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A study on the exponential Diophantine equation $a^x + (a+b)^y = b^z$

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Dedicated to Professor Masaaki Amou on the occasion of his 60th birthday

Abstract. For any given pair (a, b) of relatively prime integers greater than 1, we study the Diophantine equation in the title. First, we propose a conjecture to describe the positive integer solutions x, y, z of the equation. Second, we confirm our conjecture for certain pairs (a, b).

1. Introduction

For a fixed triple (a, b, c) of pair-wise relatively prime integers with $\min\{a, b, c\} > 1$, let us study the following equation:

$$a^x + b^y = c^z \tag{1.1}$$

in positive integers x, y and z. In the literature, there are a number of works on equation (1.1). Most of them concern the problem of determining the solutions of equation (1.1) for various given triples (a, b, c). Especially, the triples (a, b, c) satisfying $a^p + b^q = c^r$ for some positive integers p, q, r have been actively considered

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by many authors. In almost all of their works, it has been shown that equation (1.1) has no solution other than the trivial one, that is, $(x, y, z) = (p, q, r)$. A famous conjecture of Jeśmanowicz [13] in this direction, which concerns the case $(p, q, r) = (2, 2, 2)$, states that equation (1.1) has no nontrivial solution for any primitive Pythagorean triple (a, b, c) with $a^2 + b^2 = c^2$. This problem is generalized by the second author to the triples (a, b, c) satisfying $a^p + b^q = c^r$ for some integers p, q, r with $\min\{p, q, r\} > 1$ (see [27, Conjecture]). These problems are still unsolved in spite of many contributions. For example, see some recent papers [10]–[11], [17]–[19], [24], [28], and the references therein.

On the other hand, the first author considered a variant to the mentioned direction. In [15], he studied equation $c^x + b^y = a^x$ for the primitive Pythagorean triples (a, b, c) such that $a^2 + b^2 = c^2$ with b even, and raised a conjecture to describe its solutions (see [15, Conjecture 1.2]). Moreover, in [16], he proposed a similar problem for the triples (a, b, c) satisfying $a^p + b^q = c^r$ for some integers p, q, r with $\min\{p, q, r\} > 1$ (see [16, Conjecture 3]). Note that these can be regarded as relevant analogues to the mentioned conjectures of Jeśmanowicz and the second author.

In this paper, let us consider a similar problem corresponding to the case $(p, q, r) = (1, 1, 1)$. Our problem is stated as follows.

Conjecture 1. *Let a and b be fixed relatively prime integers with $\min\{a, b\} > 1$. Consider the equation*

$$a^x + (a + b)^y = b^z \tag{1.2}$$

in positive integers x, y and z. Then equation (1.2) has no solution, unless $b = a + 1$, $(a, b) = (2^j - 1, 2)$ with $j \geqslant 2$, or $(a, b) \in \{(3, 7), (5, 2), (279, 5)\}$. The solutions in these exceptional cases are given by

$$(x, y, z) = \begin{cases} (2, 1, 2) & \text{if } b = a + 1 \text{ with } a > 2, \\ (1, 2, 3), (2, 1, 2) & \text{if } (a, b) = (2, 3), \\ (1, 1, j + 1) & \text{if } (a, b) = (2^j - 1, 2) \text{ with } j > 2, \\ (1, 1, 3), (1, 3, 7), (3, 1, 5) & \text{if } (a, b) = (3, 2), \\ (5, 2, 3) & \text{if } (a, b) = (3, 7), \\ (2, 1, 5) & \text{if } (a, b) = (5, 2), \\ (2, 1, 7) & \text{if } (a, b) = (279, 5). \end{cases}$$

A simple program of Magma [5] verifies that this is true in the range $\max\{a, b\} \leqslant 10000$ and $\max\{x, y\} \leqslant 20$. Our results verify that Conjecture 1 is true in some special cases.

Theorem 1. *Conjecture 1 is true for each of its exceptional cases.*

Theorem 2. *Conjecture 1 is true for each of the following cases:*

(C1) $a = 2$ *or* a *is a power of* 4.

(C2) b *is a power of* 2.

(C3) $a + b$ *is a power of* 2.

The organization of this paper is as follows. In the next section, we quote several results related to ternary Diophantine equations to prove our theorems. In Section 3, we deal with the exceptional cases of Conjecture 1, except for the case where $b = a + 1$, which is regarded as the main part of the exceptional cases. Sections 4 and 5 are devoted to complete the case that $b = a + 1$, and the first theorem is proved, where a result of Bugeaud [6] on estimating simultaneous non-Archimedean valuations plays a crucial role. The second theorem is proved in the final section.

2. Preliminaries

The following is a direct consequence of [23, Theorem 6].

Proposition 1. *Let* A *and* B *be relatively prime integers with* $1 < A < B$. *Assume that* $(A, B) \notin \{(3, 5), (3, 13)\}$. *Then the equation*

$$A^x + B^y = 2^z$$

has at most one positive integer solution (x, y, z).

The following is a direct consequence of the combination of the results in [7] and either [14] or [26, Theorem 2].

Proposition 2. *All quadruples* (S, T, m, n) *of positive integers satisfying*

$$S^2 + 2^m = T^n, \quad \gcd(S, T) = 1, \ n \geqslant 3$$

are given by $(S, T, m, n) = (5, 3, 1, 3), (7, 3, 5, 4), (11, 5, 2, 3)$.

Proposition 3 (Corollaire of [12])**.** *All quadruples* (S, T, m, n) *of integers satisfying*

$$S^2 - 2^m = T^n, \quad S > 0, \ |T| > 1, \ \gcd(S, T) = 1, \ m \geqslant 2, \ n \geqslant 3$$

are given by $(S, T, m, n) = (13, -7, 9, 3), (71, 17, 7, 3)$.

Proposition 4 ([21]). *All pairs (S, n) of positive integers satisfying the equation*

$$S^2 + 7 = 2^n$$

are given by $(S, n) = (1, 3), (3, 4), (5, 5), (11, 7), (181, 15)$.

Proposition 5 (Theorem 2 (c) of [4]). *Let t be an integer with $t \geqslant 4$. Then all pairs (S, n) of positive integers satisfying the equation*

$$S^2 + 2^t - 1 = 2^n$$

are given by $(S, n) = (1, t), (2^{t-1} - 1, 2t - 2)$.

Proposition 6 (Corollary 1.7 ($y = 2$) of [1]). *Let D be a non-zero integer. Assume that positive integers U and l satisfy*

$$U^2 + D = 2^l, \quad l > 1.$$

Then either $(D; U, l) \in \{(-1; 3, 3), (7; 181, 15)\}$ or

$$l < \frac{50 \log |D|}{13 \log 2}.$$

3. Exceptional cases

Here, let us deal with the exceptional cases of Conjecture 1, except for the case where $b = a + 1$ with $a > 2$.

The cases $(a, b) = (2, 3)$ and $(3, 2)$ are settled by [20]. Also, the cases $(a, b) = (5, 2)$ and $(a, b) = (2^j - 1, 2)$ with $j > 2$ are settled by Proposition 1. Thus, it remains to deal with the case where $(a, b) \in \{(3, 7), (279, 5)\}$. Here let us adopt the algorithm developed by Bertók and Hajdu [3] (see also [2]).

Let $(a, b) = (3, 7)$. Then equation (1.2) is

$$3^x + 10^y = 7^z. \tag{3.1}$$

Conjecture 1 states that this Diophantine equation has no positive solution other than $(x, y, z) = (5, 2, 3)$. In order to see that $y < 3$ in equation (3.1), it suffices to show that the equation

$$3^x + 10^3 \cdot 10^y = 7^z, \quad x, y, z \geqslant 0 \tag{3.2}$$

has no solution. Suppose that equation (3.2) has some solution (x, y, z). Let us consider equation (3.2) for several moduli. Observe that

$$\mathrm{ord}_{37}(3) = 18, \quad \mathrm{ord}_{37}(10) = 3, \quad \mathrm{ord}_{37}(7) = 9.$$

Thus, if one considers equation (3.2) modulo 37, then one has congruence conditions on x modulo 18, on y modulo 3 and on z modulo 9. Indeed, there remain the following cases:

$$\begin{cases} x \equiv 2 \pmod{18}, & y \equiv 0 \pmod{3}, & z \equiv 3 \pmod{9}, \\ x \equiv 3 \pmod{18}, & y \equiv 2 \pmod{3}, & z \equiv 8 \pmod{9}, \\ x \equiv 4 \pmod{18}, & y \equiv 2 \pmod{3}, & z \equiv 4 \pmod{9}, \\ x \equiv 5 \pmod{18}, & y \equiv 2 \pmod{3}, & z \equiv 3 \pmod{9}, \\ x \equiv 8 \pmod{18}, & y \equiv 2 \pmod{3}, & z \equiv 0 \pmod{9}, \\ x \equiv 9 \pmod{18}, & y \equiv 1 \pmod{3}, & z \equiv 5 \pmod{9}, \\ x \equiv 10 \pmod{18}, & y \equiv 1 \pmod{3}, & z \equiv 1 \pmod{9}, \\ x \equiv 11 \pmod{18}, & y \equiv 1 \pmod{3}, & z \equiv 0 \pmod{9}, \\ x \equiv 14 \pmod{18}, & y \equiv 1 \pmod{3}, & z \equiv 6 \pmod{9}, \\ x \equiv 15 \pmod{18}, & y \equiv 0 \pmod{3}, & z \equiv 2 \pmod{9}, \\ x \equiv 16 \pmod{18}, & y \equiv 0 \pmod{3}, & z \equiv 7 \pmod{9}, \\ x \equiv 17 \pmod{18}, & y \equiv 0 \pmod{3}, & z \equiv 6 \pmod{9}. \end{cases} \tag{3.3}$$

In particular, since $x > 1$, taking equation (3.2) modulo 9 yields that $(-2)^z \equiv 1 \pmod{9}$, so

$$z \equiv 0 \pmod{3}. \tag{3.4}$$

Furthermore, taking equation (3.2) modulo 7 and 8 yields that $3^x \equiv 3^y \pmod{7}$ and $3^x \equiv (-1)^z \pmod{8}$, respectively. Thus

$$x - y \equiv 3 \pmod{6}, \tag{3.5}$$

$$x \equiv z \equiv 0 \pmod{2}. \tag{3.6}$$

Congruences (3.3), (3.4), (3.5) and (3.6) together yield that

$$x \equiv 8 \pmod{18}, \quad y \equiv 2 \pmod{3}, \quad z \equiv 0 \pmod{9}.$$

Thus, equation (3.2) can be written as

$$3^8(3^{18})^X + 10^5(10^3)^Y = (7^9)^Z, \quad X, Y, Z \geqslant 0.$$

It is observed that this equation does not hold by taking it modulo 3^3.

To sum up, equation (3.2) has no solution. Therefore, $y < 3$ on equation (3.1). In addition, it is easy to check that equation (3.1) with $y = 1$ does not hold by taking it modulo 4 and 5. Thus, $y = 2$, and equation (3.1) becomes

$$3^x + 100 = 7^z.$$

Based on the idea of showing $y = 2$, one can use the source file of SageMath in [9] to verify that $x < 6$ or $z < 4$. This enables us to check that $(x, z) = (5, 3)$, as desired.

Similarly, the case where $(a, b) = (279, 5)$ can be handled. In this case, equation (1.2) is

$$279^x + 284^y = 5^z.$$

Conjecture 1 states that this Diophantine equation has no positive solution other than $(x, y, z) = (2, 1, 7)$. It is easily observed that $y = 1$ by taking the equation modulo $3, 5$ and 8. Furthermore, using the source file in [9] for the above equation with $y = 1$, one can verify that $x < 3$, from which one concludes that $(x, z) = (2, 7)$, as desired.

To sum up, for the completion of the proof of Theorem 1, it remains to consider the excluded case, which is dealt with in the forthcoming sections.

4. Lemmas for Theorem 1

In this section, we prove some lemmas on equation (1.2) when $b = a+1$ with $a > 2$, that is,

$$a^x + (2a + 1)^y = (a + 1)^z \tag{4.1}$$

in positive integers x, y and z.

Let us begin by showing the following lemma concerning the parities of the solutions.

Lemma 4.1. *Let (x, y, z) be a solution of equation (4.1).*

(i) *The parities of x and y do not coincide.*

(ii) $x \equiv 2y \pmod{(a + 1)}$.

(iii) $a^{x-1} + 2y \equiv z \pmod{a}$. *In particular, if a is even with $x > 1$, then z is even.*

Proof. (i) Taking equation (4.1) modulo $(a + 1)$ implies that $(-1)^x + (-1)^y \equiv 0 \pmod{(a + 1)}$. As the modulus is greater than 2, the congruence is actually an equality, which shows the assertion.

(ii) Note that $z > 1$. Rewrite equation (4.1) as

$$(-1 + (a+1))^x = -(-1 + 2(a+1))^y + (a+1)^z.$$

Taking the above equation modulo $(a+1)^2$ enables us to find that

$$(-1)^x + (-1)^{x-1}(a+1)x \equiv (-1)^{y+1} + (-1)^y 2(a+1)y \pmod{(a+1)^2}.$$

This congruence together with (i) yields the asserted congruence.

(iii) Taking equation (4.1) modulo a^2 enables us to find that

$$a^x + 2ay + 1 \equiv az + 1 \pmod{a^2}.$$

This congruence immediately gives the asserted congruence. □

Lemma 4.2. *If $x = 2y$, then $(x, y, z) = (2, 1, 2)$.*

Proof. Assume $x = 2y$. Then

$$(a^2)^y + (2a+1)^y = (a+1)^z.$$

Let us rely on a direct consequence of an old version of the Primitive Divisor Theorem due to Zsigmondy [29], stated as follows.

Proposition 7. *Let $\mathcal{A}$ and $\mathcal{B}$ be relatively prime integers with $\mathcal{A} > \mathcal{B} \geqslant 1$. Let $\{\mathcal{V}_k\}_{k\geqslant 1}$ be the sequence defined as*

$$\mathcal{V}_k = \mathcal{A}^k + \mathcal{B}^k.$$

If $k > 1$, then $\mathcal{V}_k$ has a prime factor not dividing $\mathcal{V}_1\mathcal{V}_2\cdots\mathcal{V}_{k-1}$, whenever $(\mathcal{A}, \mathcal{B}, k) \neq (2, 1, 3)$.

Apply Proposition 7 with $(\mathcal{A}, \mathcal{B}) = (a^2, 2a+1)$ and $k = y$. Since $\mathcal{V}_1 = (a+1)^2$, it follows $y = 1$, which yields $(x, z) = (2, 2)$. □

Lemma 4.3. *If z is even, then $(x, y, z) = (2, 1, 2)$.*

Proof. Write $z = 2Z$ with some positive integer Z. Recall that $x \not\equiv y \pmod 2$ from Lemma 4.1 (i).

First, suppose that x is even. Write $x = 2X$. Then

$$(2a+1)^y = ((a+1)^Z + a^X)((a+1)^Z - a^X).$$

Since the two factors on the right-hand side are coprime, there are odd positive integers u, v such that

$$(a+1)^Z + a^X = u^y, \quad (a+1)^Z - a^X = v^y$$

with $uv = 2a+1$. Then

$$2a^X = u^y - v^y, \quad 2(a+1)^Z = u^y + v^y.$$

Since $\nu_2(u^y \pm v^y) = \nu_2(u \pm v)$ as all u, v, y are odd, one compares 2-adic valuations of both sides of each above equation to find that

$$\max\{X, Z\} \leqslant \frac{\log u}{\log 2}.$$

On the other hand, as $2a^X = u^y - v^y \geqslant 2u^{y-1}$ and $2(a+1)^Z > u^y$, one finds that

$$y \leqslant \frac{\log(a+1)}{\log u} \max\{X, Z\} + 1.$$

These inequalities together show that

$$\max\{x, 2y\} \leqslant \frac{2\log(a+1)}{\log u} \max\{X, Z\} + 2 \leqslant \frac{2}{\log 2} \log(a+1) + 2.$$

If $x \neq 2y$, then $\max\{x, 2y\} \geqslant a+1$ by Lemma 4.1 (ii), so $a+1 \leqslant \frac{2}{\log 2} \log(a+1)+2$, which implies that $a \leqslant 7$, and $\max\{x, 2y\} \leqslant 8$. However, equation (4.1) does not hold for any (a, x, y) under consideration. Thus, $x = 2y$, and so $(x, y, z) = (2, 1, 2)$ by Lemma 4.2.

Next, suppose that y is even. Write $y = 2Y$. Then $a^x = DE$, where

$$D = (a+1)^Z + (2a+1)^Y, \quad E = (a+1)^Z - (2a+1)^Y.$$

Since $D \equiv 2 \pmod{a}$ and $D > 1$, one observes that a has to be even and $D/2$ is prime to $a/2$. Then, by the equation $(a/2)^x 2^{x-1} = (D/2)E$ with $\gcd(D/2, a/2) = 1$, it follows that $E \equiv 0 \pmod{(a/2)^x}$ and $2^x \equiv 0 \pmod{D}$, in particular, $(a/2)^x \leqslant E < D \leqslant 2^x$. This contradicts the fact that $a > 2$ is even. □

Lemma 4.4. *For each a with $2 < a < 1800$, equation (4.1) has no solution (x, y, z) with $(x, y, z) \neq (2, 1, 2)$ satisfying*

$$\max\{x, y, z\} < \mathcal{C}(a), \quad \max\{x, z\} < 5y,$$

where $\mathcal{C}(a) = 1300$ if $a < 100$, and $\mathcal{C}(a) = 1000$ if $a \geqslant 100$.

PROOF. Firstly, note that one may add further restrictions on a, x, y, z from Lemmas 4.1 and 4.3 and their consequences. Moreover, it may be assumed that $x > a\log(a+1)$ if $x > z$. Indeed, if $x > z$, then, since $a^x < (a+1)^z \leqslant (a+1)^{x-1}$, it follows that $a+1 \leqslant (1+1/a)^x$, which implies that $x > a\log(a+1)$.

The verification can be done by Magma [5] in about three hours, as follows. Let $p = p(a)$ be the least prime factor of $2a+1$, and $\nu = \nu(a,x,z)$ the p-adic valuation of $(a+1)^z - a^x$, that is, $p^\nu \parallel ((a+1)^z - a^x)$. For each (a,x,z), the function `Valuation(*,p)` in Magma is used to confirm that $\nu \leqslant 10$. Since $y \leqslant \nu$ by equation (4.1), it follows that $y \leqslant 10$. Finally, a simple program enables us to verify that equation (4.1) has no solution (x,y,z) with $(x,y,z) \neq (2,1,2)$ satisfying $y \leqslant 10$ and $\max\{x,z\} < 5y \leqslant 50$. The lemma is proved. □

5. Proof of Theorem 1

In what follows, write $a+1 = M$ on equation (4.1). Then $M \geqslant 4$ and equation (4.1) is

$$(M-1)^x + (2M-1)^y = M^z. \tag{5.1}$$

Lemma 5.1. *Assume that* $M \equiv 2 \pmod 4$. *Then the only solution of equation* (5.1) *is* $(x,y,z) = (2,1,2)$.

PROOF. Let (x,y,z) be a solution of equation (5.1). From Lemma 4.3, it suffices to show that z is even. On the contrary, suppose that z is odd. Taking equation (5.1) modulo $(2M-1)$ yields that

$$(M-1)^x \equiv M^z \pmod{(2M-1)}.$$

Since x is even by Lemma 4.1 (ii), one easily sees that 2 is a quadratic residue modulo $(2M-1)$. Thus, a supplement of the quadratic reciprocity law tells us that $2M-1 \equiv \pm 1 \pmod 8$, which is absurd to the assumption that $M \equiv 2 \pmod 4$. This contradiction shows that z is even. □

Lemma 5.1 and Proposition 1 together with the result in [25] enable us to assume that $M \geqslant 7$, and $M \equiv 0 \pmod 4$ if M is even.

In what follows, let (x,y,z) be a solution of equation (5.1). Suppose that z is odd. We will observe that this leads to a contradiction.

In order to find an absolute upper bound for z, let us rely on the following result which is a simple consequence of [6, Theorem 2;$(y_1,y_2) = (1,1)$ and $\mu = 4$].

Proposition 8. *Let M be an integer with $M > 1$, and $M = \prod_{1 \leqslant i \leqslant w} p_i^{u_i}$ be the prime factorization of M. Let X_1 and X_2 be two coprime integers such that $\gcd(X_1X_2, M) = 1$ with $X_1, X_2 \neq \pm 1$. Assume that g is a positive integer satisfying*

$$\nu_{p_i}(X_1^g - 1) \geqslant u_i, \quad \nu_{p_i}(X_2^g - 1) \geqslant 1 \quad (i = 1, 2, \ldots, w),$$
$$\nu_2(X_i^g - 1) \geqslant 2 \quad (i = 1, 2) \quad \text{if } M \text{ is even.}$$

Let H_1 and H_2 be positive numbers such that

$$H_i \geqslant \max\{\log |X_i|, \log M\} \quad (i = 1, 2).$$

Then, for any positive integers b_1 and b_1 with $\gcd(b_1, b_2, M) = 1$, the exponent of the highest power of M dividing $X_1^{b_1} - X_2^{b_2}$ is at most

$$\frac{53.6gH_1H_2}{\log^4 M} \left(\max\{\log b' + \log\log M + 0.64,\ 4\log M\} \right)^2$$

with $b' = b_1/H_2 + b_2/H_1$.

In order to use this proposition, let us check that $\gcd(x, y, M) = 1$. On the contrary, suppose that x and y have some common prime divisor p. Write $x = pX$ and $y = pY$. Note that p is odd and $X \not\equiv Y \pmod 2$ by Lemma 4.1 (i). Then equation (5.1) is written as

$$R(A^X + B^Y) = M^z \tag{5.2}$$

with $(A, B) = (M - 1, 2M - 1)$ and $R = \frac{A^{pX} + B^{pY}}{A^X + B^Y}$. Note that $R > p$. Also, by elementary number theory (cf. [22, P1.2]), it is easy to see that R is not divisible by p^2. On the other hand, since $A \equiv B \equiv -1 \pmod M$, one sees that

$$\begin{aligned} R = \sum_{j=0}^{p-1} (-1)^j (A^X)^{p-1-j} (B^Y)^j &\equiv \sum_{j=0}^{p-1} (-1)^{j+(p-1-j)X+jY} \\ &\equiv \sum_{j=0}^{p-1} (-1)^{j(1-X+Y)} \equiv \sum_{j=0}^{p-1} 1 \equiv p \pmod M. \end{aligned}$$

It follows from equation (5.2) that R has no prime factor other than p. However, this contradicts the fact that $R > p$ and $R \not\equiv 0 \pmod{p^2}$.

Set $(X_1, X_2; b_1, b_2) = (1-M, 1-2M; x, y)$. Then $X_1^{b_1} - X_2^{b_2} = \pm M^z$ by Lemma 4.1. Since one can take $g := 1$, $H_1 := \log M$ and $H_2 := \log(2M-1)$, it follows from Proposition 8 that

$$z \leqslant \frac{53.6 \log(2M-1)}{\log^3 M} \left(\max\{\log b' + \log\log M + 0.64,\ 4 \log M\} \right)^2 \tag{5.3}$$

with $b' = x/\log(2M-1) + y/\log M$. From the trivial inequalities $x < \frac{\log M}{\log(M-1)}\, z$ and $y < \frac{\log M}{\log(2M-1)}\, z$, observe that

$$b' < \frac{2}{\log M}\, z.$$

Suppose that

$$\log b' + \log\log M + 0.64 > 4 \log M.$$

Since $M^4 < \mathrm{e}^{0.64} (\log M) b' \leqslant 2\mathrm{e}^{0.64} z$, it follows from inequality (5.3) that

$$\frac{z}{(\log(2z) + 0.64)^2} \leqslant \frac{53.6 \log(2M-1)}{\log^3 M} < 18.66. \tag{5.4}$$

This shows that $z < 1365$. Since $M^4 < 2\mathrm{e}^{0.64} z$, one has $M \leqslant 7$, so $M = 7$. Now, by Lemma 4.1 (i, iii), x has to be 1 and y is even, which, however, contradicts the consequence of [8].

Therefore, $\log b' + \log\log M + 0.64 \leqslant 4 \log M$, and it follows from inequality (5.3) that

$$z \leqslant \frac{857.6 \log(2M-1)}{\log M}.$$

This implies that

$$\begin{cases} x \leqslant 1226,\ y \leqslant 856,\ z \leqslant 1129 & \text{if } M \geqslant 7, \\ \max\{x, y, z\} < 1000 & \text{if } M \geqslant 100. \end{cases}$$

Since $x \neq 2y$ by Lemma 4.2, one sees from Lemma 4.1 (ii) that

$$M \leqslant 1712.$$

By Lemmas 4.1 and 4.4, it suffices to consider the case where $x \geqslant 5y$ or $z \geqslant 5y$. It is easy to verify that $x \geqslant 5y$ if $z \geqslant 5y$.

Suppose that $x \geqslant 5y$. From equation (5.1) it follows that

$$1 + \frac{(2M-1)^y}{(M-1)^x} = \frac{M^z}{(M-1)^x}.$$

Observe that

$$z\log M - x\log(M-1) = \log\left(1+\frac{(2M-1)^y}{(M-1)^x}\right) < \frac{(2M-1)^{x/5}}{(M-1)^x}.$$

Dividing both sides by $x\log M$ gives

$$(0<)\quad z/x - \xi < \frac{(2M-1)^{x/5}}{x(M-1)^x\log M}$$

with $\xi = \xi(M) = \frac{\log(M-1)}{\log M}$. Since the right-hand side is less than $1/(2x^2)$, Legendre's theorem on the theory of continued fraction tells us that z/x is a convergent to the irrational number ξ, say p_j/q_j as usual. Moreover, it is known that

$$|\xi - p_j/q_j| > \frac{1}{(a_{j+1}+2)q_j^2},$$

where a_{j+1} is the $(j+1)$-st partial quotient to ξ. Since $q_j \leqslant x$, these bounds for $|\xi - z/x|$ together yield a sharp lower bound for a_{j+1}, as follows:

$$a_{j+1} > \frac{(M-1)^x\log M}{x(2M-1)^{x/5}} - 2.$$

Finally, for each M, x under consideration, one can easily check that the above inequality does not hold for any j satisfying $q_j \leqslant x$.

To sum up, the proof of Theorem 1 is completed.

6. Proof of Theorem 2

Through the proof of Theorem 2, the following lemma is frequently used.

Lemma 6.1. *Let (A,C,p,q) be a quadruple of integers satisfying*

$$A^p + 2^q = C^2, \quad |A|>1,\ C>0,\ \gcd(A,C)=1,\ p\geqslant 1,\ q\geqslant 2.$$

Assume that $(A,C)\notin\{(-7,13),(17,71)\}$ and $(A,C,p)\neq(-7,181,1)$. Then one of the following cases holds:

(i) $(A,C,p) = (2^{q-2}-1, 2^{q-2}+1, 2)$ *and* $q\geqslant 4$.

(ii) $p=1$, *and*

$$q < \frac{50\log|A|}{13\log 2}.$$

Proof. This follows from the combination of Proposition 3 with $(S,T,m,n) = (C,A,q,p)$, Proposition 4 with $(S,n)=(C,q)$ and Proposition 6 with $(D;U,l) = (-A;C,q)$. □

In what follows, let us separately consider five cases.

6.1. The case $a = 2$. Let $a = 2$, and consider the equation:

$$2^x + (2+b)^y = b^z. \tag{6.1}$$

Let (x, y, z) be a triple of positive integers satisfying equation (6.1). Since b is odd, if both y, z are odd, then taking equation (6.1) modulo 8 implies that $2^x + 2 \equiv 0 \pmod 8$. However, this congruence does not hold. Therefore, y or z is even.

First, suppose that y is even. Note that $z > 2$ as $b^z > (2+b)^y \geqslant (2+b)^2 > b^2$. Then, Proposition 2 with $(S, T, m, n) = ((2+b)^{y/2}, b, x, z)$ tells us that $(b, x, y, z) = (3, 1, 2, 3)$.

Next, suppose that z is even. Now, apply Lemma 6.1 with $(A, C, p, q) = (2+b, b^{z/2}, y, x)$. Observe that $C - A = b^{z/2} - b - 2 \neq 2$ as $b > 1$ is odd. It follows $y = 1$. On the other hand, one takes equation (6.1) modulo $(b+1)$ to see that $(b+1)$ divides 2^x. Hence, $b = 2^t - 1$ with an integer t with $2 \leqslant t \leqslant x$. Taking equation (6.1) modulo 2^{t+1} yields that $2^x + 2^t + 1 \equiv 1 \pmod{2^{t+1}}$, and so $2^x \equiv 2^t \pmod{2^{t+1}}$. Thus, $x = t$. It holds from equation (6.1) that $2^{x+1} + 1 = (2^x - 1)^z$, from which $z = 2$ holds, and $x = 2$.

6.2. The case where a is a power of 4. Write $a = 4^k$ with a positive integer k, and consider the equation

$$4^{kx} + (4^k + b)^y = b^z. \tag{6.2}$$

Consider the case where b is congruent to 0 or 1 modulo 3. Taking equation (6.2) modulo 3 implies that either $2 \equiv 0 \pmod 3$ or $2^y \equiv 0 \pmod 3$. However, both congruences clearly do not hold. Thus, it suffices to consider the case where $b \equiv 2 \pmod 3$. Again, taking equation (6.2) modulo 3 implies that $2^z \equiv 1 \pmod 3$. Thus, z is even. Put $Z = z/2$. Apply Lemma 6.1 with $(A, C, p, q) = (4^k + b, b^Z, y, 2kx)$. One obtains that either $y = 1$ or

$$(4^k + b, b^Z, y) = (2^{2kx-2} - 1, 2^{2kx-2} + 1, 2), \quad kx > 1.$$

In the second case, it is clear that $b^Z = 2^{2kx-2} + 1$ holds with $2kx - 2 > 1$ and $Z > 1$. This is a special case of Catalan's equation. Thus, it may be concluded that this equation does not hold.

Finally, suppose $y = 1$. Equation (6.2) can be written as

$$4^k + b = (b^Z + 2^{kx})(b^Z - 2^{kx}).$$

In particular, $b + 2^{2k} \geqslant b^Z + 2^{kx}$. Thus, $x = 1$ or $(x, Z) = (2, 1)$. If $x = 1$, equation (6.2) is $2 \cdot 4^k + b = b^{2Z}$, which, however, contradicts the fact that $b > 1$ is odd. If $(x, Z) = (2, 1)$, then $b - 2^{2k} = 1$, i.e., $b = a + 1$.

This completes the proof of case (C1) in Theorem 2.

Remark 1. In case of (C1) in Theorem 2, the case where a is an odd power of 2 (> 2) is not considered. In that case, it is hard to find that at least one of y, z is even, even if it can be observed that x is even. Thus, our method does not work well.

6.3. The case $b = 2$. Let $b = 2$, and consider the equation:

$$a^x + (a+2)^y = 2^z. \tag{6.3}$$

Taking this equation modulo $(a+1)$ implies that $(-1)^x + 1 \equiv 2^z \pmod{(a+1)}$.

First, suppose that x is odd. Then $a+1$ divides 2^z. Hence, a is of the form $a = 2^j - 1$ with an integer $j \geqslant 2$. This is an exceptional case of Conjecture 1, already settled in Theorem 1.

Next, suppose that x is even. Put $X = x/2$. It is observed from equation (6.3) that y has to be odd (as $z > 1$). Now, equation (6.3) is written as

$$(-a-2)^y + 2^z = a^{2X}.$$

Lemma 6.1 with $(A, C, p, q) = (-a-2, a^X, y, z)$ implies that $y = 1$, and

$$z < \frac{50 \log(a+2)}{13 \log 2}.$$

This, together with the trivial estimate $a^{2X} < 2^z$, gives

$$X < \frac{25}{13} \frac{\log(a+2)}{\log a}.$$

First, consider the case where $a \leqslant 16$. In this case, both X, z are absolutely bounded by their upper bounds depending only on a. It is easy to verify that equation (6.3) does not hold for any triple $(a; X, z)$ under consideration.

Next, consider the case where $a \geqslant 17$. Then, $X = 1$, and $a^2 + a + 2 = 2^z$, that is,

$$(2a+1)^2 + 7 = 2^{z+2}.$$

Since a is odd with $a \geqslant 17$, Proposition 4 with $(S, n) = (2a+1, z+2)$ tells us that the above equation does not hold.

6.4. The case where $b > 2$ is a power of 2. Write $b = 2^k$ with an integer $k > 1$, and consider the equation:

$$a^x + (a + 2^k)^y = 2^{kz}. \tag{6.4}$$

Since a is odd, one takes equation (6.4) modulo 4 to find that $a^{|x-y|} \equiv -1 \pmod 4$. Thus, it suffices to consider the case where

$$a \equiv 3 \pmod 4,$$

and one may assume that

$$x \not\equiv y \pmod 2.$$

First, suppose that x is even and y is odd. Put $X = x/2$. Now, equation (6.4) is written as

$$(-a - 2^k)^y + 2^{kz} = a^{2X}.$$

Lemma 6.1 with $(A, C, p, q) = (-a - 2^k, a^X, y, kz)$ implies that $y = 1$, and

$$kz < \frac{50}{13 \log 2} \log(a + 2^k).$$

This, together with the trivial estimate $a^{2X} < 2^{kz}$, yields

$$X < \frac{25}{13} \frac{\log(a + 2^k)}{\log a}.$$

Now, let us show

$$2^k \leqslant a + 1. \tag{6.5}$$

Indeed, as a is odd, one takes equation (6.4) modulo 2^k to find that $a^{2X-1} + 1$ is divisible by 2^k. This, together with the fact that $\nu_2(a^{2X-1} + 1) = \nu_2(a + 1)$, tells us that $a + 1$ is divisible by 2^k, particularly, $2^k \leqslant a + 1$. Therefore, it is concluded that

$$X < \frac{25}{13} \frac{\log(2a+1)}{\log a} \ (< 3), \qquad kz < \frac{50}{13 \log 2} \log(2a + 1).$$

Thus, either $X = 1$, or $X = 2$ with $a < 10^8$.

Suppose $X = 1$. Then equation (6.4) is $a^2 + a + 2^k = 2^{kz}$, that is,

$$(2a + 1)^2 + (2^{k+2} - 1) = 2^{kz+2}.$$

Proposition 5 with $(S, n) = (2a + 1, kz + 2)$ yields that $kz + 2 = 2k + 2$, so $z = 2$.

Suppose that $X = 2$ with $a < 10^8$. Then, by (6.5) and the obtained upper bound for kz depending only on a, it follows that $k < 30$ and $kz < 107$. Then Magma [5] easily enables us to check that equation (6.4) (with $y = 1$) does not hold for any triple $(a, k; z)$ under consideration.

Next, suppose that x is odd and y is even. Put $Y = y/2$. Then equation (6.4) is written as

$$(-a)^x + 2^{kz} = (a + 2^k)^{2Y}.$$

Lemma 6.1 with $(A, C, p, q) = (-a, (a + 2^k)^Y, x, kz)$ implies that $x = 1$, and

$$kz < \frac{50}{13 \log 2} \log a.$$

This, together with the trivial estimate $(a + 2^k)^{2Y} < 2^{kz}$, yields

$$Y < \frac{kz \log 2}{2 \log(a + 2^k)} < \frac{25}{13},$$

and so $Y = 1$. Then equation (6.4) is $a + (a + 2^k)^2 = 2^{kz}$. Note that both k, z are odd. The obtained equation is rewritten as

$$(2a + 2^{k+1} + 1)^2 - (2^{k+2} + 1) = 2^{kz+2}.$$

Now, Proposition 6 with $(D; U, l) = (-(2^{k+2} + 1); 2a + 2^{k+1} + 1, kz + 2)$ gives us

$$kz + 2 < \frac{50}{13 \log 2} \log(2^{k+2} + 1),$$

so

$$z < \frac{50}{13k \log 2} \log(2^{k+2} + 1) - \frac{2}{k} \ (< 8).$$

This implies either $z = 3$, or $z \in \{5, 7\}$ with $k < 10$.

First, suppose that $z = 3$. Taking (6.4) modulo a implies that $2^{2k} \equiv 2^{3k} \pmod{a}$. In view of a is odd, $2^k \equiv 1 \pmod{a}$, and so $a \leqslant 2^k - 1$. On the other hand, as $a \geqslant 2^k - 1$ from (6.5), we have $a = 2^k - 1$, i.e., $b = a + 1$. This contradicts Theorem 1.

Second, suppose that $z \in \{5, 7\}$ with $k < 10$. Magma [5] easily enables us to check that equation (6.4) does not hold for any triple $(a, k; z)$ under consideration.

This completes the proof of case (C2) in Theorem 2.

6.5. The case where $a + b$ is a power of 2. Write $a + b = 2^k$ with an integer $k > 2$, and consider the equation:

$$a^x + 2^{ky} = (2^k - a)^z. \tag{6.6}$$

It suffices to show that there is no triple (x, y, z) of positive integers satisfying equation (6.6).

In what follows, suppose that (x, y, z) is a positive integer solution of equation (6.6). Since a is odd, and $k > 1$, taking equation (6.6) modulo 4 yields

$$a^x \equiv (-a)^z \pmod 4.$$

From this congruence it is observed that x or z is even according to the cases $a \equiv 3 \pmod 4$ or $a \equiv 1 \pmod 4$.

First, consider the case where $a \equiv 3 \pmod 4$. Then x is even. Proposition 2 with $(S, T, m, n) = (a^{x/2}, 2^k - a, ky, z)$ gives $z < 3$. As $z > 1$, it holds that $z = 2$, and so $y = 1$ by equation (6.6). Thus,

$$a^x + 2^k = (2^k - a)^2.$$

It is easy to see that $x \neq 1, 2$. However, by Lemma 6.1 with $(A, C, p, q) = (a, 2^k - a, x, k)$, the above equation does not hold when $x \geqslant 3$.

Next, consider the case where $a \equiv 1 \pmod 4$. Then z is even. Note that $z \neq 2$ by an observation in the previous case. Now, apply Lemma 6.1 with $(A, C, p, q) = (a, (2^k - a)^{z/2}, x, ky)$ to find $x < 3$. If $x = 2$, then $(2^k - a)^{z/2} - 2^{ky-2} = 1$ with $z/2 > 1$ and $ky \geqslant 4$. Since the resulting equation is the Catalan's equation, it follows that $(2^k - a, z/2, ky - 2) = (3, 2, 3)$, and so $(a, k, y, z) = (19, 5, 1, 4)$, where, however, equation (6.6) does not hold. Therefore, $x = 1$. Then, taking equation (6.6) modulo 2^k implies that $a^{z-1} \equiv 1 \pmod{2^k}$, which, similarly to the proof of inequality (6.5), gives rise to the inequality $a - 1 \geqslant 2^k$. However, this contradicts the trivial estimate $a < 2^k$ $(= a + b)$. This completes the proof of case (C3) in Theorem 2.

To sum up, the proof of Theorem 2 is completed.

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