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On weak quasicontractions in b-metric spaces

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Abstract. Recently, weak quasicontractions have been studied by Bessenyei [2]. The aim of the present paper is to establish fixed point results for weak quasicontractions involving comparison function in b-metric spaces. As applications of our theorems, we deduce certain well-known results as corollaries.

1. Introduction and preliminaries

Banach contraction mapping principle is a simple and powerful result with a wide range of applications, including iterative methods for solving linear, non-linear, differential, integral, and difference equations.

Theorem 1.1. *Let (X, d) be a complete metric space. Let T be a contraction mapping on X, that is, one for which exists $\lambda \in [0, 1)$ satisfying*

$$d(Tx, Ty) \leq \lambda d(x, y) \tag{1.1}$$

for all $x, y \in X$. Then T has a unique fixed point $x \in X$.

Because of its significance and simplicity, various authors have established numerous interesting extensions and generalizations of the Banach contraction principle (see, for example, the monographs of Rus [20], Kirk and Shazad [16]). In 2016, Bessenyei [2] rediscovered Theorems of Hegedűs and Szilágyi [9], and those of Walter [21], introduced weak quasicontraction involving comparison functions and also proved a theorem that generalizes the results obtained by Ćirić [7], Browder [4] and Matkowski [17].

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The orbit and the double orbit induced by T are defined in the next way:

$$O(x) := \{T^n(x) | n \in \mathbb{N} \cup \{0\}\}; \qquad O(x, y) := O(x) \cup O(y),$$

where $T^{n+1} = T \circ T^n$ and $T^0 = id$.

We mention some properties of comparison functions $\varphi : [0, \infty) \to [0, \infty)$ which are being used in metric fixed point theory:

(P_1) φ is increasing.

(P_2) φ is upper semi-continuous.

(P_3) $\varphi(0) = 0$.

(P_4) $\varphi(t) < t$, for all $t > 0$.

(P_5) $\lim_{n\to\infty} \varphi^n(t) = 0$, for each $t > 0$.

Lemma 1.2 (Bessenyei, [2])**.** $(P_1) + (P_2) + (P_3) + (P_4) \Rightarrow (P_5)$.

Lemma 1.3 (Matkowski, [18])**.** $(P_1) + (P_5) \Rightarrow (P_4)$.

Definition 1.4 ([2]). Let (X, d) be a metric space. A mapping $T : X \to X$ is called a weak quasicontraction with comparison function φ (or briefly: *a weak φ-quasicontraction*) if it induces bounded orbits, and for all $x, y \in X$,

$$d(Tx, Ty) \leq \varphi(\operatorname{diam} O(x, y)). \tag{1.2}$$

The main result of Bessenyei [2] is the below-mentioned fixed point theorem for weak quasicontractions defined on complete metric spaces.

Theorem 1.5 ([2])**.** *Let (X, d) be a complete metric space, and $T : X \to X$ a weak quasicontraction with comparison function φ that meets the conditions $(P_1), (P_2), (P_3)$ and (P_4). Then T has a unique fixed point. Moreover, the sequence of iterates at any point converges to this fixed point.*

Remark 1.6. We note that condition (1.1) implies $\operatorname{diam} O(x) \leq \frac{d(x,Tx)}{1-\lambda}$, so if we put $\varphi(t) = \lambda t$, $t \geq 0$, in Theorem 1.5, we obtain the Banach fixed point theorem.

Bakhtin [1] and Czerwik [6] defined the notion of b-metric spaces and proved some fixed point theorems for single-valued and multi-valued mappings in b-metric spaces. Successively, this notion has been reintroduced by Khamsi [14], Khamsi and Hussain [15], and Hussain *et al.* [11], [12] with the name of *metric-type space.*

Definition 1.7. Let X be a nonempty set, and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A triplet (X, d, s) is called a b-metric space with coefficient s.

Note that the class of metric spaces is a proper subset of the class of b-metric spaces with coefficient $s \geq 1$. Fixed point theory in b-metric spaces was studied by many authors (see [8], [10], [13], [16], [19]). Note also that in a b-metric space, distance function d need not be continuous, i.e, there exists a b-metric space (X, d, s) and sequences$\{x_n\}, \{y_n\}$ in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, but $\lim_{n\to\infty} d(x_n, y_n) \neq d(x, y)$.

One of the main results of [6] Czerwik is the following;

Theorem 1.8 ([6, Theorem 1]). *Let (X, d, s) be a complete b-metric space, and suppose $T : X \to X$ satisfies*

$$d(T(x), T(y)) \leq \varphi(d(x, y)), \tag{1.3}$$

for each $x, y \in X$, where mapping $\varphi : [0, \infty) \to [0, \infty)$ satisfies conditions (P_1) and (P_5). Then T has a unique fixed point $x^* \in X$, and $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$.

Remark 1.9. Note that Theorem 1.8 is a direct consequence of the main result of [3]. The recent paper [5] gives excellent overview of possible generalizations of metric spaces.

Remark 1.10. We note that due to Lemma 1.3, condition (1.3) implies that T is a continuous mapping. Also, condition (1.3) implies that T induces bounded orbits (see [2]).

The fact that d is not continuous in the b-metric space leads to the introduction of strong b-metric space.

Definition 1.11 ([16, Definition 12.7]). Let X be a nonempty set, and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a strong b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leq sd(x, y) + d(y, z)$.

A triplet (X, d, s), is called a strong b-metric space with coefficient s.

Remark 1.12. The distance function d in a strong b-metric space is continuous (see [16, Proposition 12.3]).

The aim of this paper is to obtain Theorem 1.5 in b-metric spaces using weak quasicontraction involving comparison function φ. As consequences, we derive certain known results as corollaries.

2. Main result

The proof of the next lemma is a straightforward adaptation of the reasoning from [2].

Lemma 2.1. *Let (X, d, s) be a complete b-metric space, and $T : X \to X$ a weak φ-quasicontraction where φ satisfies conditions (P_1) and (P_5). Then there exists $x^* \in X$ such that $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$.*

Proof. The boundedness of orbits implies the following:

$$\operatorname{diam} O(Tx, Ty) = \sup_{k,l\in\mathbb{N}} \{d(T^k x, T^l y), d(T^k x, T^l x), d(T^k y, T^l y)\}. \tag{2.1}$$

From condition (1.2), we obtain

$$d(T^k x, T^l y) \leq \varphi(\operatorname{diam} O(T^{k-1}x, T^{l-1}y))) \leq \varphi(\operatorname{diam} O(x, y)).$$

Similarly, we have

$$d(T^k x, T^l x) \leq \varphi(\operatorname{diam} O(T^{k-1}x, T^{l-1}x)) \leq \varphi(\operatorname{diam} O(x)) \leq \varphi(\operatorname{diam} O(x, y)),$$

which implies

$$d(T^k y, T^l y) \leq \varphi(\operatorname{diam} O(x, y)). \tag{2.2}$$

According to the foregoing, we conclude that

$$\operatorname{diam} O(Tx, Ty) \leq \varphi(\operatorname{diam} O(x, y)). \tag{2.3}$$

From inequality (2.3), we have the following:

$$\operatorname{diam} O(T^2x, T^2y) \leq \varphi(\operatorname{diam} O(Tx, Ty)) \leq \varphi(\varphi(\operatorname{diam} O(x, y))) = \varphi^2(\operatorname{diam} O(x, y)).$$

Using induction, we obtain

$$\operatorname{diam} O(T^n x, T^n y) \leq \varphi^n(\operatorname{diam} O(x, y)). \tag{2.4}$$

Choose $x \in X$, we show that $\{T^n x\}$ is a Cauchy sequence. The boundedness of orbits and condition (P_5) imply that for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(\operatorname{diam} O(x)) < \frac{\epsilon}{2s}$. Therefore, using inequality (2.4), we obtain, for all $n \geq n_0$,

$$d(T^{n_0}x, T^n x) \leq \varphi^{n_0}(\operatorname{diam} O(x, T^{n-n_0}x)) \leq \varphi^{n_0}(\operatorname{diam} O(x)) \leq \frac{\epsilon}{2s}.$$

Using inequality (3) in Definition 1.7, we conclude that for all $m, n \geq n_0$,

$$d(T^m x, T^n x) \leq s[d(T^m x, T^{n_0}x) + d(T^{n_0}x, T^n x)] \leq s\left[\frac{\epsilon}{2s} + \frac{\epsilon}{2s}\right] = \epsilon.$$

So, $\{T^n x\}$ is a Cauchy sequence, and hence there exists $x^* \in X$ such that $\lim\limits_{n\to\infty} T^n x = x^*$. □

Theorem 2.2. *Let (X, d, s) be a complete b-metric space, and $T : X \to X$ a weak φ-quasicontraction such that function φ satisfies conditions (P_1) and (P_5). Then $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$, and x^* is a unique fixed point of T, provided one of the following conditions is satisfied:*

(i) *T is continuous at x^*;*

(ii) *d is continuous.*

Proof. Lemma 2.1 implies that $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$. Let us prove that x^* is a unique fixed point of T.

(i) Suppose that T is continuous at $x^* \in X$.

Then we have

$$x^* = \lim_{n\to\infty} T^{n+1}x = T \lim_{n\to\infty} T^{n+1}x = Tx^*.$$

(ii) Let d be continuous. If x^* is not a fixed point of T, then $\operatorname{diam} O(x^*) > 0$. The methods of [2] provide $n_0 \in \mathbb{N}$ such that $\operatorname{diam} O(x^*) = d(x^*, T^{n_0}x^*)$ holds. By Lemma 1.3, we obtain that

$$\varphi(\operatorname{diam} O(x^*)) < \operatorname{diam} O(x^*).$$

Since d is continuous, we obtain

$$\begin{aligned}\operatorname{diam} O(x^*) &= d(x^*, T^{n_0}x^*) = \lim_{n\to\infty} d(T^{n+n_0}x^*, T^{n_0}x^*)\\ &\leq \varphi^{n_0}(\operatorname{diam} O(T^n x^*, x^*)) \leq \varphi(\operatorname{diam} O(x^*)) < \operatorname{diam} O(x^*).\end{aligned}$$

Consequently, $\operatorname{diam} O(x^*) = 0$. Thus x^* is the fixed point of the mapping T.

For uniqueness, let y^* be another fixed point of T. Since,

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq \varphi(\operatorname{diam} O(x^*, y^*)) < \operatorname{diam} O(x^*, y^*) = d(x^*, y^*),$$

which implies T has exactly one fixed point. □

Remark 2.3. Since $d(x,y) \leq \operatorname{diam} O(x,y)$, for all $x, y \in X$ and φ, satisfies condition (P_4) (see Lemma 1.3), so from Theorem 2.2 we obtain Theorem 1.8.

Corollary 2.4. *Let (X, d, s) be a complete b-metric space, and let mapping $T : X \to X$ induce bounded orbits. Suppose that for all $x, y \in X$,*

$$d(Tx, Ty) \leq \operatorname{diam} O(x,y) - \psi(\operatorname{diam} O(x,y)), \tag{2.5}$$

where function $\psi : [0,\infty) \to [0,\infty)$ satisfies the below conditions:

(a) *ψ is a decreasing,*

(b) *$id - \psi$ satisfy condition (P_5).*

Then there exists $x^* \in X$ such that $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$, and x^* is a unique fixed point of T, provided one of the following conditions is satisfied:

(i) *T is continuous at $x^* \in X$,*

(ii) *d is continuous.*

3. Some applications

In this section, we present certain consequences of Theorem 2.2 in b-metric spaces.

Lemma 3.1. *Let (X, d, s) be a complete b-metric space, and $T : X \to X$ a map such that for all $x, y \in X$ and some $\lambda \in [0,1)$, we have*

$$d(Tx, Ty) \leq \lambda d(x,y). \tag{3.1}$$

Then T induces bounded orbits.

Proof. Since $\lim\limits_{n\to\infty} \lambda^n = 0$, there exists a natural number n_0 such that

$$0 < \lambda^{n_0} \cdot s^2 < 1. \tag{3.2}$$

Let $O_n(x) = \{x, Tx, \ldots, T^n x\}$. Then, we conclude that $\operatorname{diam} O_n(x) = d(T^k x, T^l x)$ for some $k, l \in \{1, 2, \ldots, n\}$, or $\operatorname{diam} O_n(x) = d(x, T^k x)$ for some $k \in \{1, 2, \ldots, n\}$. If $\operatorname{diam} O_n(x) = d(T^k x, T^l x)$, we obtain (where it is understood that $T^0 x = x$)

$$d(T^k x, T^l x) \leq \lambda d(T^{k-1}x, T^{l-1}x) < d(T^{k-1}x, T^{l-1}x) \leq \operatorname{diam} O_n(x).$$

Therefore, we conclude that $d(x, T^k x) = \operatorname{diam} O_n(x)$ for some $k \in \{1, 2, \ldots, n\}$. Applying inequality (3) in Definition 1.7, we obtain

$$\begin{aligned}d(x,T^kx) &\le s[d(x,T^{n_0}x)+d(T^{n_0}x,T^kx)]\\&\le s[d(x,T^{n_0}x)+s(d(T^{n_0}x,T^{n_0+k}x)+d(T^{n_0+k}x,T^kx))]\\&\le sd(x,T^{n_0}x)+s^2[\lambda^{n_0}d(x,T^kx)+\lambda^kd(T^{n_0}x,x)]\\&\le (s+s^2)d(x,T^{n_0}x)+s^2\lambda^{n_0}d(x,T^kx)].\end{aligned}$$

Therefore, we get

$$\operatorname{diam} O_n(x) \le \frac{s+s^2}{1-\lambda^{n_0}s^2}d(x,T^{n_0}x). \tag{3.3}$$

Since $\operatorname{diam} O(x) = \sup\{\operatorname{diam} O_n(x) : n \in \mathbb{N}\}$, we obtain that T induces bounds orbits. □

Theorem 3.2 (The Banach contraction principle in b-metric spaces, Dung [8, Theorem 2.1])**.** *Let (X,d,s) be a complete b-metric space, and let $T : X \to X$ be a continuous map such that for all $x,y \in X$ and some $\lambda \in [0,1)$,*

$$d(Tx,Ty) \le \lambda d(x,y). \tag{3.4}$$

Then T has a unique fixed point x^*, and $\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$.

Proof. If we put $\varphi(t) = \lambda t$, then the assertion follows from Theorem 2.2 and Lemma 3.1. □

Theorem 3.3. *Let (X,d,s) be a complete b-metric space, and let $T : X \to X$ be a quasicontraction inducing bounded orbits, i.e, there exists $\lambda \in [0,1)$ such that*

$$d(Tx,Ty) \le \lambda \operatorname{diam} O(x,y), \tag{3.5}$$

for all $x,y \in X$. Then there exists $x^* \in X$ such that $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$, and T has a unique fixed point x^*, provided one of the following conditions are satisfied:

(i) *T is continuous at $x^* \in X$;*

(ii) *d is continuous.*

Proof. The proof follows from Theorem 2.2, if we put $\varphi(t) = \lambda t$. □

Remark 3.4. (1) If T is a quasicontraction on b-metric space (X,d,s) with $\lambda \in [0,\frac{1}{s})$, then similar to the proof of Lemma 3.1, there exists some $k \in \{1,2,\dots,n\}$, such that $d(x,T^kx) = \operatorname{diam} O_n(x)$. Since,

$$\operatorname{diam} O_n(x) \le s[d(x,Tx) + \operatorname{diam} O_{n-1}(Tx)] \le s[d(x,Tx) + \lambda \operatorname{diam} O_n(x)],$$

which implies

$$\operatorname{diam} O(x) \le \frac{s}{1-\lambda s} d(x, Tx). \tag{3.6}$$

(2) If (X, d, s) is a strong b-metric space, then

$$\operatorname{diam} O_n(x) \le sd(x, Tx) + \operatorname{diam} O_{n-1}(Tx) \le sd(x, Tx) + \lambda \operatorname{diam} O_n(x),$$

which implies

$$\operatorname{diam} O(x) \le \frac{s}{1-\lambda} d(x, Tx). \tag{3.7}$$

From Lemma 2.1, Theorem 3.3 and Remark 3.4, we obtain the following quasi-contraction principle in b-metric and strong b-metric spaces.

Corollary 3.5 (Version of the fixed point theorem of Ćirić in b-metric spaces)**.** *Let (X, d, s) be a complete b-metric space and let $T : X \to X$ be a map such that for all $x, y \in X$ and some $\lambda \in [0, 1/s)$,*

$$d(Tx, Ty) \le \lambda \max\left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty)}{2s}, \frac{d(y, Tx)}{2s} \right\}.$$

Then T has a unique fixed point x^*.

Proof. From Lemma 2.1 and Remark 3.4, there exists $x^* \in X$ such that $\lim_{n\to\infty} T^n(x) = x^*$ for each $x \in X$. Let us show that x^* is a fixed point. We have the following:

$$\begin{aligned} d(x^*, Tx^*) &\le s[d(x^*, T^{n+1}x) + d(T^{n+1}x, Tx^*)] \\ &\le sd(x^*, T^{n+1}x) + s\lambda \max\Big\{ d(T^n x, x^*), d(T^n x, T^{n+1}x), \\ &\qquad d(x^*, Tx^*), \frac{d(x^*, T^{n+1}x)}{2s}, \frac{d(T^n x, Tx^*)}{2s} \Big\}. \end{aligned}$$

Since,

$$\begin{aligned} \frac{d(x^*, T^{n+1}x)}{2s} &\le \frac{d(x^*, T^n x) + d(T^n x, T^{n+1}x)}{2} \\ &\le \max\{d(x^*, T^n x), d(T^n x, T^{n+1}x)\}, \end{aligned}$$

and

$$\frac{d(T^n x, Tx^*)}{2s} \le \frac{d(T^n x, x^*) + d(x^*, Tx^*)}{2} \le \max\{d(x^*, T^n x), d(x^*, Tx^*)\},$$

so we obtain

$$d(x^*, Tx^*) \leq sd(x^*, T^{n+1}x) + s\lambda \max\{d(T^n x, x^*), d(T^n x, T^{n+1}x), d(x^*, Tx^*)\}.$$

Since $\lim_{n\to\infty} T^n x = x^*$ and $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$, this shows that $(1 - \lambda s)d(x^*,\, Tx^*) = 0$, which implies that x^* is a fixed point of T. The uniqueness follows from the quasi-contractivity of T. □

Corollary 3.6 (Version of the fixed point theorem of Ćirić in strong b-metric spaces)**.** *Let (X, d, s) be a complete strong b-metric space, and let $T : X \to X$ be a map such that for all $x, y \in X$ and some $\lambda \in [0, 1)$,*

$$d(Tx, Ty) \leq \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Then T has a unique fixed point x^*.

Proof. The continuity of d follows directly from Theorem 3.3 and Remark 3.4. □

Remark 3.7.

(i) The conclusion of Corollary 3.6 does not hold in the setting of b-metric spaces for $\lambda \in [0, 1)$ (see [8, Example 2.6]).

(ii) Corollary 3.5 improves the result of Jovanović *et al.* ([13, Corollary 3.12]).

(ii) Corollary 3.6 improves the results of Dung (see [8, Corollaries 2.4 and 2.5]).

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