C(X) \rightarrow C(Y)$, then there is a continuous function $h \in C(Y)$ with $|h(y)| = 1$ ($y \in Y$) and a homeomorphism $\tau : Y \rightarrow X$ such that*

$$(Tf)(y) = h(y) \cdot f(\tau(y)) \quad (y \in Y, f \in C(X)). \quad (3)$$

PROOF. The proof will be divided into two parts. In the first part, we show that it is true if $T1 = 1$. Then, making use of this result, we show in the second part that the result holds for a general ring isomorphism in norm T .

Part I. Assume that $T1 = 1$. Since T is a bijection satisfying (2), [8, Corollary 1] shows that T is additive, and hence we have $T(\alpha f) = \alpha(Tf)$ for all $\alpha \in \mathbb{Q}$.

In what follows, we show that if $f(x) \geq 0$ for all $x \in X$, then

$$(Tf)(y) \geq 0 \quad \text{for all } y \in Y. \quad (4)$$

Since

$$\left\| T \left(\left(\frac{f}{n^2} \right)^{\frac{1}{2}} \right) \right\| = \frac{\|T(f^{\frac{1}{2}})\|}{n}, \quad (5)$$

we can choose n large enough such that

$$\left\| T \left(\left(\frac{f}{n^2} \right)^{\frac{1}{2}} \right) \right\| \leq 1. \quad (6)$$

Then

$$\begin{aligned} \left\| T \left(1 - \frac{f}{n} \right) \right\| &= \left\| T \left(\left(1 - \left(\frac{f}{n} \right)^{\frac{1}{2}} \right) \left(1 + \left(\frac{f}{n} \right)^{\frac{1}{2}} \right) \right) \right\| \\ &= \left\| T \left(1 - \left(\frac{f}{n} \right)^{\frac{1}{2}} \right) \cdot T \left(1 + \left(\frac{f}{n} \right)^{\frac{1}{2}} \right) \right\| \\ &= \left\| \left(1 - T \left(\left(\frac{f}{n} \right)^{\frac{1}{2}} \right) \right) \left(1 + T \left(\left(\frac{f}{n} \right)^{\frac{1}{2}} \right) \right) \right\| \\ &= \left\| 1 - \left(T \left(\left(\frac{f}{n} \right)^{\frac{1}{2}} \right) \right)^2 \right\| \leq 1. \end{aligned} \quad (7)$$

Therefore, $\|1 - T(f/n)\| \leq 1$, which implies $(T(f/n))(y) \geq 0$ for all $y \in Y$, and hence (4) holds.

Fix $f \in C(X)$. Put $f_+ = \max\{f, 0\}$, and $f_- = \max\{-f, 0\}$. Clearly, $f = f_+ - f_-$. By (1), we get $\|Tf_+ \cdot Tf_-\| = 0$, and then

$$\text{supp}(Tf_+) \cap \text{supp}(Tf_-) = \emptyset. \quad (8)$$

It follows that

$$\|Tf\| = \|Tf_+ - Tf_-\| = \max\{\|Tf_+\|, \|Tf_-\|\} = \|Tf_+ + Tf_-\| = \|T(|f|)\|. \quad (9)$$

We next show that

$$\|Tf\| \leq \|f\| \quad (10)$$

for all $f \in C(X)$. Let λ be any rational number so that $\|f\| \leq \lambda$. Then (4) implies $T(\lambda - |f|) \geq 0$, and hence $0 \leq T(|f|) \leq \lambda$. Therefore, $\|T(|f|)\| \leq \lambda$, and then (9) shows $\|Tf\| \leq \lambda$. Since λ is an arbitrary rational number satisfying $\|f\| \leq \lambda$, we obtain (10).

Now the additivity of T and (10) imply that T is continuous and linear. Since T is a continuous linear bijection, by the open mapping theorem, T is an isomorphism. Therefore, there is $\beta > 0$ such that

$$\beta\|f\| \leq \|Tf\| \leq \|f\| \quad (11)$$

for all $f \in C(X)$.

We claim that β in (11) can be chosen to be 1. Suppose not. Then there exists $f \in C(X)$ so that $\|f\| = 1$ and $\|Tf\| < \|f\|$. For any positive integer n , by (1),

$$\begin{aligned} \|T(f^{2^n})\| &= \|T(f^{2^{n-1}}) \cdot T(f^{2^{n-1}})\| = \|T(f^{2^{n-1}})\|^2 \\ &= \|T(f^{2^{n-2}})\|^4 = \dots = \|Tf\|^{2^n}. \end{aligned} \quad (12)$$

Since $\|Tf\| < \|f\| = 1$, choose n large enough so that $\|Tf\|^{2^n} < \beta$. (12) shows that $\|T(f^{2^n})\| < \beta$. On the other hand, it is clear that $\|f^{2^n}\| = 1$, since $\|f\| = 1$. Then (11) shows that $\|T(f^{2^n})\| \geq \beta\|f^{2^n}\| = \beta$, which leads to a contradiction.

Now (11) becomes $\|Tf\| = \|f\|$ for all $f \in C(X)$. Since T is linear, we have proved that T is a surjective linear isometry with $T1 = 1$. By the Banach–Stone theorem, there is a homeomorphism $\tau : Y \rightarrow X$ such that

$$(Tf)(y) = f(\tau(y)) \quad (13)$$

for all $f \in C(X)$ and $y \in Y$.

Part II. We claim that $|(T1)(y)| = 1$ for all $y \in Y$. Assume not. There is an open subset U of Y and $\varepsilon > 0$ such that $|(T1)(y)| \geq 1 + \varepsilon$ for all $y \in U$ or $|(T1)(y)| \geq 1 - \varepsilon$ for all $y \in U$. Without loss of generality, we assume that

$$|(T1)(y)| \geq 1 - \varepsilon \quad (14)$$

for all $y \in U$. By Urysohn’s lemma, there is a nonzero function $h \in C(Y)$ such that $\text{supp}(h) \subset U$. Let f be an element in X such that $Tf = h$ (note that T is surjective). Then

$$\|h\| = \|Tf\| = \|T(1 \cdot f)\| = \|T1 \cdot Tf\| = \|T1 \cdot h\| \leq (1 - \varepsilon)\|h\|. \quad (15)$$

We get a contradiction.

Clearly, $T/T1$ also satisfies (1) and (2), since $|(T1)(y)| = 1$. Replacing T by $T/T1$ in Part I, we get that there exists a homeomorphism $\tau : Y \rightarrow X$ such that

$$(Tf)(y) = (T1)(y) \cdot (f(\tau(y))) \quad (16)$$

for all $f \in C(X)$ and $y \in Y$. Letting $h = T1$ finishes the proof. $\square$

Remark 2.2. In the proof of the above theorem, we see that if T is a unital ring isomorphism in norm, then T has the form $Tf = f \circ \tau$, and hence T is a ring isomorphism (indeed, an algebraic isomorphism).

Remark 2.3. A direct consequence of Theorem 2.1 is that the existence of a ring isomorphism in norm between $C(X)$ and $C(Y)$ implies a homeomorphism between X and Y . The referee told us this consequence can also be obtained by the main result of [6].

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References

- [1] S. BANACH, *Théorie des Opérations Linéaires*, Reprint of the 1932 edition, *Chelsea Publishing, New York*, 1963.
- [2] M. I. GARRIDO and J. A. JARAMILLO, Variations on the Banach–Stone theorem, *Extracta Math.* **17** (2002), 351–383.
- [3] I. GELFAND and A. KOLMOGOROFF, On rings of continuous functions on topological spaces, *Dokl. Akad. Nauk. SSSR* **22** (1939), 11–15.
- [4] L. GILLMAN and M. JERISON, *Rings of Continuous Functions*, *Springer-Verlag, New York – Heidelberg*, 1976.
- [5] H. HOLSZTYŃSKI, Continuous mappings induced by isometries of spaces of continuous functions, *Studia Math.* **26** (1966), 133–136.
- [6] T. KANIA and M. RMOUTIL, Recovering a compact Hausdorff space X from the compatibility ordering on $C(X)$, *Fund. Math.* **242** (2018), 187–205.
- [7] M. H. STONE, Applications of the theory of boolean rings to General Topology, *Trans. Amer. Math. Soc.* **41** (1937), 375–481.

- [8] J. TABOR, Stability of the Fischer–Muszély functional equation, *Publ. Math. Debrecen* **62** (2003), 205–211.

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