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Some special subclasses of univalent starlike functions

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Abstract. Let $0 < \lambda < 1$, $\alpha > 1$, $\frac{2(\alpha-1)}{\lambda} < 1$ and $T(\lambda,\alpha)$ the class of homomorphic functions in the unit disc for which $f(0) = f'(0) - 1 = 0$ and $\mathrm{Re}\Big[(1-\lambda)z\,\frac{f'(z)}{f(z)} + \lambda\Big(1 + \frac{z(f''(z))}{f'(z)}\Big)\Big] < \alpha$. We find sharp inequalities for the quantities $\mathrm{Re}\Big\{z\frac{f'(z)}{f(z)}\Big\}$, $\mathrm{Re}\Big\{\frac{f(z)}{zf'(z)}\Big\}$, $\Big|\frac{zf'(z)}{f(z)}\Big|$, $|f(z)|$ for $f \in T(\lambda,\alpha)$. A special result is that if $f \in T(\lambda,\alpha)$ then f is a univalent starlike function.

Introduction

If $\mathbf{H}(\mathbf{U})$ is the class of holomorphic functions defined in the unit disc $\mathbf{U} = \{z : |z| < 1\}$ then we denote by:

(i) $\mathbf{A}$ the class of functions $f \in \mathbf{H}(\mathbf{U})$ for which $f(0) = f'(0) - 1 = 0$.

(ii) $\mathbf{T}(\lambda,\alpha)$ the class of functions $f \in \mathbf{A}$ for which $\mathrm{Re} f_\lambda < \alpha$ where

$$f_\lambda(z) = (1-\lambda)z\,\frac{f'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right), \quad 0 < \lambda < 1,\ \alpha > 1.$$

(iii) $\mathbf{P}$ the class of functions $f \in \mathbf{H}(\mathbf{U})$ for which $f(0) = 1$ and $\mathrm{Re}\, f > 0$.

Nunokawa [1] has proved the following Theorem.

Theorem 1. *If $f \in \mathbf{T}\left(1, \frac{3}{2}\right)$, then*

$$0 < Re\left[\frac{zf'(z)}{f(z)}\right] < \frac{4}{3} \quad in \quad \mathbf{U}.$$

The inequalities in Nunokawa's Theorem are best possible. In the present paper we prove the following Theorem:

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Theorem 2. *If $f \in \mathbf{T}(\lambda, \alpha)$, $\frac{2(a-1)}{\lambda} < 1$ and*

$$P(\lambda, \alpha, z) = \frac{1}{\lambda} \int_0^1 \left(\frac{1+tz}{1+z} \right)^{\frac{2(a-1)}{\lambda}} t^{\frac{1}{\lambda}-1} \, dt,$$

then

(i) $\frac{1}{P(\lambda,\alpha,-r)} < \left| \frac{zf'(z)}{f(z)} \right| < \frac{1}{P(\lambda,\alpha,r)}$ *in* $\mathbf{U}_r = \{z : |z| < r\}$.

(ii) $0 < Re \left[\frac{zf'(z)}{f(z)} \right] < P(\lambda, \alpha, 1)$ *in* $\mathbf{U}$.

(iii) $P(\lambda, \alpha, r) < Re \left[\frac{f(z)}{zf'(z)} \right] < P(\lambda, \alpha, -r)$ *in* $\mathbf{U}_r$.

(iv) $|f(z)| < u(\lambda, \alpha, r)$ *in* U_r, *where*

$$u(\lambda, \alpha, r) = r \cdot \exp \int_0^r \left[P^{-1}(\lambda, \alpha, \xi) - 1 \right] \xi^{-1} d\xi.$$

All the above inequalities are best possible.

Remark. If $a = \frac{1}{2} + 1$ then

$$P(\lambda, \alpha, z) = \frac{1}{1+z} + \frac{z}{(\lambda+1)(z+1)}, u(\lambda, \alpha, r) = r \cdot \frac{(\lambda+1+r)^\lambda}{(\lambda+1)^\lambda}.$$

If $\lambda = 1$ then

$$\begin{aligned} P(\lambda, \alpha, z) &= \frac{1}{(2\alpha-1)z} \left[(1+z) - \frac{1}{(1+z)^{2\alpha-1}} \right], u(\lambda, \alpha, r) \\ &= \frac{1}{2\alpha-1} \left[(1+r)^{2\alpha-1} - 1 \right]. \end{aligned}$$

In the special case $\lambda = 1, \ \alpha = \frac{3}{2}$ it is now obvious that (ii) coincides with Nunokawa's Theorem. In the same case we also have that

$$|f(z)| < r \left(1 + \frac{r}{2} \right) \text{ in } \mathbf{U}_r, \quad \text{and} \quad |f(z)| < \frac{3}{2} \text{ in } \mathbf{U}.$$

We now prove the following Lemma.

Lemma. *If $f \in \mathbf{A}$ then $f_\lambda = q, q(0) = 1$ if and only if*

$$\frac{zf'(z)}{f(z)} = \left[\frac{1}{\lambda} \int_0^1 \frac{F(tz)}{F(z)} t^{\frac{1}{\lambda}-1} \, dt \right]^{-1} \quad \textit{where}$$

$$F(z) = \exp \frac{1}{\lambda} \int_0^z [q(\omega) - 1] \omega^{-1} d\omega.$$

Proof. We consider a real intervall $(0,\varepsilon)$ such that $\operatorname{Re} f(x) > 0$ in $(0,\varepsilon)$. We shall first prove the required relation in this interval. Then by the uniqueness Theorem for holomorphic functions we get the result in the general case.

From the relation

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left[1+\frac{zf''(z)}{f'(z)}\right] = q(z)$$

we obtain

$$(1-\lambda)\left[\frac{f'(z)}{f(z)} - \frac{1}{z}\right] + \lambda\frac{f''(z)}{f'(z)} = \frac{q(z)-1}{z}$$

or

$$\left[(1-\lambda)\mathrm{Log}\left(\frac{f(z)}{z}\right) + \lambda\mathrm{Log} f'(z)\right]' = \frac{q(z)-1}{z}$$

or

$$(*) \qquad \left(\frac{f(z)}{z}\right)^{1-\lambda}(f'(z))^{\lambda} = c\cdot\exp\int_0^z \frac{q(z_1)-1}{z_1}dz_1.$$

Since $f'(0)=1$ for $z\to 0$ we get $c=1$. Therefore from the relation $(*)$ we have

$$(**) \qquad \lambda(f^{\frac{1}{\lambda}})' = f^{\frac{1}{\lambda}-1}\cdot f' = z^{\frac{1}{\lambda}-1}F(z)$$

and

$$(***) \qquad f^{\frac{1}{\lambda}} = \frac{1}{\lambda}\int_0^z u^{\frac{1}{\lambda}-1}F(u)du = \frac{1}{\lambda}z^{\frac{1}{\lambda}}\int_0^1 t^{1-\lambda}F(tz)dt.$$

Dividing $(*)^{\frac{1}{\lambda}}$ by $(***)$ we obtain the required result.

Conversely, if we set

$$Q(z) = \int_0^z u^{\frac{1}{\lambda}-1}F(u)du$$

we have

$$z\frac{f'(z)}{f(z)} = \lambda z\frac{Q'(z)}{Q(z)} \qquad \forall z\in U.$$

Let $f_\lambda = q_1$. From q_1 we define, the functions F_1 and Q_1, in the same manner as F and Q were defined from q. It is now obvious that:

$$z\frac{Q'(z)}{Q(z)} = z\frac{Q_1'(z)}{Q_1(z)} \qquad \forall z\in \mathbf{U}.$$

From the above relation we obtain succesively:

$$\left(\frac{Q}{Q_1}\right)' = 0, Q' = cQ_1',\ f = cF_1.$$

Since $F(0) = F_1(0) = 1$ then $c = 1$ and

$$\int_0^z (q(\omega) - 1)\omega^{-1} d\omega = \int_0^z (q_1(\omega) - 1)\omega^{-1} d\omega \qquad \forall z \in \mathbf{U}.$$

Differentiating both sides of the above relation we have: $q = q_1$

Proof of Theorem 2. If

$$p_n(z) = \sum_{k=1}^{n} \lambda_k \left(\frac{1 + \varepsilon_k z}{1 - \varepsilon_k z}\right), \quad |\varepsilon_k| = 1, \quad \lambda_k \geq 0 \quad \text{and} \sum_{k=1}^{n} \lambda_k = 1$$

then we will prove the Theorem in case $f_\lambda = (1 - \alpha)p_n + \alpha$. In this case by simple calculations we obtain

$$F(z) = \prod_{k=1}^{n} (1 - z \cdot \varepsilon_k)^{\frac{2(a-1)}{\lambda} \cdot \lambda_k} . \tag{1}$$

Since the set

$$\left\{\frac{1 + tz}{1 + z} : |z| < r\right\}$$

coincides with the open disc $S(R_0, R)$, where

$$R_0 = \frac{1}{2}\left(\frac{1 + tr}{1 + r} + \frac{1 - tr}{1 - r}\right),\ R = \frac{1}{2}\left(\frac{1 - tr}{1 - r} - \frac{1 + tr}{1 + r}\right),$$

from the relation (1) follows

$$\left(\frac{1 + tr}{1 + r}\right)^{\frac{2(a-1)}{\lambda}} < \left|\frac{F(tz)}{F(z)}\right| < \left(\frac{1 - tr}{1 - r}\right)^{\frac{2(a-1)}{\lambda}} . \tag{2}$$

The conclusion (i) follows from Lemma and (2).

Since

$$|Arg\omega| < \frac{\pi}{2} \quad \forall \omega \in S(R_0, R), \quad \sum_{k=1}^{n} \lambda_k = 1 \quad \text{and} \quad \frac{2(a-1)}{\lambda} < 1$$

it follows

$$\operatorname{Re}\left[\frac{F(tz)}{F(z)}\right] > 0 \quad \text{in } \mathbf{U}. \tag{3}$$

The conclusion (ii) follows from (2) and (3).

The convexity of the function $\tau(\omega)=(R_0+R\omega)^{\frac{2(a-1)}{\lambda}}$ in U implies

$$\mathrm{Re}\left(\frac{1+tz}{1+z}\right)^{\frac{2(a-1)}{\lambda}} > \left(\frac{1+tr}{1+r}\right)^{\frac{2(a-1)}{\lambda}}. \tag{4}$$

If $\tau_1(\omega)=\mathrm{Log}\tau(\omega)$ then

$$\mathrm{Re}\left[1+\omega\frac{\tau_1''(\omega)}{\tau_1'(\omega)}\right]=\mathrm{Re}\left[\frac{R_0}{\tau(\omega)}\right]>0 \quad \text{in } U.$$

The convexity of the function $\mathrm{Log}\tau_1(\omega)$ implies hat for every $z\in\mathbf{U}_r$ there exists $\omega(z)\in\mathbf{U}_r$ such that

$$\frac{F(tz)}{F(z)}=\left(\frac{1+t\omega(z)}{1+\omega(z)}\right)^{\frac{2(a-1)}{\lambda}}. \tag{5}$$

The conlusion (iii) follows from (4), (5) and Lemma.

If $\frac{zf'(z)}{f(z)}=h(z)$ from the relation $(*)$ of the Lemma for $\lambda=1$ then we have

$$f(z)=z\exp\int_0^z (h(\omega)-1)\omega^{-1}d\omega$$

or

$$|f(z)|=|z|\exp\int_0^1 \mathrm{Re}[h(tz)-1]t^{-1}dt$$

Since $\mathrm{Re}h(tz)<\frac{1}{P(\lambda,\alpha,tr)}$ the relation (6) implies the conclusion (iv).

If $f_\lambda(z)=(1-\alpha)(1+z)(1-z)^{-1}+\alpha$ then $\frac{zf'(z)}{f(z)}=P(\lambda,\alpha,z)$.

It is now obvious that the inequalities of the theorem are best possible.

If $p\in\mathbf{P}$ then it is known that there exists a sequence of functions $p_n\in\mathbf{P}$ having the form we used such that $\lim_{n\to\infty}p_n(z)=p(z)$ in $\mathbf{U}$.

In the general case where $\mathrm{Re}f_\lambda>\alpha$ or $f_\lambda=(1-\alpha)p+\alpha$, $p\in\mathbf{P}$ we consider the sequence

$$(1-\lambda)\frac{zf_n'(z)}{f_n(z)}+\lambda\left[1+z\frac{f_n''(z)}{f_n'(z)}\right]=(1-\alpha)p_n(z)+\alpha.$$

If $F(z)$ and $F_n(z)$ are the functions of Lemma corresponding to f and f_n, respectively, then from the relation $\left|\frac{(1-\alpha)p_n(z)+\alpha-1}{z}\right|\le\frac{2(a-1)}{r}$ in $\mathbf{U}_r$ it follows that $\lim_{n\to\infty}F_n(z)=F(z)$ in $\mathbf{U}_r$. Continuiting in this manner we prove that

$$\lim_{n\to\infty}\left[\frac{zf_n'(z)}{f_n(z)}\right]=\frac{zf'(z)}{f(z)} \quad \text{and} \quad \lim_{n\to\infty}f_n'(z)=f(z).$$

References

[1] M. Nunokawa, A sufficient condition for Univalence and Strarlikeness, *Proc. Japan Acad.* **65** Ser. A no. 6 (1989), 163–164.

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