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Division rings with power commuting semi-linear additive maps

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Abstract. Hereafter, R denotes a noncommutative division ring with centre Z, and $f\colon R \to R$ is a semi-linear additive map of R (in the sense given by N. Jacobson, or a more general condition given in the Introduction). In this article, we show that if f is power commuting, that is, (i) there is a positive integer m such that $[f(x), x^m] = 0$, all $x \in R$, then f is, in fact, commuting, that is, $[f(x), x] = 0$, all $x \in R$. More generally, suppose that (ii) for a fixed pair of positive integers m and n, $[f(x), x^m]_n = 0$, all $x \in R$. Again, we will show that f is commuting. Now, a doubly more liberal version of the latter condition is Condition (C), which asserts that for each x in R, $[f(x), x^{m(x)}]_{n(x)} = 0$, where $m(x)$ and $n(x)$ are both positive integers depending on x. Unless we are ready to condition appropriately the carrier R, the status of Condition (C) remains totally unknown. Granted R is algebraic over Z, in particular if R is finite dimensional over Z, we show here that if f is an endomorphism or anti-endomorphism of R, then from Condition (C) follows again that f is commuting.

1. Introduction

Let R be any (associative) ring. For x, y elements of R, set: $[x, y]_0 := x$, $[x, y]_1 := [x, y] = xy - yx$, and $[x, y]_k := \big[[x, y]_{k-1}, y\big]$ for k any positive integer. We shall say the map $f\colon R \to R$ is *power commuting* (or, if there is a need to specify, f is *m-power commuting*) if there is a fixed positive integer m such that

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$[f(x), x^m] = 0$, all $x \in R$. Thus a 1-power commuting map f satisfies the defining functional identity $[f(x), x] = 0$, all $x \in R$. Now, power commuting additive maps of prime or semiprime rings R have been studied by a varied class of scholars (see notably [22], [3], [2], [4], [19], [5], [20], [21], [14], [7], [6], [8], [9], [17], [10]). In this article, we focus on a given division ring R with its centre written as $Z = Z(R) \neq R$. In a recent paper, the authors proved that every power commuting automorphism or anti-automorphism f of R must be commuting (see [10]). As a consequence, the authors answered [17, Question 2.11] in the affirmative. One might wonder as to whether this result carries over to an additive map f of R which is *semi-linear* over Z in the classical sense given by N. Jacobson, that is to say, that there is an automorphism τ of the field Z such that (1) $f(zx) = \tau(z)f(x)$, all $z \in Z$, $x \in R$. In fact, in this paper, we subscribe to a more liberal choice of the map τ, namely, τ is any map from Z to R subject to the only defining condition (1). We will show herein that the power commuting condition on f implies the commuting conclusion. In effect, we will show the following theorem, which is our main theorem.

Theorem A (Theorem 8)**.** *If f is any power commuting semi-linear additive map of the division ring R, then f is commuting.*

As a consequence of the theorem in the above, we will show the following theorem, which extends in parts [10, Theorem 2.13].

Theorem B (Theorem 10)**.** *Let $f\colon R \to R$ be a nonzero endomorphism or anti-endomorphism of the division ring $R \neq Z$. If f is power commuting, then f is commuting. Moreover, exactly one of the following trichotomy must occur:*

(i) *$f(R) \subseteq Z$, or*

(ii) *$f = \mathrm{id}(R)$, the identity map of R, or else*

(iii) *f is a normal involution of R, in which case, R must be a quaternion division algebra.*

In accordance with [9], we say that the additive map $f\colon R \to R$ satisfies a generalized Engel condition if there are fixed positive integers m and n such that $[f(x), x^m]_n = 0$, all $x \in R$. As shown in [2, Theorem 1.1], if f is not commuting, then R has a prime characteristic $p > 0$. By a standard argument, f is then a power commuting map. Thus both Theorem A and Theorem B carry over to the liberal assumption that f satisfies a generalized Engel condition (all other things unchanged).

We are left with the considered *quasi-generalized Engel condition* (notation: Condition (C)), which asserts that for each element x of R, $[f(x), x^{m(x)}]_{n(x)} = 0$,

where $m(x)$ and $n(x)$ are given positive integers both depending on x. As a preamble, we repeat that the status of Condition (C) is totally unknown in the context of a general division ring R, while the case R is algebraic was studied in several papers, notably [8], [9], [10]. Now, in [9, Theorem 4.3], the first author showed that, regardless of the algebraic assumption, if f is an anti-automorphism of finite order of the current division ring R satisfying Condition (C), then, in fact, f is a commuting involution of R. Actually, the finite order assumption forces $[R : Z] < \infty$ and, hence, certainly R is algebraic. In [10], the authors dropped the finite order assumption and showed the following improvement of the cited [9, Theorem 4.3].

Theorem C (see [10, Theorem 2.12]). *Let $R(\neq Z)$ be a division ring algebraic over the field Z, and let f be an anti-automorphism of R satisfying Condition (C). Then f must be a commuting (or a normal) involution of R and then, consequently, R is a quaternion division algebra.*

Using Theorem A, we will show the following generalizations of Theorem C.

Theorem D (Corollary 17). *If R is an algebraic division ring over Z, if f is an injective semi-linear additive map of R such that $f(1)$ is a scalar, and if f satisfies Condition (C), then f is commuting.*

Theorem E (Theorem 18). *If R is a noncommutative algebraic division ring over Z, if f is a ring endomorphism or anti-endomorphism of R, and if f satisfies Condition (C), then f is commuting. Consequently, exactly one of the following trichotomy holds true:*

(i) *$f(R) \subseteq Z$, or*

(ii) *$f = \mathrm{id}(R)$, or else*

(iii) *f is a normal involution of R, in which case, R must be a quaternion division algebra.*

2. Proof of the Main Theorem

First, let us recall the current definition of a semi-linear map f of the given division ring $R(\neq Z)$ with associated map τ. Here, $\tau\colon Z \to R$ is subject to just the defining identity $f(zx) = \tau(z)f(x)$, all $z \in Z$ and $x \in R$.

Proposition 1. *If f is a non-null additive map of the division ring R, which is semi-linear, then the associated map τ is an injective ring homomorphism from the field Z to the ring R.*

PROOF. Let $z_i \in Z$, $i = 1, 2$. There is $x \in R$ such that $f(x) \neq 0$. Now

$$f(z_1z_2x) = \tau(z_1z_2)f(x) = f(z_1(z_2x)) = \tau(z_1)f(z_2x) = \tau(z_1)\tau(z_2)f(x).$$

Cancelling the invertible factor $f(x)$ in the third and last equations in the above, we get $\tau(z_1z_2) = \tau(z_1)\tau(z_2)$. Similarly, one can show the additive property $\tau(z_1 + z_2) = \tau(z_1)+\tau(z_2)$. It follows that $\ker(\tau)$, the kernel of the ring homomorphism τ, is an ideal of the field Z so that $\ker(\tau) = 0$ or Z. In the latter case, for any $x \in R$, clearly $x = 1x$, so that $f(x) = \tau(1)f(x) = 0$, since, in particular, $\tau(1) = 0$. Hence, f is the null additive map, contrary to hypothesis. □

More is to be said about the associated map τ once we establish a key theorem.

Theorem 2 (see [1, Theorem 4.4])**.** *Let R be a division ring satisfying no polynomial identity. Assume that $f_0, f_1, \ldots, f_m$ are additive maps of R such that*

$$f_0(x)x^m + \cdots + x^i f_i(x)x^{m-i} + \cdots + x^m f_m(x) = 0,$$

all $x \in R$. Then, in particular, there is a fixed element $a_0 \in R$ such that $f_0(x) = xa_0 + \mu_0(x)$, all $x \in R$, where μ_0 is an additive map from R to Z.

PROOF. Since R satisfies no polynomial identity, it is clear that R cannot be algebraic (over Z) of bounded degree. Hence, $\deg(R) = \infty$ in the sense given in [1], whence, the asserted description of the initial additive map f_0. □

Theorem 3 (Key Theorem)**.** *If f is an m-power commuting additive map of the division ring R which is not commuting, then*

(1) $\operatorname{char} R = p$, *a positive prime number, and*

(2) $[R\colon Z] < \infty$.

PROOF. (1) Since f is additive and R is evidently a prime ring, [2, Theorem 1.1] applies and yields R has nonzero characteristic written as p.

(2) Deny the asserted inequality. In particular, R satisfies no polynomial identity. Define the additive maps $f_i, 0 \leq i \leq m$, as follows: $f_0 := f$, $f_i := 0$, all i such that $0 < i < m$; $f_m := -f$. Now

$$f_0(x)x^m + \cdots + x^i f_i(x)x^{m-i} + \cdots + x^m f_m(x) = [f(x), x^m] = 0,$$

all $x \in R$. Hence, by Theorem 2, there is a fixed element a in R such that $f(x) = xa + \mu(x)$, all $x \in R$, where μ is an additive map from R to Z. Thus

$$0 = f(x)x^m - x^m f(x) = xax^m - x^{m+1}a.$$

Left cancellation by x in the above equation yields $[a, x^m] = 0$, all $x \in R$. Invoking [16, Theorem, p. 19], we get $a \in Z$. Hence, $[f(x), x] = [ax + \mu(x), x] = [ax, x] = 0$, all $x \in R$, contrary to the noncommuting hypothesis. □

Proposition 4. *Consider the square system written as* $\sum_{j=1}^{n} z_{i,j} b_j = 0$, $i, j = 1, \ldots, n$ *with unknown* b_j*'s and known* $z_{i,j} := (z_j)^i \in Z \setminus \{0\}$. *If the* z_j*'s,* $j = 1, \ldots, n$, *are distinct in pairs, then all* $b_j = 0$.

Proof. For the matrix of known $(z_{i,j})_{i,j}$ is a Vandermonde $n \times n$ matrix over the field Z and, consequently, it has a nonzero determinant. Hence, all unknown $b_i = 0$. □

Theorem 5. *If* f *is a power commuting semi-linear additive map of the current division ring* R, *which is not commuting, then both* f *and its associate map* τ *stabilize the field* Z, *that is,* (1) $f(Z) \subseteq Z$ *and* (2) $\tau(Z) \subseteq Z$. *Consequently,* τ *can be viewed as an injective endomorphism of the field* Z.

Proof. (1) Invoking the Key Theorem (Theorem 3), R is certainly algebraic over its centre Z, and the field Z has prime subfield precisely GF(p), the Galois field of p elements, where $p = \operatorname{char} R > 0$. Thus, if Z were algebraic over GF(p), then R would be algebraic over GF(p). Therefore, if $a \in R$, then GF(p)[a] is a finite field, implying $a^{n(a)} = a$ for some $n(a) > 1$. It follows from Jacobson's theorem [13, Theorem 3.1.2] that R is commutative, contrary to the hypothesis saying f is not commuting. Hence, there must be some $z \in Z$ not algebraic over GF(p) which is fixed thereafter. Now, given a running central element c of R, then $\big[f(zx + c), (zx + c)^m\big] = 0$. Hence, $\big[[f(zx + c), (zx + c)^m], (zx)^m\big] = 0$. And, since, evidently $(zx + c)^m$ and $(zx)^m$ commute, they can be interchanged, so that $\big[[f(zx + c), (zx)^m], (zx + c)^m\big] = 0$. Since $[f(zx), (zx)^m] = 0$, it follows that $\big[[f(c), (zx)^m], (zx + c)^m\big] = 0$, and hence, $\big[[f(c), (zx + c)^m], (zx)^m\big] = 0$. Write $C(m, j)$ for the combinatorial number of subsets of j elements of the set of m elements $\{1, 2, \ldots, m\}$, where j ranges from 0 to m. Expansion of the preceding commutation equation, taking into account that evidently $\big[[f(c), c^m], (zx)^m\big] = 0$, yields

$$\sum_{k=0}^{m-1} z^{2m-k} \Big(C(m, k) \big[[f(c), c^k x^{m-k}], x^m\big] \Big) = 0.$$

For the purpose of the rest of the proof, view the above equation to be a homogeneous linear equation with all known z^{2m-k}, $0 \le k < m$, and all unknown parenthesized elements of R appearing on the right of the just indicated known. Owing to the choice of z, the known are distinct in pairs and are evidently all nonzero. And, for each fixed natural number j, the substitution z^j for z gives

rise to an equation with same unknown and initial known all elevated to their j-powers. Hence, successive substitutions $z^2, \ldots, z^m$ for z give rise to an $m \times m$ Vandermonde system. Thus Proposition 4 applies and yields all unknown must be zeros. In particular, at $k = 0$, we get $[f(c), x^m]_2 = 0$. Hence, if $p = 2$ then $[f(c), x^{2m}] = [f(c), x^m]_2 = 0$. If, on the other hand side, $p > 2$, then from $[f(c), x^m]_2 = 0$ readily follows $[f(c), x^m]_p = 0$ so that $[f(c), x^{pm}] = 0$, all $x \in R$. In view of [16, Theorem, p. 19], it follows that $f(c) \in Z$, all $c \in Z$.

(2) Again, z denotes a fixed central element not algebraic over $\mathrm{GF}(p)$, c denotes a running nonzero central element of R, x denotes a given element in $\ker(f)$, the kernel of f in R, and y denotes a given element of $R \setminus \ker(f)$ (possible since, by hypothesis, f is not commuting, and hence, f is non null). Now,

$$\begin{aligned} 0 &= [f(c(zx+y)), (c(zx+y))^m] \\ &= c^m[\tau(c)f(zx+y), (zx+y)^m] = c^m\big[\tau(c)f(y), (zx+y)^m\big]. \end{aligned} \tag{1}$$

Notice that in the expansion of $(zx+y)^m$ as a polynomial expression in z with decreasing powers of z and with coefficients in R, the constant term is precisely y^m. We contend that this term can be neglected in the initial commutation equations (1) for $c^m[\tau(c)f(y), y^m] = [f(cy), (cy)^m] = 0$. Writing $S(i)$ for the sum of all monic monomials in x, y of degrees $m-i$, i respectively, where i ranges from 0 to $m-1$, we then have

$$\begin{aligned} 0 &= \big[\tau(c)f(y), (zx+y)^m\big] \\ &= z^m[f(cy), S(0)] + z^{m-1}[f(cy), S(1)] + \cdots + z[f(cy), S(m-1)]. \end{aligned} \tag{2}$$

Note that $S(0) = x^m$. By an argument similar to the proof of (1), it follows in particular that $[f(cy), x^m] = 0$, for all $c \in Z \setminus \{0\}$, $x \in \ker(f)$, and $y \in R \setminus \ker(f)$.

Choosing $c = 1$, we get (i) $[f(y), x^m] = 0$. Hence, $0 = [f(cy), x^m] = [\tau(c)f(y), x^m] = [\tau(c), x^m]f(y) = 0$, whence, since $f(y) \neq 0$, (ii) $[\tau(c), x^m] = 0$, all $x \in \ker(f)$. It remains to show the complementary result that (iii) $[\tau(c), y^m] = 0$, all $y \in R \setminus \ker(f)$, which is relatively easier. The equality $[f(cy), (cy)^m] = 0$ gives

$$0 = [\tau(c)f(y), c^m y^m] = c^m[\tau(c)f(y), y^m] = c^m[\tau(c), y^m]f(y),$$

and hence, since $f(y) \neq 0$, $[\tau(c), y^m] = 0$. Therefore, by the cited [16, Theorem, p. 19], $\tau(c) \in Z$, all $c \in Z \setminus \{0\}$. Therefore, $\tau(Z) \subseteq Z$. Hence, τ can be thought of as an endomorphism of the field Z, one which is injective owing to Proposition 1. □

Notation. In all that will follow, the power p^k, where p is a given positive integer and k is a given nonnegative integer, is denoted by $p[k]$.

Proposition 6. *Let R be any ring with prime characteristic p, and let c, x be nonzero elements of R such that $[c,x]=x$. If n is any positive integer, then*

$$(c+x)^{p[n]} = c^{p[n]} + \sum_{i=0}^{n} x^{p[i]}.$$

PROOF. The proof will proceed by way of induction on n. In the case $n=1$, the asserted equation is then $(c+x)^p = c^p + x + x^p$, which is an immediate consequence of [15, Equation (66), p. 187]. Notice that as a quick consequence of the basic equation $[c,x]=x$, one has $c^k x = x(c+1)^k$, all positive integers k, and hence, $[c^p,x]=x$, whence $[c^p,x^p]=px^p=0$ resulting in $[c^p+x,x^p]=0$. Consequently,

$$\begin{aligned}(c+x)^{p[n]} &= \big((c^p+x)+x^p\big)^{p[n-1]} \\ &= (c^p+x)^{p[n-1]} + (x^p)^{p[n-1]} = c^{p[n]} + \sum_{i=0}^{n} x^{p[i]}.\end{aligned}$$ □

Theorem 7. *If R has nonzero characteristic p, if f is a semi-linear additive map of R, and if f satisfies a special power commuting condition, that is, there is a nonnegative integer n such that f is $p[n]$-power commuting, then f is commuting.*

PROOF. Deny the conclusion. There is $x \in R$ such that $[f(x),x] \neq 0$. Now $\big[f(x),x^{p[n]}\big] = [f(x),x]_{p[n]} = 0$. If r is a positive integer which is minimum for the equation $[f(x),x]_r = 0$, clearly $r \geq 2$. Set

$$c := [f(x),x]_{r-2}\big([f(x),x]_{r-1}\big)^{-1}x.$$

In view of [10, Lemma 2.9], $[c,x]=x$, and hence, $[c,zx]=zx$, all nonzero central elements z of R. Thus, by the proposition just proved, we can write

$$(c+zx)^{p[n]} = c^{p[n]} + \sum_{i=0}^{n} z^{p[i]}x^{p[i]}.$$

Thus

$$\left[f(c+zx), c^{p[n]} + \sum_{i=0}^{n} z^{p[i]}x^{p[i]}\right] = 0. \tag{3}$$

Now, by Theorem 5, τ is an endomorphism of the field Z. And, taking into account that $[f(c), c^{p[n]}] = 0$ and $[f(zx), z^{p[n]}x^{p[n]}] = 0$, we can rewrite (3) as

$$\sum_{i=0}^{n} z^{p[i]}\big[f(c), x^{p[i]}\big] + \tau(z)[f(x), c^{p[n]}] + \tau(z)\sum_{j=0}^{n-1} z^{p[j]}\big[f(x), x^{p[j]}\big] = 0. \tag{4}$$

Now, add to equation (4) the specialization of (4) at $z = 1$, using the equality $\tau(1) = 1$. Subtract the resulting equation from the version of (4) obtained by replacing z by $z + 1$. A simple calculation shows that the end-result equation is then the following:

$$\tau(z)a + zb_0 + z^p b_1 + \cdots + z^{p[i]}b_i + \cdots + z^{p[n-1]}b_{n-1} = 0, \tag{5}$$

all $z \in Z$, where $a := \sum_{i=0}^{n-1}[f(x), x^{p[i]}]$ and $b_j := [f(x), x^{p[j]}]$ for $j = 0, \ldots, n-1$. Now, by the Key Theorem, there is $z \in Z$ not algebraic over GF(p), which we fix for the rest of the proof. If k is an arbitrary positive integer, then

$$\begin{aligned}&\tau(z)^k a + z^k b_0 + (z^p)^k b_1 + \cdots + (z^{p[n-1]})^k b_{n-1}\\ &\quad = \tau(z^k)a + z^k b_0 + (z^k)^p b_1 + \cdots + (z^k)^{p[n-1]}b_{n-1} = 0.\end{aligned} \tag{6}$$

We contend that we may assume $a \neq 0$, for otherwise (6) would reduce to the system of equations

$$\sum_{i=0}^{n-1} z^{kp[i]}b_i = 0,$$

where k ranges from 1 to n. Since z is not algebraic over GF(p), it is clear that the $z^{p[i]}$ are distinct in pairs. In view of Proposition 4, all $b_j = 0$. In particular, $b_0 = [f(x), x] = 0$, a contradiction to the assumption $a = 0$.

Next, view equations (6) to be a homogenous system of linear equations with unknown $a, b_0, \ldots, b_{n-1}$ and corresponding known $\tau(z^k), z^k, z^{kp}, \ldots, z^{kp[n-1]}$ with k ranging from 1 to $n + 1$. Assume further that the known at $k = 1$ are distinct in pairs. Then by Proposition 4, all unknown must be zeros contrary to the inequality $a \neq 0$. Thus there must be a repetition of the known at $k = 1$ and, since z is not algebraic over GF(p), there must be a unique repetition $\tau(z) = z^{p[r]}$, $0 \leq r \leq n-1$. Then, evidently, $\tau(z^k) = z^{kp[r]}$. Assume that $r \neq 0$. Adding $\tau(z^k)a$ to $z^{kp[r]}b_r$, this gives $z^{kp[r]}(a + b_r)$ so that equations (6)) become

$$z^k b_0 + \cdots + z^{kp[r]}(a + b_r) + \cdots + z^{kp[n-1]}b_{n-1} = 0 \quad (1 \leq k \leq n).$$

In view of Proposition 4, all $b_i = 0$ and, in particular, $b_0 = 0$, resulting in $[f(x), x] = 0$, which is ruled out. This shows that necessarily $r = 0$, that is,

$\tau(z) = z$ regardless of the fixed z not algebraic over GF(p). Hence, $\tau(z) = z$, for all z not algebraic over GF(p). On the other hand, if z_0 is a nonzero central element of R which is algebraic over GF(p) and z is, as before, a fixed central element not algebraic over GF(p), then zz_0 is clearly not algebraic over GF(p), and hence, $\tau(zz_0) = zz_0$, whence, $\tau(z_0) = \frac{\tau(zz_0)}{\tau(z)} = \frac{zz_0}{z} = z_0$. Therefore, $\tau = \mathrm{id}(Z)$, the identity automorphism of the field Z. In other words, $f(zy) = zf(y)$, all $z \in Z$, and $y \in R$. Thus in view of [19, Theorem 1.1], f is commuting, a contradiction. With this, the proof is complete. □

We can now show our Main Theorem.

Theorem 8 (Main Theorem)**.** *If R is any division ring, if f is a semi-linear additive map of R, and if f is m-power commuting, then, in fact, f is commuting.*

Proof. Deny the affirmation f is commuting. By the Key Theorem, $\mathrm{char} R = p > 0$ and $[R\colon Z] < \infty$. Also, by Theorem 5, both f and τ stabilize Z. In view of upcoming Theorem 14, part (3), f satisfies the special power commuting condition $[f(x), x^{p[w]}] = 0$, all $x \in R$, where $p[w]$ is the highest power of p dividing the degree of R $(= \sqrt{[R : Z]})$. It suffices then to quote Theorem 7.

For convenience to the reader, hereafter, we show directly the special power commuting condition $\big[f(x), x^{p[r]}\big] = 0$, all $x \in R$, where $p[r]$ is the highest power of p dividing precisely m. In effect, since $p[r]$ divide m, we can write $m = sp[r]$, where s is a positive integer, and since $p[r]$ is the highest power of p dividing m, s is not divisible by p. Now, if $s = 1$, there is nothing more to show. Assume $s > 1$, let z be a central element not algebraic over GF(p) (possible by the Key Theorem), and let $x \in R$. Because $f(z) \in Z$, we can write

$$0 = [f(z + x), (z + x)^m] = [f(z) + f(x), (z + x)^{p[r]s}] = \big[f(x), (z^{p[r]} + x^{p[r]})^s\big].$$

Write $c(s, k)$ for the combinatorial number of subsets of k elements of a set of s elements. Then

$$(z + x)^{p[r]s} = (z^{p[r]} + x^{p[r]})^s = \sum_{k=0}^{s} z^{(s-k)p[r]}\big(c(s, k)x^{kp[r]}\big).$$

Left commutation with $f(x)$ of the third member of the above equations combined with the commutation equations $[f(x), z^{sp[r]}] = 0$ and $[f(x), x^{sp[r]}] = [f(x), x^m] = 0$ yield the following equation:

$$z^{(s-1)p[r]}\Big(s\big[f(x), x^{p[r]}\big]\Big) + \cdots + z^{(s-k)p[r]}\Big(c(s, k)\big[f(x), x^{kp[r]}\big]\Big) + \cdots + z^{p[r]}\Big(s\big[f(x), x^{(s-1)p[r]}\big]\Big) = 0. \tag{7}$$

For the purpose of the proof, view (7) to be a homogeneous linear equation with known indicated powers of z appearing on the left-hand side and unknown corresponding parenthesized elements of R. Then the usual successive substitutions $z^2, \ldots, z^{s-1}$ yield a square system of $s-1$ linear equations with Vandermonde matrix as the known appearing in the first equation (7) are distinct (recall, z is not algebraic over $\mathrm{GF}(p)$). Therefore, all unknown must be zeros, in particular, $s\big[f(x), x^{p[r]}\big] = 0$. Recalling that $p \nmid s$, we get $\big[f(x), x^{p[r]}\big] = 0$, as wished. With this, the proof is complete. □

3. Applications

A first immediate application of the Main Theorem is in the case where the current map f of the division ring R is a power commuting non-null endomorphism or anti-endomorphism of R. For then, f can be viewed to be a semi-linear additive map with associated map τ precisely $\tau := f_{|Z}$. Notice, in passing, that since $\ker(f)$ is an ideal of the division ring R, necessarily $\ker(f) = 0$, in other words, f must be injective. Precisely, we have

Theorem 9. *If f is a nonzero power commuting endomorphism or anti-endomorphism of the division ring R, then f is an injective commuting map.*

A well-known theorem of Brešar asserts that every additive commuting map f of the noncommutative division ring R to itself (or a more general result) is expressible as follows:

$$f(x) = \lambda x + \mu(x), \tag{8}$$

all $x \in R$, where λ is a uniquely determined central element of R, referred as to the *associated parameter* of f, and μ is a uniquely determined additive map from R to Z (see [3, Theorem A]). Using (8), it is evident that if $\lambda = 0$, then $f(R) \subseteq Z$. Conversely, if $f(R) \subseteq Z$ but $\lambda \neq 0$, then by (8) $\lambda x \in Z$, all $x \in R$, resulting in $R = Z$, which is, by hypothesis, ruled out. Therefore, to say that $f(R) \subseteq Z$ is to say that the associated parameter λ is zero. We proceed to a fairly complete description of the considered case of the map f.

Theorem 10. *Let $R \neq Z$, and let f be a nonzero power commuting endomorphism or anti-endomorphism of the division ring R. Exactly one of the following trichotomy holds:*

(i) $f(R) \subseteq Z$;

(ii) $f = \mathrm{id}(R)$;

(iii) *f is a normal involution of R, in which case R must be a quaternion division algebra.*

PROOF. By Theorem 9, f is commuting. Assume that f is an endomorphism and $f(R) \not\subseteq Z$. Then (and only then) f is expressible in the form $f(x) = \lambda x + \mu(x)$, where $\lambda \in Z \setminus \{0\}$ and μ is an additive map from R to Z. Now, given $x, y \in R$, we can write

$$f(xy) = \lambda xy + \mu(xy) = f(x)f(y) = (\lambda x + \mu(x))(\lambda y + \mu(y)).$$

And, a simple computation shows that

$$(\lambda^2 - \lambda)xy + \lambda\big(\mu(x)y + \mu(y)x\big) + \mu(x)\mu(y) - \mu(xy) = 0. \tag{9}$$

Commutating equation (9) on the left-hand side with x, we get

$$\big((\lambda^2 - \lambda)x + \lambda\mu(x)\big)[x, y] = 0 \quad (\text{all } y \in R). \tag{10}$$

Thus, if $x \notin Z$, then there is y such that $[x, y] \neq 0$, and hence, by (10), $(\lambda^2 - \lambda)x + \lambda\mu(x) = 0$, resulting in $(\lambda^2 - \lambda)x \in Z$. Consequently, $\lambda^2 = \lambda$, resulting in $\lambda = 1$, since $\lambda \neq 0$. Going back to (10), we get, $\mu(x) = 0$. On the other hand, choose some $t \in R \setminus Z$. If $x \in Z$, it is clear that $x + t \notin Z$. Consequently, $\mu(x + t) = \mu(t) = 0$, and hence, since μ is additive, $\mu(x) = 0$. Therefore $\mu = 0$, giving $f = \mathrm{id}(R)$.

Assume next that f is an anti-endomorphism of R. Using $f(x) = \lambda x + \mu(x)$ for $x \in R$, we have

$$f^2(x) = f(\lambda x + \mu(x)) = \lambda(\lambda x + \mu(x)) + \mu\big(\lambda x + \mu(x)\big) = \lambda^2 x + \mu'(x)$$

for all $x \in R$, where μ' is a readily found additive map from R into Z. Thus f^2 is commuting. Clearly, we may assume $\lambda \neq 0$. Since f^2 is an endomorphism of R, by the preceding, $f^2 = \mathrm{id}(R)$. In other words, f is an involution of R. Since f is commuting, by [7, Theorem 1.3], f is of the first kind and R is a quaternion division algebra. □

In accordance with [9], we say that the additive map f of R satisfies a *generalized Engel condition* if there are fixed positive integers m and n such that $[f(x), x^m]_n = 0$. Now, if f is not commuting, then by [2, Theorem 1.1], $\mathrm{char}\, R = p > 0$, where R is merely a prime ring. This readily gives the following theorem.

Theorem 11. *If f is a semi-linear additive map of the division ring R, and if f satisfies a generalized Engel condition, then f is commuting.*

PROOF. Deny the conclusion. By the just cited result in the above, R has a positive prime characteristic p. If k is a sufficiently large integer so that $p[k] \geq n$, then, evidently, $[f(x), x^m]_{p[k]} = 0$. By the well-known identity $[u, v]_{p[k]} = \left[u, v^{p[k]}\right]$, it follows that $\left[f(x), x^{mp[k]}\right] = 0$, showing thereby f is power commuting. It suffices then to apply the Main Theorem. □

We are left with the considered quasi-generalized Engel condition written as Condition (C). Recall that for each x in R, there corresponds a pair of positive integers $m(x)$ and $n(x)$ both depending on x such that $[f(x), x^{m(x)}]_{n(x)} = 0$. An important special case of Condition (C) is when all the integers $n(x)$ can be taken to be equal to 1. In other words, f is a weak (or local) power commuting map (e.g., for each x in R, there corresponds a positive integer $m = m(x)$ depending on x such that $[f(x), x^m] = 0$). Inspired from the earlier stronger case of a generalized Engel condition, can one say that analogously to the generalized Engel condition, Condition (C) implies that f is a weak power commuting map? Below, we show that this is indeed the case whenever we are given an algebraic carrier R, by showing that in the presence of the algebraic assumption, if the additive map f fails to be commuting, necessarily R has a nonzero characteristic. Of course, it would be a major advance if one could show the nonzero characteristic conclusion regardless of the algebraic assumption.

Proposition 12. *If R is algebraic (over Z), if f is any additive map of R and if satisfies Condition (C), then f is a weak power commuting map.*

PROOF. Negate the conclusion. Hence, there must be $x \in R$ such that (i) $[f(x), x^k] \neq 0$, all positive integers k. Now, from hypothesis, there are positive integers $m(x)$ and $n(x)$ such that (ii) $[f(x), x^{m(x)}]_{n(x)} = 0$. In view of inequality (i), one has $[f(x), x^{m(x)}] \neq 0$. Hence, $n(x) \neq 1$, hence, $n(x) \geq 2$. Set $u := x^{m(x)}$. Since $[f(x), u] = [f(x), x^{m(x)}] \neq 0$, it follows that u is not central. If r is the first positive integer such that $[f(x), u]_r = 0$, then $2 \leq r \leq n(x)$. Setting $c := [f(x), u]_{r-2}([f(x), u]_{r-1})^{-1}u$, we know from [10, Lemma 2.9], $[c, u] = u$. Now, since $u \notin Z$, the minimal polynomial of u over Z written as $m_{u/Z} = m_{u/Z}(t)$ (t is a central indeterminate over R) has degree written as k with $k \geq 2$. Consider the minimal equation satisfied by u over Z written below:

$$m_{u/Z}(u) = u^k + z_1u^{k-1} + \cdots + z_{k-1}u + z_k = 0. \tag{11}$$

Next, by straight induction on q, $[c, u^q] = qu^q$. Commutating equation (11) on the left-hand side with c yields

$$ku^k + (k-1)z_1u^{k-1} + \cdots + z_{k-1}u = 0. \tag{12}$$

Cancelling u (as a common factor) in equation (12), we get $m'_{u/Z}(u) = 0$, where $m'_{u/Z}$ is the usual derivative of $m_{u/Z}$. As is well known, this can happen only if $\operatorname{char} R$ is a positive prime p and $m_{u/Z}$ is a polynomial in t^p. Now, choose k large enough so that $p[k] \geq r\,(= n(x))$. Evidently, $[f(x), u]_{p[k]} = 0$, and hence, $[f(x), u^{p[k]}] = 0$, giving $[f(x), x^{m(x)p[k]}] = [f(x), u^{p[k]}] = 0$, a contradiction to the opening choice of x. □

Proposition 13. *If R is a noncommutative division ring, if f is an injective semi-linear additive map of R, and if f is a weak power commuting map, then $\tau(Z) \subseteq Z$. Consequently, f stabilizes Z (i.e. $f(Z) \subseteq Z$) if and only if $f(1)$ is a scalar.*

Proof. For a fixed nonzero scalar written as z, and for a general $x \in R\backslash\{0\}$, there corresponds a positive integer k (depending on z, x) such that $[f(x), x^k] = [f(zx), (zx)^k] = 0$. Now,

$$0 = [f(zx), (zx)^k] = [\tau(z)f(x), z^k x^k] = z^k[\tau(z)f(x), x^k] = z^k[\tau(z), x^k]f(x).$$

Since f is injective, $f(x) \neq 0\,(= f(0))$. Hence, $[\tau(z), x^k] = 0$. In other words, the division ring R is a *radical extension* of $C_R(\tau(z))$ (i.e. for each x in R, there is some power $x^{m(x)}$ falling in $C_R(\tau(z))$, where $m(x)$ is a positive integer depending on x). Since R is noncommutative, by [11, Theorem B, p. 46] and/or [12, Theorem 2], $C_R(\tau(z)) = R$, that is, $\tau(z) \in Z$, as asserted. The rest of the proposition is evident from the equation $f(Z) = \tau(Z)f(1)$. □

We also need the following two results.

Theorem 14 ([9, Theorem 2.4])**.** *If R is an algebraic division ring over Z, if f is an additive map of R stabilizing the centre Z of R, and if f satisfies Condition (C); but, yet, f is not commuting, then:*

(1) *R has a positive prime characteristic p.*

(2) *f satisfies a special weak power commuting condition, namely, for each x in R, there corresponds a nonnegative integer $n = n(x)$ depending on x such that $[f(x), x^{p[n]}] = 0$.*

(3) *If $[R\colon Z] < \infty$, then p divides $d\,(= \sqrt{[R\colon Z]})$, the degree of R, and, if $p[k]$ is the highest power of p dividing d, then f satisfies the special power commuting condition $[f(x), x^{p[k]}] = 0$, all $x \in R$.*

Theorem 15 ([10, Proposition 2.3 and Theorem 2.4])**.** *If $R \neq Z$ is an algebraic division ring over Z with prime characteristic $p > 0$, if f is a semi-linear additive map of R with associated map τ such that $\tau(Z) \subseteq Z$, and if f*

satisfies a special weak power commuting condition, then $[f(x), x] = 0$ *if* $C_R(x)$, *the centralizer of* x *in* R, *is noncommutative.*

PROOF. We follow the pattern of proofs of [10, Proposition 2.3 and Theorem 2.4]. Concerning the referred proposition, once we equate the expression $f(zx)$ with $\tau(z)f(x)$ (in lieu of $f(z)f(x)$ as in the proof of the quoted proposition) where, by hypothesis, $\tau(z)$ is central, then exactly as in the conclusion of the referred proposition, for given x and y in R, there is a scalar z and an integer k such that

(i) $\tau(z) \neq z^{p[k]}$, and

(ii) $\big[f(x+zy), (x+yz)^{p[k]}\big] = [f(x), x^{p[k]}] = [f(y), y^{p[k]}] = \big[f(x+y), (x+y)^{p[k]}\big] = 0.$

Concerning the referred theorem, we show first that $C_R(x)$ is a radical extension of the subring $C_R(x) \cap C_R(f(x))$. In effect, given $y \in C_R(x)$, there is a scalar z for which x, y, z verify both (i) and (ii) as in the above. Then

$$\big[f(x+y), (x+y)^{p[k]}\big] = \big[f(x), y^{p[k]}\big] + \big[f(y), x^{p[k]}\big] = 0; \tag{13}$$

$$\big[f(x+zy), (x+zy)^{p[k]}\big] = z^{p[k]}\big[f(x), y^{p[k]}\big] + \tau(z)\big[f(y), x^{p[k]}\big] = 0. \tag{14}$$

Multiplying through equation (13) by $\tau(z)$, and subtracting equation (14) from the resulting equation, this gives

$$(\tau(z) - z^{p[k]})[f(x), y^{p[k]}] = 0, \quad \text{and hence } [f(x), y^{p[k]}] = 0,$$

showing thereby the asserted radical extension property. In view of [11, Theorem B, p. 46], necessarily $C_R(x) = C_R(x) \cap C_R(f(x))$, meaning that $C_R(x) \subseteq C_R(f(x))$. Then, by the well-known Double Centralizer Theorem, $Z[x]$ contains $Z[f(x)]$ or, it is the same, $f(x)$ is a polynomial expression in x with central coefficients, and hence, $[f(x), x] = 0$. □

Theorem 16. *If* R *is an algebraic division ring over* Z, *if* f *is a semi-linear additive map of* R *satisfying Condition (C), and if both* f *and the associate map* τ *stabilize* Z, *then* f *is commuting.*

PROOF. Deny the conclusion. By Proposition 12, f is a weak power commuting map. Now, by hypothesis, $f(Z) \subseteq Z$. In view of Theorem 14, parts (1) and (2), $\operatorname{char} R = p > 0$, and f satisfies a special local power commuting condition. Since, also by hypothesis, $\tau(Z) \subseteq Z$, Theorem 15 applies and yields that for a given $x \in R$, if $C_R(x)$ is not commutative, necessarily $[f(x), x] = 0$. Equivalently, if $[f(x), x] \neq 0$, then $C_R(x)$ is necessarily commutative. And, since the commuting property of f was denied at the opening, there must be x such that

$[f(x), x] \neq 0$, and, consequently, $C_R(x)$ is commutative. Hence, $C_R(x)$ is a maximal subfield of R. Now, since x is algebraic, the centre of $C_R(x)$ is precisely $Z[x]$, so that $C_R(x) = Z[x]$ (see [10, Lemma 2.2]) is finite dimensional over Z. Hence, $[R : Z] = [Z[x] : Z]^2 < \infty$. Then Theorem 14, part (3), applies and yields f to be a power commuting, and, consequently, by the Main Theorem, f would be commuting, contrary to the inequality $[f(x), x] \neq 0$. Hence, the opening noncommuting assumption must be discarded. □

Now we show Theorem D announced in the Introduction.

Corollary 17. *If R is an algebraic division ring over Z, if f is an injective semi-linear additive map of R such that $f(1)$ is a scalar, and if f satisfies Condition (C), then f is commuting.*

Proof. By Propositions 12 and 13, both f and the associate map τ stabilize Z. It suffices then to quote Theorem 16. □

We have all the pieces to show our concluding theorem:

Theorem 18. *If R is algebraic (over Z), if f is a nonzero endomorphism or anti-endomorphism of the current division ring R, and if f satisfies Condition (C), then f must be commuting. Consequently, exactly one of the following trichotomy holds:*

(i) $f(R) \subseteq Z$, *or*

(ii) $f = \mathrm{id}(R)$, *or else*

(iii) f *is a normal involution of R, in which case, R must be a quaternion division algebra.*

Proof. To begin, since $\ker(f) \neq R$ is an ideal of the division ring R, necessarily $\ker(f) = 0$, that is, f is injective. Also, since $(f(1))^2 = f(1^2) = f(1) \neq 0$, it follows that $f(1) = 1$ is certainly a scalar. In view of Corollary 17, f is commuting. The rest of the conclusion was already dealt with (see Theorem 10 and the proof). □

Finally, we conclude the paper with two open questions concerning Condition (C).

Questions 19. (a) If R is a division ring, if f is a semi-linear additive map satisfying Condition (C), say, f is a weak power commuting map, must both f and the associate map τ stabilize Z?

(b) If we are granted the stabilizing assumptions about f and τ, must f be, in fact, a commuting map?

One could assume R to be algebraic, yet, Question 19 (a) remains open, but for the particular case f is injective and $f(1)$ is a scalar (see Proposition 13). As for Question 19 (b), the answer is definitely yes, owing to Theorem 16.

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