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On Baker's explicit *abc*-conjecture

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Dedicated to the memory of Professor Alan Baker

Abstract. We derived from Baker's explicit *abc*-conjecture that $a+b=c$, where a, b and c are relatively prime positive integers, implies that $c < N^{1.72}$ for $N \geq 1$ and $c < 32N^{1.6}$ for $N \geq 1$. This sharpens an estimate of Laishram and Shorey. We also show that it implies $c < \frac{6}{5}N^{1+G(N)}$ for $N \geq 3$, and $c < \frac{6}{5}N^{1+G_1(N)}$ for $N \geq 297856$, where $G(N)$ and $G_1(N)$ are explicitly given positive valued decreasing functions of N tending to zero as N tends to infinity. Finally, we give applications of our estimates on triples of consecutive powerful integers and generalized Fermat equation.

1. Introduction

The well-known *abc*-conjecture was formulated by Joseph Oesterlé [9] and David Masser [6] in 1988. It states that

Conjecture 1.1. *For any given $\epsilon > 0$, there exists a number K_ϵ depending only on ϵ such that if*

$$a+b=c \tag{1}$$

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where a, b and c are relatively prime positive integers, then

$$c \le K_\epsilon \Big(\prod_{p|abc} p \Big)^{1+\epsilon},$$

where the product is taken over all primes p dividing abc.

The name *abc*-conjecture derives from letters a, b, c used in the statement. There are several works on the *abc*-conjecture and its variations.

For a positive integer ν, we define the radical $N(\nu)$ of ν by the product of primes dividing ν and $\omega(\nu)$ for the number of distinct prime divisors of ν. The letter p always denotes a prime number in this paper except in Theorem 1.6 and its proof. We denote the radical of abc by

$$N = N(abc) = \prod_{p|abc} p \tag{2}$$

unless otherwise specified. Further, we write $\omega = \omega(N)$ for the number of distinct prime divisors of N. We see when $\omega = 0$ or N is odd, then (1) does not hold, and trivially $1+1 = 2$ when $\omega = 1$. Therefore, we always have N being even and $\omega \ge 2$ unless $(a, b, c) = (1, 1, 2)$ when $\omega = 1$. We understand that $\log_2 x = \log\log x$ for $x \ge 2$, and $\log_3 x = \log\log\log x$ for $x \ge 3$. We observe that Conjecture 1.1 is not explicit in the sense that K_ϵ is not explicit. Alan Baker [1] in 2004 formulated the following explicit version of Conjecture 1.1.

Conjecture 1.2. *Let a, b and c be relatively prime positive integers satisfying* (1) *with $N > 2$. Then*

$$c < \frac{6}{5} N \frac{(\log N)^\omega}{\omega!}, \tag{3}$$

where $N = N(abc)$ and $\omega = \omega(N)$.

We refer to Conjecture 1.1 as *abc*-conjecture and Conjecture 1.2 as an explicit version of the *abc*-conjecture. For integer $N > 2$, let

$$A(N) = \log_2 N - \log_3 N, A_1(N) = A(N) + \log A(N) - 1.076869,$$

and

$$G(N) = \frac{1 + \log A(N)}{A(N)}. \tag{4}$$

Further, we define $G(x) = G([x])$ for $x > 2$. We observe that $G(N)$ is a positive-valued function that tends to zero as N tends to infinity. It is decreasing if $A'(N)\log A(N) > 0$, which is the case when $N \geq 16$, since

$$A'(N) = \frac{1}{N\log N}\Big(1 - \frac{1}{\log_2 N}\Big). \tag{5}$$

Thus $G(N)$ is decreasing for $N \geq 16$. Further, for integer $N \geq 40$, let

$$G_1(N) = \frac{1 + \log A_1(N)}{A_1(N)} \tag{6}$$

and $G_1(x) = G_1([x])$ for $x \geq 40$. We observe that $G_1(N)$ is positive for $N \geq 574$ and tends to zero as N tends to infinity. Further, $G_1(N)$ is decreasing if $A_1'(N)\log A_1(N) > 0$. Let $N \geq 297856$. Then $A_1(N) > 1$. Further, $A(N) > 0$ and $A'(N) > 0$ by (5). Since

$$A_1'(N) = A'(N) + \frac{A'(N)}{A(N)} = \frac{A'(N)}{A(N)}(1 + A(N)),$$

we see that $A_1'(N)\log A_1(N) > 0$. Hence, $G_1(N)$ is decreasing whenever $N \geq 297856$.

We compare these functions. For this, we observe that the function $F(x) = \frac{1+\log x}{x}$ is decreasing for $x > 1$ and

$$1 < A(N) < A_1(N) \quad \text{for } N \geq 1.5 \times 10^{36},$$

since $A(N) > e^{1.076869}$ for $N \geq 1.5 \times 10^{36}$. Therefore,

$$G(N) = F(A(N)) \geq F(A_1(N)) = G_1(N) \quad \text{for } N \geq 1.5 \times 10^{36},$$

and similarly, we derive that

$$G(N) \leq G_1(N) \quad \text{for } 297856 \leq N \leq 10^{36}. \tag{7}$$

Conjecture 1.2 implies the following sharper and explicit version of *abc*-conjecture in which we allow ϵ to be a function of N tending to zero as N tends to infinity.

Theorem 1.3. *Let a, b and c be relatively prime positive integers satisfying* (1). *Then* (3) *implies that*

$$c < \frac{6}{5}N^{1+G(N)} \quad \textit{for } N > 2, \tag{8}$$

and

$$c < \frac{6}{5}N^{1+G_1(N)} \quad \textit{for } N \geq 297856. \tag{9}$$

On the other hand, STEWART and TIJDEMAN [11] showed that there are infinitely many relatively prime positive integers a, b, c satisfying (1) such that for $\delta > 0$, we have

$$c > N^{1+\frac{4-\delta}{\sqrt{\log N}\log\log N}}.$$

The constant $4-\delta$ was improved by VAN FRANKENHUYSEN in [13].

LAISHRAM and SHOREY [5] showed that Conjecture 1.2 implies that for $N > 2$, we have

$$c < N^{1+\theta} \quad \text{with } \theta = \frac{3}{4}. \tag{10}$$

Further, they also derived under Conjecture 1.2 that for $0 < \theta < 3/4$, (10) holds when $N \geq N_\theta$, where N_θ is an effectively computable number depending only on θ. Theorem 1.3 provides a value of N_θ for every $0 < \theta < 1$ determined by an explicitly given function; we do not have to compute for every θ. Now we prove the following Theorem with a sharper exponent than (10).

Theorem 1.4. *Let a, b and c be relatively prime positive integers satisfying* (1). *Then* (3) *implies that for $N > 2$, we have*

$$c < N^{1.72}. \tag{11}$$

Further,

$$c < 10N^{1.62991} \tag{12}$$

and

$$c < 32N^{1.6}. \tag{13}$$

Remark 1.1. The exponent 1.72 in (11) has been improved to 1.7 by CHIM, NAIR and SHOREY in [2].

E. REYSSAT [3] considered (1.1) with $a = 2$, $b = 3^{10} \times 109$, $c = 23^5$ and $N = 15042$. This implies $c > N^{1.62991}$, which we may compare with (12).

The following theorem follows immediately from (11), (13) and (9).

Theorem 1.5. *Let a, b and c be relatively prime positive integers satisfying* (1). *Then* (3) *implies that*

$$c < \begin{cases} N^{1.72}, & \text{if } N > 2, \\ 32N^{1.6}, & \text{if } N \geq 10^{12.55}, \\ \frac{6}{5}N^{1+G_1(N)}, & \text{if } N \geq 10^{80.53}. \end{cases}$$

The result can be applied to give an explicit bound for the magnitude of solutions of the generalized Fermat equation. Let $(p,q,r) \in \mathbb{Z}_{\geq 2}$ with $(p,q,r) \neq (2,2,2)$. The equation

$$x^p + y^q = z^r, \quad (x,y,z) = 1 \quad \text{with integers } x > 0, y > 0, z > 0 \tag{14}$$

is called the *generalized Fermat equation*. It is conjectured that there are no nontrivial solutions to (14) once $\min\{p,q,r\} \geq 3$. We consider (14) with $p \geq 3$, $q \geq 3$, $r \geq 3$. For solving (14), there is no loss of generality in assuming $x > 1$, $y > 1$ and $z > 1$, since otherwise (14) is completely solved by Mihăilescu [7].

Let $[p,q,r]$ denote all permutations of the ordered triple (p,q,r). Let

$$Q = \{[3,5,p] : 7 \leq p \leq 23, p \text{ prime}\} \cup \{[3,4,p] : p \text{ prime}\}. \tag{15}$$

Then Laishram and Shorey [5] proved that (14) with $x > 1$, $y > 1$, $z > 1$, $p \geq 3$, $q \geq 3$, $r \geq 3$ implies that $[p,q,r] \in Q$ such that $\max(x^p, y^q, z^r) < e^{1758.3353}$ whenever (3) holds. This implies that $\max\{p,q,r\} < \frac{1758.3353}{\log 2} < 2537$ trivially. Let $Q_1 = \{[3,5,p] : 7 \leq p \leq 19, p \text{ prime}\} \cup \{[3,4,p] : 11 \leq p < 253, p \text{ prime}\}$. We sharpen the above result as follows.

Theorem 1.6. *Assume* (3). *Then* (14) *with* $x > 1$, $y > 1$, $z > 1$, $p \geq 3$, $q \geq 3$ *and* $r \geq 3$ *implies that* $[p,q,r] \in Q_1$. *Further, for each* $[p,q,r] \in Q_1$, *we have the following upper bound for* $\max(x^p, y^q, z^r)$.

$[p,q,r]$	$\max(x^p,y^q,z^r) <$	$[p,q,r]$	$\max(x^p,y^q,z^r) <$
$[3,4,p], 37 \leq p < 253$, p *prime*	8.1×10^{75}	$[3,4,11]$	2.2×10^{599}
$[3,4,31]$	1.3×10^{123}	$[3,5,19]$	1.6×10^{61}
$[3,4,29]$	4.3×10^{130}	$[3,5,17]$	6.7×10^{69}
$[3,4,23]$	1.2×10^{167}	$[3,5,13]$	3.9×10^{107}
$[3,4,19]$	9.8×10^{217}	$[3,5,11]$	3.9×10^{155}
$[3,4,17]$	1.2×10^{263}	$[3,5,7]$	6.6×10^{645}
$[3,4,13]$	1.5×10^{481}		

Next, we give some applications of our theorems to powerful numbers. An integer ν is called powerful if $\nu > 0$ and $p^2 | \nu$ whenever $p | \nu$ for every prime p. Golomb [4] proved in 1970 that there are infinitely many pairs of consecutive powerful integers and there exists no four (or more) consecutive powerful integers. Erdős conjectured that there does not exist three consecutive powerful integers. Trudgian [12] proved, under Conjecture 1.2, that $t < 10^{20000}$ whenever

$(t-1,t,t+1)$ is a triple of consecutive powerful integers. MOLLIN and WALSH [8] obtained the following results. Assume $t-1$, t, $t+1$ are powerful. Put

$$P = t, \qquad Q = (t-1)(t+1) = my^2,$$

where m is squarefree. Then $m \equiv 7 \pmod 8$, and (t, y) is a solution of $x^2 - my^2 = 1$. For the case when $m = 7$, Mollin and Walsh [8] proved that

$$t > 10^{10^8}. \tag{16}$$

Hence, together with the result by Trudgian [12], there is no triple $(t-1,t,t+1)$ of consecutive powerful integers such that $t^2 - 7y^2 = 1$ under Conjecture 1.2. By following the arguments given in Mollin and Walsh [8], we have checked that if $m = 7$ is replaced by $m \in \{15, 23, 31, 39, 47, 55, 87\}$, then (16) can be replaced by

$$t > 10^{3\times 10^{13}}.$$

Therefore, combining with the result by Trudgian [12], there is no triple $(t-1, t, t+1)$ of consecutive powerful integers such that $t^2 - my^2 = 1$ with $m \in \{7, 15, 23, 31, 39, 47, 55, 87\}$ under Conjecture 1.2.

Next, we prove the following result on triples of $(a+kd, a+(k+1)d, a+(k+2)d)$ of consecutive powerful integers in arithmetic progression.

Theorem 1.7. *Let $a > 0$, $d > 0$ and $k \geq 0$ be integers such that $(a, d) = 1$. Assume that $a_{k+i} := a + (k+i)d$, $0 \leq i \leq 2$ are all powerful integers. Then* (3) *implies the following:*

(1) *Let $\varepsilon > 0$. There exists an effectively computable number k_0 depending only on ε such that for $k \geq k_0$, we have*

$$a_{k+1} < (1.2d)^{2+\varepsilon}. \tag{17}$$

(2) *We have*

$$a_{k+1} < \max\{2.31 \times 10^{158} d^{2666}, 10^{51075}\}. \tag{18}$$

If $(t-1, t, t+1)$ is a triple of powerful integers, then $N(t(t^2-1)) < t^{3/2}$. In the next result we show that $N(t(t^2-1)) > t^{3/2}$ for all sufficiently large t whenever (3) holds.

Theorem 1.8. *If $t > 10^{51075}$, then* (3) *implies that*

$$N > t^{1.52},$$

where N is the square free part of $t(t^2-1)$.

We use SAGE for calculation and, in particular, for extracting values of a, b, c that fulfill specified conditions to come to the conclusion that (11) holds for $5 \leq \omega \leq 9$ when proving Theorem 1.4.

2. Preliminaries

For any real number $x > 0$, let $\theta(x) = \sum_{p \le x} \log p$. In 1983, G. Robin [10] proved the following lemma for $\theta(x)$.

Lemma 2.1. *Let p_n be the n-th prime. Then*

$$\theta(p_n) \ge n\Big(\log n + \log_2 n - 1.076869\Big) \quad \text{for } n > 1. \tag{19}$$

Lemma 2.2. *For $N \ge 4$, the function $u(x) = (\frac{e \log N}{x})^x$ is increasing in $1 \le x < \log N$.*

Lemma 2.3. *Let $\omega = \omega(N) \ge 13$. Then*

$$\log N > \omega \log \omega.$$

Proof. Let $N = Q_1 Q_2 \cdots Q_\omega$, where $Q_1 < Q_2 < \cdots < Q_\omega$ are prime numbers. Now if p_i denotes the i-th prime, then we have

$$N = \prod_{i=1}^{\omega} Q_i \ge \prod_{i=1}^{\omega} p_i.$$

This gives

$$\log N \ge \sum_{i=1}^{\omega} \log p_i = \theta(p_\omega).$$

Therefore, it suffices to show that $\theta(p_\omega) > \omega \log \omega$ for $\omega \ge 13$. This follows by Lemma 2.1 for $\omega \ge 19$, since $\log_2 \omega - 1.07869$ is positive. Further, we check that $\theta(p_\omega) > \omega \log \omega$ for $13 \le \omega \le 18$ by direct computation. □

Lemma 2.4. *Assume that $\log N > \omega \log \omega$. Then*

$$\omega < \frac{\log N}{A(N)}.$$

Proof. Let $\log N > \omega \log \omega$. Then we have

$$\omega < \frac{\log N}{\log \omega}. \tag{20}$$

Without loss of generality, we may assume $\omega > \frac{\log N}{\log_2 N}$. Then

$$\log \omega > \log_2 N - \log_3 N = A(N). \tag{21}$$

By combining (20) and (21), we get $\omega < \frac{\log N}{A(N)}$. □

Lemma 2.5. *Equation* (1) *with* (3) *implies that* $c < \frac{6}{5}N^{1+G(N)}$ *for* $\log N > \omega \log \omega$ *where* $G(N)$ *is given by* (4).

PROOF. Let $N < 16$ and $\log N > \omega \log \omega$. Then $\omega = 2$ and $N = 2p$ with $p \in \{3, 5, 7\}$. Now we re-write (1.1) as $2^x - p^y = \pm 1$, where $x \geq 1$ and $y \geq 1$ are integers. We may suppose that $x > 1$ and $y > 1$, otherwise the assertion follows. Mihăilescu [7] proved that Catalan equation $x^p - y^q = 1$ with $p > 1$, $q > 1$ has unique integral solution $(x, y, p, q) = (3, 2, 2, 3)$, and this implies that the solutions of (1) are given by $(a, b, c) \in \{(8, 1, 9), (1, 8, 9)\}$, and the assertion follows for each of these triplets.

Thus we may assume that $N \geq 16$. Let $\log N > \omega \log \omega$. Since $\omega! \geq \omega^\omega e^{-\omega}$ by induction on ω, we derive from (3) that

$$c < \frac{6}{5}N\frac{(\log N)^\omega}{\omega!} \leq \frac{6}{5}N\Big(\frac{e \log N}{\omega}\Big)^\omega. \tag{22}$$

Since $A(N) > 1$ for $N \geq 16$, we derive from Lemma 2.4 that

$$\omega < \frac{\log N}{A(N)} < \log N.$$

Then Lemma 2.2 implies that

$$\Big(\frac{e \log N}{\omega}\Big)^\omega \leq (eA(N))^{\frac{\log N}{A(N)}} = N^{G(N)}.$$

Thus, by (22), we get $c < \frac{6}{5}N^{1+G(N)}$. □

Corollary 2.6. *Equation* (1) *with* (3) *implies that* $c < \frac{6}{5}N^{1+G(N)}$ *for* $\omega \geq 13$ *where* $G(N)$ *is given by* (4).

PROOF. The assertion follows from Lemma 2.3 and 2.5. □

Lemma 2.7. *Equation* (1) *with* (3) *implies that* $c < \frac{6}{5}N^{1+G(N)}$ *for* $N > 2$.

PROOF. By Corollary 2.6 and Lemma 2.5, we have to consider $2 \leq \omega \leq 12$ and $\log N \leq \omega \log \omega$. Let $\omega = 2$. Then $6 \leq N \leq 4$, which is not possible. Let $\omega = 3$. Then $N \leq 27$, which is not possible, since the product of the first three prime numbers is equal to 30. Thus $\omega \geq 4$ and $N \geq 210$. Therefore $G(N)$ is decreasing. We check that $G(10^{23}) > \frac{3}{4}$, and therefore $G(N) > \frac{3}{4}$ for $N \leq 10^{23}$, since $G(N)$ is decreasing. Hence the assertion follows for $N \leq 10^{23}$ by (10). Thus we may assume that $N > 10^{23}$. Then $\omega^\omega \geq N > 10^{23}$, which implies that $\omega > 12$. This is a contradiction. □

For given $0 < \theta < 1$, $m \geq 2$ and $K > 0$, let

$$f(x) = \frac{(\log x)^m}{m!} - Kx^\theta.$$

Then

$$g(x) = x^{1-\theta}(m-1)!f'(x) = \frac{(\log x)^{m-1}}{x^\theta} - K\theta(m-1)!,$$

and

$$g'(x) = \frac{(\log x)^{m-2}}{x^{1+\theta}}\big(m - 1 - \theta \log x\big). \tag{23}$$

Then we have the following Lemma.

Lemma 2.8. *Assume that there exist positive numbers x_0 and x_1 with $1 < x_1 \leq x_0$ such that*

$$f(x_0) < 0, \quad g(x_0) < 0 \quad \textit{and} \quad g'(x_1) < 0. \tag{24}$$

Then $f(x) < 0$ for $x \geq x_0$.

Proof. Since $g'(x_1) < 0$, we see from (23) that $g'(x) < 0$ for $x \geq x_1$. Therefore, g is a decreasing function for $x \geq x_1$. Then, since $g(x_0) < 0$ and $x_0 \geq x_1$, we derive that $g(x) < 0$ for $x \geq x_0$, which implies that $f'(x) < 0$ for $x \geq x_0$. Thus $f(x)$ is decreasing for $x \geq x_0$. Hence the assertion follows, since $f(x_0) < 0$. □

Lemma 2.9. *Let a, b and c be relatively prime positive integers satisfying* (1). *Then* (3) *implies that*

$$c < 32N^{1.6} \quad \textit{for } N > 2.$$

Proof. Following the same proof as in [5, Theorem 1], we have $\omega_1 = \omega_\epsilon = 42$ for $\epsilon = 0.6$ such that

$$\epsilon \geq \frac{1 + \log X_0(i)}{X_0(i)} \quad \text{for } i \geq \omega_1, \quad \text{and} \quad \frac{i!\Theta(p_i)^\epsilon}{\theta(p_i)^i} > \sqrt{2\pi i} \quad \text{for } i \geq \omega_\epsilon \tag{25}$$

holds. Here $X_0(i) = \log i + \log_2 i - 1.076869$ and $\frac{i!N^\epsilon}{(\log N)^i} > \frac{i!\Theta(p_i)^\epsilon}{\theta(p_i)^i}$. We check that for $35 \leq \omega < 42$, we have

$$\frac{\omega!\Theta(p_\omega)^\epsilon}{\theta(p_\omega)^\omega} > \frac{6}{5}. \tag{26}$$

Then

$$\frac{(\log N)^\omega}{\omega!} < \frac{5}{6}N^{0.6} \quad \text{for } N > 2, \omega \geq 35,$$

and the assertion follows from (3). Let $2 \leq \omega \leq 34$. We check that, for all ω, we may choose x_0, x_1 as in Lemma 2.8 with $x_1 = x_0 = \prod_{p \leq p_\omega} p$, $K = 80/3$ and $\theta = 0.6$ so that (24) is satisfied. Thus $f(x) < 0$ for $x \geq x_0$. Therefore $f(N) < 0$, since $N \geq \prod_{p \leq p_\omega} p = x_0$. Hence Lemma 2.9 follows. □

Lemma 2.10. *Let a, b and c be relatively prime positive integers satisfying* (1). *Then* (3) *implies that*

$$c < 10N^{1.62991} \quad \textit{for } N > 2.$$

Proof. Let $\epsilon = 0.62991$. As in Lemma 2.9, we have $\omega_1 = 33$, $\omega_\epsilon = 32$ such that (25) holds. We check that for $26 \leq \omega < 32$, we have (26). Therefore, $c < 10N^{1.62991}$ for $N > 2$ with $\omega \geq 26$. Let $2 \leq \omega \leq 25$. We may choose $x_1 = x_0 = \prod_{p \leq p_\omega} p$ with $K = 25/3$ and $\theta = 0.62991$ in Lemma 2.8, we get $f(x) < 0$ for $x \geq x_0$, which implies that $f(N) < 0$ for $N \geq \prod_{p \leq p_\omega} p = x_0$. Hence Lemma 2.10 follows. □

3. Proof of Theorem 1.3

By Lemma 2.7, we have (8). Now by (7), we have

$$c < \frac{6}{5}N^{1+G(N)} \leq \frac{6}{5}N^{1+G_1(N)} \quad \text{for } 297856 \leq N \leq 10^{36}.$$

Therefore, we may assume that $N > 10^{36}$. By Lemma 2.1 with $n = \omega$, we have

$$\omega \leq \frac{\log N}{\log\omega + \log_2\omega - 1.076869}. \tag{27}$$

Let $\omega \geq \frac{\log N}{\log_2 N}$. Then $\log\omega \geq A(N)$, $\log_2\omega \geq \log A(N)$. Thus (27) gives $\omega \leq \frac{\log N}{A_1(N)}$. Therefore

$$\omega \leq \max\Big(\frac{\log N}{\log_2 N}, \frac{\log N}{A_1(N)}\Big) < \frac{\log N}{A_1(N)} < \log N, \tag{28}$$

since $A_1(N) \leq \log_2 N - 1.076869 < \log_2 N$ and $A_1(N) > 1$ by $N \geq 297856$. Then we derive from (3), (28) and Lemma 2.2 that

$$c < \frac{6}{5}N\Big(\frac{e\log N}{\omega}\Big)^\omega \leq \frac{6}{5}N(eA_1(N))^{\frac{\log N}{A_1(N)}} = \frac{6}{5}N^{1+G_1(N)}.$$

4. Proof of Theorem 1.4

Assertions (12) and (13) follow from Lemmas 2.10 and 2.9, respectively. We proceed with the proof of assertion (11).

As in Lemma 2.9, we have $\omega = 18$ and $\omega_\epsilon = 17$ for $\epsilon = 0.72$ such that (25) holds. We check that for $10 \le \omega < 17$, we have (26). Thus we get

$$\frac{(\log N)^\omega}{\omega!} < \frac{5}{6} N^{0.72} \quad \text{for } N > 2, \omega \ge 10.$$

Let $\omega \le 9$. We apply Lemma 2.8 with $x_1 = x_0$, $K = 5/6$ and $\theta = 0.72$. Then N's lie in the range $\left[\prod_{p \le p_\omega} p, x_0\right)$.

We observe that for $\omega \le 4$, we may choose $x_1 = x_0 = \prod_{p \le p_\omega} p$ so that (24) is satisfied. Then (11) follows by Lemma 2.8 with $K = 5/6$.

For $5 \le \omega \le 9$, we choose $x_1 = x_0$ as given in Table 1 so that they satisfy (24), and we extract all square free N with $\omega(N) = \omega$ that lie in the range $\left[\prod_{p \le p_\omega} p, x_0\right)$. Hence we obtain Table 1.

ω	$\prod_{p \le p_\omega} p$	$x_0,\ x_1$	$N \in \left[\prod_{p \le p_\omega} p, x_0\right)$
5	2310	4100	2310, 2730, 3570, 3990.
6	30030	87900	30030, 39270, 43890, 46410, 51870, 53130, 62790, 66990, 67830, 71610, 72930, 79170, 81510, 82110, 84630, 85470.
7	510510	1510000	510510, 570570, 690690, 746130, 870870, 881790, 903210, 930930, 1009470, 1067430, 1111110, 1138830, 1193010, 1217370, 1231230, 1272810, 1291290, 1345890, 1360590, 1385670, 1411410, 1438710, 1452990, 1504230.
8	9699690	24500000	9699690, 11741730, 13123110, 14804790, 15825810, 16546530, 17160990, 17687670, 18888870, 20030010, 20281170, 20930910, 21111090, 21411390, 21637770, 21951930, 23130030, 23393370, 23993970.
9	223092870	391000000	223092870, 281291010, 300690390, 340510170, 358888530, 363993630, 380570190.

Table 1. Data for $5 \le \omega \le 9$.

By (3), for each $N = Q_1Q_2\cdots Q_\omega$ where $Q_1, Q_2, \ldots, Q_\omega$ are distinct primes and $5 \le \omega \le 9$, it suffices to restrict $c \in \left[N^{1.72}, \frac{6}{5}N\frac{(\log N)^\omega}{\omega!}\right)$, otherwise (11) holds. We perform searching of c with SAGE by identifying all integers falling in this interval having only prime factors in the set $\{Q_1, \ldots, Q_\omega\}$. For each c with $\mathrm{rad}(c) < N$, we construct all possible choices of a satisfying $a < b$, which we may assume without loss of generality, so that $a < \frac{c}{2}$ and the property that a has only prime factors in $\{Q_1, \ldots, Q_\omega\}$ and $(a, c) = 1$. Then for each pair of (c, a) obtained with $\mathrm{rad}(ac) < N$, we construct the corresponding b by (1). We note that $(a, b, c) = 1$. We check that for each case, there does not exist any a, b, c such that the radical of abc is equal to N. Besides, it is clear that if $\mathrm{rad}(c) = N$ or $\mathrm{rad}(ac) = N$, then there exist no relatively prime positive integers a, b, c satisfying (1) with $\mathrm{rad}(abc) = N$. Hence (11) holds.

To illustrate, for $\omega = 5$, $N = 3990 = 2 \times 3 \times 5 \times 7 \times 19$, the only c extracted is $1562500 = 2^2 \times 5^8$, and there are a total of 117 a's. For $\omega = 7$, $N = 1504230 = 2 \times 3 \times 5 \times 7 \times 13 \times 19 \times 29$, the only c's extracted are

$$42168581000, 42169420800, 42174006784, 42174732915, 42176295000,$$
$$42178070844, 42182400000, 42185786580 \quad \text{and} \quad 42185937500.$$

For each c in the above list, the number of corresponding a's is 22, 54, 599, 181, 10, 71, 186, 147 and 115, respectively. Table 2 lists the number of c extracted for some selected cases of ω and N.

ω	N	No. of c extracted
5	2310	32
	3570	9
6	30030	631
	85470	18
7	510510	4565
	1452990	183

ω	N	No. of c extracted
8	9699690	25548
	23993970	648
9	223092870	98273
	380570190	4885

Table 2. Number of c extracted in selected cases of ω and N.

5. Proof of Theorem 1.6

We may assume that each of p, q, r is either 4 or an odd prime. Let $[p, q, r]$ denote all permutations of the ordered triple (p, q, r). Laishram and Shorey [5]

proved that (14) with the assumptions in Theorem 1.6 implies that $[p,q,r] \in Q$, where Q is in (15). Therefore we shall restrict our attention to those $(p,q,r) \in Q$. Note that

$$x < z^{r/p}, \qquad y < z^{r/q}.$$

We observe that $N(x^p y^q z^r) = N(xyz)$, and we always write $N = N(xyz)$ in the proof of Theorem 1.6. Then by using (11), we get

$$z^r < N^{1.72} \le (xyz)^{1.72} < z^{1.72(1+r/p+r/q)},$$

implying

$$\frac{1}{1.72} < \frac{1}{p} + \frac{1}{q} + \frac{1}{r}.$$

Thus we need to consider $(p,q,r) \in Q_2 = \{[3,5,p] : 7 \le p \le 19,\ p \text{ prime}\} \cup \{[3,4,p] : 11 \le p < 2537,\ p \text{ prime}\}$. For $N < 297856$, we apply (11) to get

$$\max(x^p, y^q, z^r) < N^{1.72} < 297856^{1.72} < 2.7 \times 10^9.$$

Therefore, we may assume that $N \ge 297856$. We deduce the upper bound for each case of $[p,q,r]$ separately. We present the proof of $[3,4,p]$ with $37 \le p < 2537$, p prime as follows. Let $N > e^{107.07}$, where we observe that $\prod_{p \le p_{30}} p < e^{107.07}$. By following the proof as in [5, Theorem 1], we have $\omega_1 = 31$, $\omega_\epsilon = 30$ for $\epsilon = 173/271$ such that (25) holds and

$$z^r < \frac{6}{5\sqrt{2\pi\omega_\epsilon}} N^{1+\epsilon} \le (xyz)^{1+\epsilon}.$$

Then $z^r < z^{(1+\epsilon)(1+r/p+r/q)}$, implying

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > \frac{1}{1+\epsilon} = \frac{271}{444} = \frac{1}{3} + \frac{1}{4} + \frac{1}{37}.$$

This is a contradiction. Therefore, we may suppose that $N < e^{107.07}$. By (13), we have

$$\max(x^p, y^q, z^r) < 32N^{1.6} < 32e^{107.07(1.6)} < 8.1 \times 10^{75}.$$

This also implies that $\max\{p,q,r\} < \frac{\log(8.1\times 10^{75})}{\log 2} < 253$ trivially.

Let $[p,q,r] = [3,5,7]$. First, we consider $N \ge e^{1004.763}$. We apply [5, Theorem 1] with $\epsilon = 34/71$. We observe that $\omega_\epsilon = 175$ and $\prod_{p \le p_{175}} p < e^{1004.763}$. Therefore, by [5, Theorem 1], we have $z^r < N^{1+\epsilon} \le (xyz)^{1+\epsilon} < z^{(1+\epsilon)(1+r/p+r/q)}$. This implies that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > \frac{1}{1+\epsilon} = \frac{71}{105} = \frac{1}{3} + \frac{1}{5} + \frac{1}{7},$$

which is a contradiction. Therefore, we may suppose that $N < e^{1004.763}$. Now we apply Theorem 1.5 repetitively to obtain upper bound for z^r as follows:

(1) For $N < 10^{12.55}$, we have $z^r < N^{1.72} < 10^{12.55(1.72)} < 3.9 \times 10^{21}$.

(2) For $10^{12.55} \le N < 10^{80.53}$, $z^r < 32N^{1.6} < 32\left(10^{80.53}\right)^{1.6} < 2.3 \times 10^{130}$.

(3) For $10^{80.53} \le N < e^{900}$, we use $G_1(10^{80.53}) \le 0.61771$ to get

$$z^r < \frac{6}{5}N^{1+G_1(10^{80.53})} < \frac{6}{5}e^{900(1.61771)} < e^{1457}.$$

(4) For $e^{900} \le N < e^{984}$, we use $G_1(e^{900}) \le 0.49781$ to get

$$z^r < \frac{6}{5}N^{1+G_1(e^{900})} < \frac{6}{5}e^{984(1.49781)} < e^{1475}.$$

(5) For $e^{984} \le N < e^{1004.763}$, we observe that $\prod_{p \le p_{172}} p < e^{984}$. By following the proof as in [5, Theorem 1] with $\epsilon = 0.48$, $\omega_1 = 173$ and $\omega_\epsilon = 172$, we get

$$z^r < \frac{6}{5\sqrt{2\pi\omega_\epsilon}}N^{1+\epsilon} < e^{1004.763(1.48)} < e^{1488}.$$

Now we combine all the above estimates. We get $\max(x^p, y^q, z^r) < e^{1488} < 6.6 \times 10^{645}$.

The proof of $[3, 4, 11]$ is similar. In this case, we suppose $N < e^{928.667}$, by following the proof of [5, Theorem 1] with $\epsilon = 43/89$ and observing that $\omega_\epsilon = 164$, $\prod_{p \le p_{164}} p < e^{928.667}$. We apply Theorem 1.5 repetitively to obtain $\max(x^p, y^q, z^r) < e^{1380} < 2.2 \times 10^{599}$.

We now present the proof of the case $[3, 5, 19]$ with $r = 3$. We first suppose that $z < 1.21 \times 10^{15} =: Z_{[3,5,19]}$. By (13),

$$\begin{aligned} z^r &< 32N^{1.6} \le 32(xyz)^{1.6} < 32z^{1.6(1+r/p+r/q)} \\ &< 32{Z_{[3,5,19]}}^{1.6(1+3/5+3/19)} < 8.5 \times 10^{43} =: A_{[3,5,19]}. \end{aligned}$$

Next, suppose that $z \ge Z_{[3,5,19]}$. From (9) we have

$$z^r < \frac{6}{5}(xyz)^{1+G_1(N)} < \frac{6}{5}z^{(1+r/p+r/q)(1+G_1(N))} < z^{0.00525+r(1/r+1/p+1/q)(1+G_1(N))},$$

giving

$$\frac{1}{1+G_1(N)} < \frac{r}{r-0.00525}\left(\frac{1}{r}+\frac{1}{p}+\frac{1}{q}\right) = \frac{3}{3-0.00525}\left(\frac{167}{285}\right). \tag{29}$$

If $N \geq 2 \times 10^{37} =: N_{[3,5,19]}$, we use the fact that G_1 is decreasing to get $G_1(N) \leq 0.7036 =: G_{1[3,5,19]}$. Then $\frac{1}{1+G_1(N)}$ exceeds the right-hand side of (29). Thus, we may assume $N < 2 \times 10^{37}$, and hence

$$z^r < 32\big(2 \times 10^{37}\big)^{1.6} < 1.6 \times 10^{61} =: B_{[3,5,19]}.$$

For $r = 5$ and $r = 19$, the proofs are similar with the corresponding parameters $Z_{[3,5,19]}$, $A_{[3,5,19]}$, $G_{1[3,5,19]}$, $N_{[3,5,19]}$ and $B_{[3,5,19]}$ as shown in Table 4. Hence we conclude

$$\max\left(x^p, y^q, z^r\right) < 1.6 \times 10^{61} =: C_{[3,5,19]}.$$

The proofs for the remaining cases of $[p, q, r]$ can be deduced similarly. The results for all cases of $[p, q, r]$ are shown in Table 3 and Table 4.

$[p, q, r]$	Upper bound for $\max\left(x^p, y^q, z^r\right)$
$[3, 4, p], 37 \leq p < 253$, p prime	8.1×10^{75}
$[3, 5, 7]$	6.6×10^{645}
$[3, 4, 11]$	2.2×10^{599}

Table 3. Upper bound for $\max\left(x^p, y^q, z^r\right)$ for $[3, 4, p]$ $(37 \leq p < 253$, p prime), $[3, 5, 7]$ and $[3, 4, 11]$.

$[p, q, r]$	r	$Z_{[p,q,r]}$	$A_{[p,q,r]}$	$G_{1[p,q,r]}$	$N_{[p,q,r]}$	$B_{[p,q,r]}$	$C_{[p,q,r]}$
$[3, 5, 19]$	3	1.21×10^{15}	8.5×10^{43}	0.7036	2×10^{37}	1.6×10^{61}	1.6×10^{61}
	5	1.12×10^{9}	8.5×10^{43}	0.7036	2×10^{37}	1.6×10^{61}	
	19	241	8.7×10^{43}	0.7036	2×10^{37}	1.6×10^{61}	
$[3, 5, 17]$	3	6.8×10^{21}	3.7×10^{63}	0.6867	5×10^{42}	6.7×10^{69}	6.7×10^{69}
	5	1.26×10^{13}	3.7×10^{63}	0.6867	5×10^{42}	6.7×10^{69}	
	17	7125	3.7×10^{63}	0.6867	5×10^{42}	6.7×10^{69}	
$[3, 5, 13]$	3	5.2×10^{29}	3.6×10^{88}	0.6372	2×10^{66}	3.9×10^{107}	3.9×10^{107}
	5	6.8×10^{17}	3.7×10^{88}	0.6372	2×10^{66}	3.9×10^{107}	
	13	7.21×10^{6}	3.6×10^{88}	0.6372	2×10^{66}	3.9×10^{107}	
$[3, 5, 11]$	3	7.9×10^{44}	1.1×10^{136}	0.601	2×10^{96}	3.9×10^{155}	3.9×10^{155}
	5	8.7×10^{26}	1.1×10^{136}	0.601	2×10^{96}	3.9×10^{155}	
	11	1.8×10^{12}	1.5×10^{136}	0.601	2×10^{96}	3.9×10^{155}	
$[3, 4, 31]$	3	4.72×10^{40}	4.9×10^{121}	0.6234	10^{76}	1.3×10^{123}	1.3×10^{123}
	4	3.2×10^{30}	4.9×10^{121}	0.6234	10^{76}	1.3×10^{123}	
	31	8635	5×10^{121}	0.6234	10^{76}	1.3×10^{123}	

Table 4 – *Continued on next page*

Continued from previous page

$[p,q,r]$	r	$Z_{[p,q,r]}$	$A_{[p,q,r]}$	$G_{1[p,q,r]}$	$N_{[p,q,r]}$	$B_{[p,q,r]}$	$C_{[p,q,r]}$
$[3,4,29]$	3	3.4×10^{42}	4.3×10^{127}	0.6176	5×10^{80}	4.3×10^{130}	4.3×10^{130}
	4	7.9×10^{31}	4.3×10^{127}	0.6176	5×10^{80}	4.3×10^{130}	
	29	25065	4.1×10^{127}	0.6176	5×10^{80}	4.3×10^{130}	
$[3,4,23]$	3	1.3×10^{48}	1.9×10^{146}	0.5945	3×10^{103}	1.2×10^{167}	1.2×10^{167}
	4	1.2×10^{36}	1.8×10^{146}	0.5945	3×10^{103}	1.2×10^{167}	
	23	1.9×10^{6}	2.2×10^{146}	0.5945	3×10^{103}	1.2×10^{167}	
$[3,4,19]$	3	1.4×10^{58}	1.1×10^{179}	0.5717	2×10^{135}	9.8×10^{217}	9.8×10^{217}
	4	4.1×10^{43}	1.1×10^{179}	0.5717	2×10^{135}	9.8×10^{217}	
	19	1.52×10^{9}	1.1×10^{179}	0.5717	2×10^{135}	9.8×10^{217}	
$[3,4,17]$	3	3×10^{74}	1.2×10^{231}	0.5567	3×10^{163}	1.2×10^{263}	1.2×10^{263}
	4	7.2×10^{55}	1.2×10^{231}	0.5567	3×10^{163}	1.2×10^{263}	
	17	1.4×10^{13}	1.4×10^{231}	0.5567	3×10^{163}	1.2×10^{263}	
$[3,4,13]$	3	1.3×10^{110}	3.1×10^{350}	0.5142	6×10^{299}	1.5×10^{481}	1.5×10^{481}
	4	3.8×10^{82}	2.9×10^{350}	0.5142	6×10^{299}	1.5×10^{481}	
	13	2.6×10^{25}	3.5×10^{350}	0.5142	6×10^{299}	1.5×10^{481}	

Table 4. Upper bound for $\max(x^p, y^q, z^r)$ for the remaining cases of $[p,q,r]$.

6. Proof of Theorem 1.7

We denote $M = N(a_k a_{k+1} a_{k+2})$ and $M_1 = N(d a_k a_{k+1} a_{k+2})$. Note that

$$2a_{k+1} = a_k + a_{k+2} \tag{30}$$

and $a_k \equiv a_{k+2} \pmod 2$. First, we obtain a lower bound for M and M_1 in terms of a_k by using (13). We consider the cases $2 \nmid a_k$ and $2|a_k$ separately.

Case 1. $2 \nmid a_k$. Then $(2a_{k+1}, a_k) = 1$, implying $(2a_{k+1}, a_k, a_{k+2}) = 1$. Thus, by (13), after taking $a = a_k$, $b = a_{k+2}$ and $c = 2a_{k+1}$ in (30), we obtain

$$2a_{k+1} < 32\,(N(2a_k a_{k+1} a_{k+2}))^{1.6} \le 98M^{1.6}.$$

Case 2. $2|a_k$. Then $2|a_{k+2}$, so from (30), we have

$$a_{k+1} = \frac{a_k}{2} + \frac{a_{k+2}}{2}, \tag{31}$$

where $a_{k+1}, \frac{a_k}{2}, \frac{a_{k+2}}{2} \in \mathbb{Z}$ and $\left(a_{k+1}, \frac{a_k}{2}, \frac{a_{k+2}}{2}\right) = 1$. We observe that d is odd, since $(a,d) = 1$, and therefore a_{k+1} is odd. This time, by taking $a = \frac{a_k}{2}$, $b = \frac{a_{k+2}}{2}$ and $c = a_{k+1}$ in (31), we obtain from (13) that

$$a_{k+1} < 32\left(N\left(\frac{1}{4}a_k a_{k+1} a_{k+2}\right)\right)^{1.6} \le 32M^{1.6}.$$

Hence, in both cases, we get $a_{k+1} < 49M^{1.6}$, which implies that

$$M_1 \geq M > \Big(\frac{a_{k+1}}{49}\Big)^{1/1.6}. \tag{32}$$

Next, we note that $a_k a_{k+2} = a_{k+1}^2 - d^2 < a_{k+1}^2$ and $(d^2, a_k a_{k+2}, a_{k+1}^2) = 1$. Assume

$$M \geq 297856. \tag{33}$$

Then (9) holds. Since G_1 is decreasing, we have $G_1(M_1) \leq G_1(M)$. By applying (9) with $a = a_k a_{k+2}$, $b = d^2$ and $c = a_{k+1}^2$, we obtain

$$a_{k+1}^2 < \frac{6}{5} M_1^{1+G_1(M_1)} \leq \frac{6}{5} M_1^{1+G_1(M)}.$$

Further,

$$M_1 \leq N(d)M \leq N(d)\,(a_k a_{k+1} a_{k+2})^{1/2} < d a_{k+1}^{3/2},$$

since a_k, a_{k+1} and a_{k+2} are powerful. Thus we get

$$a_{k+1}^2 < \frac{6}{5}\Big(d a_{k+1}^{3/2}\Big)^{1+G_1(M)},$$

that is

$$a_{k+1} < 1.2^{\frac{2}{1-3G_1(M)}}\, d^{\frac{2(1+G_1(M))}{1-3G_1(M)}}, \tag{34}$$

implying

$$a_{k+1} < (1.2d)^{\frac{2(1+G_1(M))}{1-3G_1(M)}}. \tag{35}$$

(1) Let $\varepsilon > 0$. We take $\varepsilon_1 = \frac{\varepsilon}{8+3\varepsilon}$. We may assume that $k \geq k_0$, where k_0 is a sufficiently large effectively computable number depending only on ε such that from (32) the assumption (33) is satisfied and $G_1(M) < \varepsilon_1$, using the facts that G_1 is decreasing and tends to 0. From (35) we have

$$a_{k+1} < (1.2d)^{\frac{2(1+G_1(M))}{1-3G_1(M)}} < (1.2d)^{\frac{2(1+\varepsilon_1)}{1-3\varepsilon_1}} = (1.2d)^{2+\varepsilon}.$$

(2) Suppose on the contrary that (18) does not hold. Then we have

$$a_{k+1} \geq \max\{2.31 \times 10^{158} d^{2666}, 10^{51075}\}. \tag{36}$$

Applying (36) to (32), we have $M_1 \geq M > \Big(\frac{a_{k+1}}{49}\Big)^{1/1.6} \geq e^{73500}$, and so assumption (33) is satisfied. Further, we derive that $G_1(M_1) \leq G_1(M) \leq 0.333$ by (6) and the monotonicity of G_1. Now we derive from (34) to give

$$a_{k+1} < 1.2^{2000} d^{2666} < 2.31 \times 10^{158} d^{2666}.$$

This is a contradiction.

7. Proof of Theorem 1.8

We assume (3) and write

$$t^2 = (t^2 - 1) + 1.$$

By (1) with $a = 1$, $b = t^2 - 1$ and $c = t^2$ and (13), we have

$$10^{2\times 51075} < t^2 < 32N^{1.6}, \tag{37}$$

which implies that $N > 10^{63842}$. Then

$$G_1(N) < 0.317315. \tag{38}$$

Thus we obtain a sharper upper bound for t^2, and we can revise (37) with the use of (9) and (38) to give

$$10^{2\times 51075} < t^2 < \frac{6}{5}N^{1.317315}. \tag{39}$$

This time we have $N > 10^{77544}$. Then, by following as above, we obtain $G_1(N) < 0.313229$ and $N > 10^{77785}$. Then

$$G_1(N) < 0.313165. \tag{40}$$

Finally, we apply (9) and (40) to derive that

$$t^2 < \frac{6}{5}N^{1.313165},$$

which implies that

$$N > 0.87t^{1.523037} > t^{1.52}.$$

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