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On the factors of CNS polynomials with dominant constant term

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Abstract. It is proved that every monic integer expansive polynomial without positive real roots is a factor of an effectively computable CNS polynomial with dominant constant term.

1. Introduction

A monic integer polynomial f is a CNS polynomial if every integer polynomial is congruent to some polynomial in $\{0, \dots, |f(0)| - 1\}[X]$ modulo f. The concept of a CNS polynomial and the notion of a canonical number system (CNS) were introduced by A. Pethő [9] and extended in the sequel (see, for example, [1], [4], [13], [10]). Compared to other number systems, canonical number systems seem to be rather exceptional (e.g., see [10]). Detailed background information on the historical development and relations of CNS polynomials to other areas such as shift radix systems, finite automata or fractal tilings can be found in the survey by P. Kirschenhofer and J. M. Thuswaldner [7] and the literature cited there.

Many years ago, A. Pethő [8] asked whether each monic integer polynomial all of whose roots lie outside the closed unit disk and are non-positive is a factor of a CNS polynomial. In this note we give an affirmative answer to this question by a constructive proof based on a classical result by Akiyama–Rao [3] and Scheicher–Thuswaldner [12].

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2. Real expansive polynomials without positive roots

Our principal interest lies in the set $\mathcal{E}$ of real monic expansive polynomials of positive degree which do not have a real positive root. Remember that a real polynomial f is called expansive if all its roots lie outside the closed unit disk. We often need the sum of the absolute values of the coefficients of f, known as the length of f and abbreviated by $\mathrm{L}(f)$. Further, it turns out that

$$G_n(f) := \prod_{\alpha \in \mathrm{Z}_f} (X^n - \alpha^n) \qquad (n \in \mathbb{N})$$

are helpful auxiliary functions[1]; here Z_f denotes the multiset of roots of f.

Now we formulate our result which implies in particular that every $f \in \mathcal{E}$ is a factor of a monic polynomial with non-negative coefficients and strictly dominant constant term. Recall that f is said to have a strictly dominant constant term if its length is less than $2\,|f(0)|$ (cf. [6]).

Theorem 1. *Let f be a real monic expansive polynomial of positive degree which does not have a real positive root. For every $\rho > 0$, we can effectively compute an odd natural number n and a monic polynomial h with integer coefficients such that the product $hG_n(f)$ has only non-negative coefficients and*

$$\mathrm{L}(h\,G_n(f)) < (1+\rho)\,(h\,G_n(f))(0).$$

The proof of this result is provided at the end of the following section. Here we recall a fundamental classical result on CNS polynomials.

Proposition 2 (Akiyama–Rao [3, Theorem 3.2][2], Scheicher–Thuswaldner [12, Theorem 5.8])**.** *Every monic polynomial with non-negative integer coefficients and strictly dominant constant term is a CNS polynomial.*

These two results allow a positive answer to Pethő's question mentioned above.

Theorem 3. *Every monic expansive integer polynomial without positive real roots is a factor of an effectively computable CNS polynomial with non-negative coefficients and strictly dominant constant term.*

[1] $\mathbb{N}$ is the set of positive rational integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

[2] Note that the assumption "expanding" is not used in the respective part of the proof.

PROOF. Take $f \in \mathcal{E} \cap \mathbb{Z}[X]$. Theorem 1 with $\rho := 1$ yields some effectively computable $n \in \mathbb{N}$ and a monic polynomial $h \in \mathbb{Z}[X]$ such that $F := hG_n(f)$ has only non-negative coefficients and

$$\mathrm{L}(F) < 2F(0).$$

This means that F has a strictly dominant constant term. Observe that the coefficients of $G_n(f)$ are, up to signs, the values of the elementary symmetric functions in the variables α^n where α runs through the roots of f. Thus, $G_n(f)$ has integer coefficients, and we conclude $F \in \mathbb{N}_0[X]$. Clearly, f divides $G_n(f)$, hence there is some (monic) $g \in \mathbb{Z}[X]$ with $fg = G_n(F)$. We resume that

$$F = f(gh)$$

satisfies the prerequisites of Proposition 2, hence F is a CNS polynomial. □

For completeness' sake let us state a trivial consequence of Theorem 3. Recall that for integer polynomials $f \in \mathcal{E}$ we defined [5, Section 3]

$$\gamma(f) = \inf\{\deg(g) : g \in \mathbb{Z}[X],\ gf \text{ is a CNS polynomial}\},$$

and see the example in Section 4 below.

Corollary 4. *For every $f \in \mathcal{E} \cap \mathbb{Z}[X]$, we have $\gamma(f) < \infty$.*

3. Proof of Theorem 1

Let us first collect some facts on the function G_n and the length of a polynomial.

Lemma 5. (1) *For $f, g \in \mathbb{R}[X]$ and $n \in \mathbb{N}$, we have*

$$G_n(fg) = G_n(f)G_n(g).$$

(2) *Let $\alpha \in \mathbb{C} \setminus \mathbb{R}$, $n \in \mathbb{N}$ and $q_n := (X - \alpha^n)(X - \overline{\alpha}^n)$. Then we have*

$$G_n(q_1)(X) = q_n(X^n).$$

PROOF. (i) Clear by the definition of G_n.
(ii) We verify

$$G_n(q_1)(X) = (X^n - \alpha^n)(X^n - \overline{\alpha}^n) = q_n(X^n).$$ □

For convenience, we write

$$\overline{\mathrm{L}}\,(f) := \mathrm{L}\,(f)/\,|f(0)|$$

for $f \in \mathbb{R}[X]$ with $f(0) \neq 0$.

Lemma 6. *Let $f \in \mathbb{R}[X]$ with $f(0) \neq 0$.*

(1) $\overline{\mathrm{L}}\,(f(X^n)) = \overline{\mathrm{L}}\,(f) \qquad (n \in \mathbb{N})$.

(2) *If all coefficients of f are non-negative, we have $\overline{\mathrm{L}}\,(f) = f(1)/f(0)$.*

(3) *For $g \in \mathbb{R}[X]$ with $g(0) \neq 0$, we have $\overline{\mathrm{L}}\,(fg) \leq \overline{\mathrm{L}}\,(f)\,\overline{\mathrm{L}}\,(g)$.*

Proof. Clear by well-known properties of the length function (e.g., see [5, Lemma 2]) and the definition. □

Now we gather some rather technical auxiliary means to establish Theorem 1 for quadratic real expansive polynomials with negative discriminant. For simplicity, we concentrate on multipliers which are powers of linear integer polynomials.

Our first lemma can immediately be verified.

Lemma 7. *For $b, c, r \in \mathbb{R}$ and $m \in \mathbb{N}$, we have*

$$\begin{aligned}&(X+r)^m \cdot (X^2 - bX + c)\\ &\quad = r^m c + r^{m-1}(cm - br)X + \sum_{k=2}^{m} p_{r,m}(k)X^k + (mr - b)X^{m+1} + X^{m+2}\end{aligned}$$

with

$$\begin{aligned}p_{r,m}(k) :&= \binom{m}{k-2} r^{m-k+2} - b\binom{m}{k-1} r^{m-k+1} + c\binom{m}{k} r^{m-k}\\ &= \binom{m}{k-1} r^{m-k} f_{r,m}(k)\end{aligned}$$

and

$$f_{r,m}(k) := \frac{m+1-k}{k}c - br + \frac{k-1}{m+2-k}r^2 \qquad (2 \leq k \leq m).$$

After these preparations, we present our first main lemma on particular quadratic polynomials, where we use the set

$$\begin{aligned}\mathcal{D}_\rho := \big\{ f \in \mathbb{R}[X] \,:\, &f(0) \neq 0,\ f \text{ monic with only}\\ &\text{non-negative coefficients and } \overline{\mathrm{L}}(f) < 1 + \rho \big\}.\end{aligned}$$

Lemma 8. *Let* $\rho > 0,\ 0 < \eta \le \log(1+\rho),\ 0 < \varepsilon \le \min\{1, \eta/2\}$, *and assume*

$$0 \le b < \varepsilon c,\ b^2 < 4c \quad \text{and} \quad c \ge \frac{1}{\exp(\eta/2) - 1}.$$

For $r > \max\{b,\ c/(\varepsilon c - b)\}$, *we have*

$$(X + r)^{\lfloor \varepsilon r \rfloor}(X^2 - bX + c) \in \mathcal{D}_\rho \cap \mathbb{R}_{>0}[X],$$

where $\mathbb{R}_{>0}$ *denotes the set of positive reals.*

Proof. First we show that for $m := \lfloor \varepsilon r \rfloor$, we have

$$p := (X + r)^m(X^2 - bX + c) \in \mathbb{R}_{>0}[X]. \tag{3.1}$$

Indeed, we observe

$$\varepsilon r - 1 < m \le \varepsilon r \le r, \tag{3.2}$$

and

$$m > \frac{br}{c}, \tag{3.3}$$

because the following implications hold:

$$(\varepsilon c - b)r > c \implies \varepsilon r - \frac{br}{c} > 1 \implies \frac{br}{c} < \varepsilon r - 1 < m.$$

Thus, trivially $m \ge 1$ by (3.3) and $r > b/m$, and by Lemma 7, the linear and the second highest coefficients of p are positive.

Therefore, (3.1) drops out if $m = 1$, and we let $m \ge 2$. For $2 \le k \le m$, we have

$$(k-1)r \ge m - (k-2)b$$

by (3.2), and using the notation of Lemma 7, we verify

$$\begin{aligned} k(m-k+2)f_{r,m}(k) &= kr((k-1)r - (m-(k-2)b)) + ((m-k)^2 + 3(m-k) + 2)c \\ &\ge ((m-k)^2 + 3(m-k) + 2)c \ge 2c > 0. \end{aligned}$$

Now (3.1) is clear by Lemma 7.

Exploiting the well-known fact that the map

$$x \mapsto \log(1 + 1/x) \qquad (x \in [1, \infty))$$

takes its values in $(0, 1)$, our prerequisites and (3.2), we check

$$m \log\left(1 + \frac{1}{r}\right) + \log\left(1 + \frac{1}{c}\right) \le \varepsilon\left(r \log\left(1 + \frac{1}{r}\right)\right) + \frac{\eta}{2} < \varepsilon + \frac{\eta}{2} \le \eta \le \log(1+\rho),$$

and Lemma 6 (iii) yields

$$\overline{\mathrm{L}}(p) = (1+r)^m(1 - b + c)\frac{1}{r^m c} \le \left(1 + \frac{1}{r}\right)^m \left(1 + \frac{1}{c}\right) < 1 + \rho. \qquad \square$$

For $\alpha \in \mathbb{C} \setminus \mathbb{R}$, we set

$$q_\alpha := (X - \alpha)(X - \bar{\alpha}).$$

Without loss of generality, we tacitly assume $\theta := \arg(\alpha) \in (0, \pi)$ if not mentioned otherwise. For convenience, we further introduce the following notation. If θ is a rational multiple of π, we write

$$\theta = \frac{u}{v}\pi \qquad (u, v \in \mathbb{N},\ u < v,\ \gcd(u, v) = 1)$$

and set $v(\alpha) := v$; otherwise we set $v(\alpha) := 0$.

Lemma 9. *For $\alpha \in \mathbb{C} \setminus \mathbb{R}$, $n \in \mathbb{N}$ and*

$$q_{\alpha,n} := X^2 - b_n X + c_n := (X - \alpha^n)(X - \bar{\alpha}^n), \tag{3.4}$$

the following statements hold:

(1) *$b_n = 2\Re(\alpha^n)$, $c_n = |\alpha|^{2n}$, and for the discriminant of $q_{\alpha,n}$, we have*

$$\operatorname{discr}(q_{\alpha,n}) = -4\,|\alpha|^{2n} \sin^2(n\theta) \le 0.$$

(2) *$\operatorname{discr}(q_{\alpha,n}) = 0$ if and only if $v(\alpha)$ divides n.*

Proof. (i) The first two statements are well-known, and we have

$$\begin{aligned}\operatorname{discr}(q_{\alpha,n}) &= (2\,|\alpha|^n \cos(n\theta))^2 - 4\,|\alpha|^{2n} \\ &= 4\,|\alpha|^{2n} (\cos^2(n\theta) - 1) = -4\,|\alpha|^{2n} \sin^2(n\theta)) \le 0.\end{aligned}$$

(ii) From (i), we deduce

$$\operatorname{discr}(q_{\alpha,n}) = 0 \iff \sin(n\theta)) = 0 \iff n\theta = m\pi$$

for some $m \in \mathbb{N}$, or

$$\operatorname{discr}(q_{\alpha,n}) = 0 \iff \theta = \frac{m}{n}\pi = \frac{u}{v(\alpha)}\pi$$

for some $m, u \in \mathbb{N}$ with $\gcd(u, v(\alpha)) = 1$.

The proof can now easily be completed. □

Lemma 10. *Let $\alpha \in \mathbb{C} \setminus \mathbb{R}$ with $|\alpha| > 1$. For every $\varepsilon > 0$, $B \geq 0$ and $C > 0$, we have*

$$|b_n| < \varepsilon c_n - B \quad \text{and} \quad c_n > C$$

for all

$$n \geq 1 + \max\left\{ \left\lfloor \frac{\log((1+\sqrt{1+\varepsilon B})/\varepsilon)}{\log|\alpha|} \right\rfloor, \left\lfloor \frac{\log C}{2\log|\alpha|} \right\rfloor \right\},$$

where we use the notation (3.4).

Proof. From the implication

$$n > \frac{\log C}{2\log|\alpha|} \implies 2n\log|\alpha| > \log C,$$

we deduce $c_n = |\alpha|^{2n} > C$. By our prerequisites, $|\alpha|^n$ exceeds the largest root $(1+\sqrt{1+\varepsilon B})/\varepsilon$ of the quadratic equation $\varepsilon x^2 - 2x - B = 0$. Thus we have

$$\varepsilon|\alpha|^{2n} - 2|\alpha|^n - B > 0,$$

yielding

$$|b_n| = 2|\alpha|^n|\cos(n\theta)| \leq 2|\alpha|^n < \varepsilon c_n - B. \qquad \square$$

Since we are interested in the quadratic polynomials $q_{\alpha,n}$ which have a negative discriminant, we find it convenient to introduce the set

$$\mathcal{N}_f := \bigcap_{\alpha \in Z_f} M(\alpha)$$

for the real monic polynomial f, where we put

$$M(\alpha) = \begin{cases} \{n \in \mathbb{N} : n \text{ odd}\} & (\alpha \in \mathbb{R} \text{ or } (\alpha \notin \mathbb{R} \text{ and } v(\alpha) = 0)), \\ \{n \in \mathbb{N} : n \text{ odd and } v(\alpha) \not| \ n\} & (\alpha \notin \mathbb{R} \text{ and } v(\alpha) \neq 0). \end{cases}$$

Let us give some simple examples.

Example 11. (1) Let $a \in \mathbb{R} \setminus \{0\}$ and ζ a root of unity different from ± 1. For $\alpha := a\zeta$ Lemma 9 yields

$$\mathcal{N}_{q_\alpha} = \{n \in \mathbb{N} : n \text{ odd and } v(\alpha) \not| \ n\}.$$

(2) Every odd prime which does not divide $v(\alpha)$ for any $\alpha \in Z_f \setminus \mathbb{R}$ belongs to $\mathcal{N}_f$.

The following properties of the set $\mathcal{N}_f$ can immediately be verified.

Lemma 12. (1) *For $\alpha \in \mathbb{C} \setminus \mathbb{R}$, we have*

$$\mathcal{N}_{q_\alpha} = \{n \in \mathbb{N} : n \text{ odd}\} \setminus \{nv(\alpha) : n \in \mathbb{N}\}.$$

(2) *If $g \in \mathbb{R}[X]$ is a monic non-constant divisor of f, then $\mathcal{N}_f$ is contained in $\mathcal{N}_g$.*

Now we are in a position to formulate our second main lemma.

Lemma 13. *Let $\rho > 0$ and $\alpha \in \mathbb{C} \setminus \mathbb{R}$ with $|\alpha| > 1$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ with $n \in \mathcal{N}_{q_\alpha}$, there is some $t \in \mathbb{Z}[X]$ with the property*

$$t\, q_{\alpha,n} \in \mathcal{D}_\rho \cap \mathbb{R}_{>0}[X]. \tag{3.5}$$

Moreover, N and t can effectively be computed, and we can choose

$$t = (X + r)^m \tag{3.6}$$

with $m \in \mathbb{N}_0$ and $r \in \mathbb{N}$.

Proof. Set $\eta := \log(1+\rho)$ and pick $0 < \varepsilon < \min\{1,\ \rho,\ \eta/2\}$. By Lemma 10 we compute an $N \in \mathbb{N}$ such that

$$|b_n| < \varepsilon c_n \quad \text{and} \quad c_n \geq \max\left\{\frac{1}{\rho - \varepsilon},\ \frac{1}{\exp(\eta/2) - 1}\right\}$$

for all $n \geq N$, where we use the notation (3.4).

Let $n \geq N$ with $n \in \mathcal{N}_{q_\alpha}$, and set $q := q_{\alpha,n}$, $b := b_n$ and $c := c_n$.

If $b < 0$, we trivially have $q \in \mathbb{R}_{>0}[X]$, and with $t := 1$ we immediately check

$$\overline{\mathrm{L}}\,(tq) = \frac{1 + |b| + c}{c} < \frac{1}{c} + \varepsilon + 1 \leq 1 + \rho,$$

thus (3.5) holds.

Now let $b \geq 0$. Note that by our choice of n and Lemma 9, the discriminant of q is negative. Take an integer r with the property $r > \max\{b,\ c/(\varepsilon c - b)\}$, put $m := \lfloor \varepsilon r \rfloor$ and $t := (X + r)^m$. Then Lemma 8 yields

$$tq \in \mathcal{D}_\rho \cap \mathbb{R}_{>0}[X],$$

and we are done. □

For convenience, we reformulate a result from [5] in our surroundings here.

Lemma 14. *Let f be a real monic expansive polynomial of positive degree, $\rho > 0$ and*

$$N := 1 + \left\lceil -\frac{\log\left((1+\rho)^{1/\deg(f)} - 1\right)}{\log|\mu|} \right\rceil, \tag{3.7}$$

where $\mu \in \mathbb{C}$ is a root of minimal modulus of f.

(1) *We have*

$$\overline{\mathrm{L}}(G_n(f)) < 1 + \rho$$

for all natural $n \geq N$.

(2) *If all roots of f are real, then we have $G_n(f) \in \mathcal{D}_\rho$ for all odd natural integers $n \geq N$.*

Proof. (i) See [5, Lemma 5].
(ii) Since n is odd, $G_n(f)$ is a product of polynomials with non-negative coefficients, and then our assertion is clear by (i). □

Let us now establish a more technical version of Theorem 1.

Proposition 15. *If $f \in \mathcal{E}$ and $\rho > 0$, then there exists $N \in \mathbb{N}$ with the following property: If $n \geq N$ is odd and not divisible by $v(\alpha)$ for any $\alpha \in \mathrm{Z}_f \setminus \mathbb{R}$, then there exists some $h \in \mathbb{Z}[X]$ such that*

$$h\, G_n(f) \in \mathcal{D}_\rho.$$

Moreover, N and h can effectively be computed, and we can choose

$$h = \left(\prod_{r\in R}(X+r)\right) \cdot \left(\prod_{s\in S}(X^n+s)\right) \tag{3.8}$$

with (possibly empty) multisets R, S of positive integers.

Proof. We proceed by induction on the number c_f of pairs of complex conjugate roots of f. If $c_f = 0$, then our claim is clear by Lemma 14 (ii) with $h := 1$ and N given by (3.7).

Now we let $c_f > 0$, pick a non-real root α of f, set $q := q_\alpha$, $g := f/q$ and

$$\sigma := \sqrt{1+\rho} - 1,$$

thus

$$0 < \sigma < \rho.$$

Lemma 13 yields an effectively computable $K \in \mathbb{N}$ such that for every odd $n \geq K$ which is not divisible by $v(\alpha)$, we can determine some $t \in \mathbb{Z}[X]$ of the form (3.6) such that

$$t\, q_{\alpha,n} \in \mathcal{D}_\sigma \subseteq \mathcal{D}_\rho,$$

which by Lemmas 5 and 6 implies

$$t(X^n)\, G_n(q) = t(X^n)\, q_{\alpha,n}(X^n) = (t\, q_{\alpha,n})(X^n) \in \mathcal{D}_\rho. \tag{3.9}$$

Thus, if g is constant, we are done, because $f = q$ and $t(X^n)$ has the form (3.8).

Otherwise, we have $g \in \mathcal{E}$ with $c_g < c_f$, and by induction hypothesis, we compute some $M \in \mathbb{N}$ such that for all odd $n \geq M$ not divisible by $v(\beta)$ for any $\beta \in \mathrm{Z}_g \setminus \mathbb{R}$, we can determine some $s \in \mathbb{Z}[X]$ of the form (3.8) such that

$$s\, G_n(g) \in \mathcal{D}_\sigma. \tag{3.10}$$

Put

$$N := \max\{K, M, v(\gamma) \ : \ \gamma \in \mathrm{Z}_f \setminus \mathbb{R}\}$$

and consider an odd $n \geq N$ not divisible by $v(\gamma)$ for any $\gamma \in \mathrm{Z}_f \setminus \mathbb{R}$. By the above and Lemma 12, we find $s, t \in \mathbb{Z}[X]$ satisfying (3.8), (3.9) and (3.10), and with

$$h(X) := s(X) \cdot t(X^n)$$

we infer

$$h\, G_n(f) = (s(X)\, G_n(g)) \cdot (t(X^n) G_n(q)) \in \mathcal{D}_\sigma \cdot \mathcal{D}_\sigma \subseteq \mathcal{D}_\rho$$

from Lemmas 5 and 6 and [5, Lemma 4]. Clearly, h has the form (3.8), and the proof is terminated. □

Now we straightforwardly provide a proof of Theorem 1. Let $f \in \mathcal{E}$ and $\rho > 0$. Exploiting Proposition 15, we take a suitable odd $n \in \mathbb{N}$ (e.g., we may choose a prime larger than $\max\{2, v(\alpha) \ : \ \alpha \in \mathrm{Z}_f \setminus \mathbb{R}\}$) and compute some $h \in \mathbb{Z}[X]$ such that

$$h\, G_n(f) \in \mathcal{D}_\rho,$$

i.e., all coefficients of $h\, G_n(f)$ are non-negative and

$$\mathrm{L}\,(h\, G_n(f)) < (1 + \rho)\, (h\, G_n(f))(0).$$

The proof is completed. □

4. Concluding remarks

With some more effort, the method of our proofs above allows the algorithmic determination of a CNS multiple of a polynomial f which satisfies the prerequisites of Theorem 3. Obviously, Proposition 15 exploits rather crude bounds to construct a very special CNS multiple of f. Therefore we might expect that a construction of other multiples of f which are better adapted to the structure of f can be easier in favorable cases. The reader may check this by [5, Example 13 (iii)]; here we illustrate this observation by another example.

Consider

$$f := X^5 - 4X^4 + 3X^3 + 12X^2 - 26X + 20,$$

which factorizes in the form

$$f = (X+2)(X^2 - 2X + 2)(X^2 - 4X + 5),$$

hence its root are -2, $1 \pm \mathrm{i}$ and $2 \pm \mathrm{i}$. Thus $f \in \mathcal{E}$, and by [2, Lemma 2] f is not a CNS polynomial, since $f(1) < f(0)$.

Similarly as in the proof of Theorem 3 (Lemma 13, respectively), we set

$$\rho := 2^{1/3} - 1, \ \eta := (\log 2)/3 \quad \text{and} \quad \varepsilon := (\log 2)/6.$$

By Lemma 14 we have

$$X^n + 2^n = G_n(X+2) \in \mathcal{D}_\rho \qquad (n \text{ odd}, n \geq 3).$$

For $\alpha := 1 + \mathrm{i}$ with $\arg(\alpha) = \pi/4$ we find $n \geq 9$, and we check that this lower bound for n also suffices for the root $2 + \mathrm{i}$ of f. Again exploiting the proof of Lemma 13, we set $r := 24$ and $m := 2$, and then we convince ourselves that

$$(X+24)^2 \cdot G_9(f) = (X+24)^2 \cdot G_9(X+2) \cdot G_9(X^2-2X+2) \cdot G_9(X^2-4X+5) \in \mathcal{D}_1,$$

hence it is a CNS polynomial by Proposition 2 and therefore

$$\gamma(f) \leq 42.$$

On the other hand, again applying Proposition 2, we check that $(X+24)^2 \cdot G_3(f)$ is a CNS polynomial, because it can be written as

$$\begin{aligned} X^{17} &+ 48X^{16} + 576X^{15} + 8X^{14} + 384X^{13} + 4608X^{12} + 117X^{11} + 5616X^{10} \\ &+ 67392X^9 + 1404X^8 + 67392X^7 + 808704X^6 + 4744X^5 + 227712X^4 \\ &+ 2732544X^3 + 8000X^2 + 384000X + 4608000, \end{aligned}$$

yielding the sharper bound

$$\gamma(f) \leq 12.$$

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