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Some recurrent normal Jacobi operators on real hypersurfaces in complex two-plane Grassmannians

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Abstract. In this paper, we prove that there are no Hopf hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ such that the normal Jacobi operator is generalized $\mathfrak{F}$-recurrent, where $\mathfrak{F} = \operatorname{span}\{\xi, \xi_1, \xi_2, \xi_3\}$. We also prove that there are no Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ such that the normal Jacobi operator is $\mathfrak{D}^\perp$-recurrent and the Hopf principal curvature is invariant along the Reeb flow, where $\mathfrak{D}^\perp = \operatorname{span}\{\xi_1, \xi_2, \xi_3\}$.

1. Introduction

A complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is defined as the set of all two-dimensional linear subspaces in $\mathbb{C}^{m+2}$ which is identified with the homogeneous space $SU(m+2)/S(U(2) \times U(m))$. It is known as a compact irreducible Hermitian symmetric space of rank two equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ with a canonical basis $\{J_1, J_2, J_3\}$ which does not contain J (see [2]). When $m = 1$, $G_2(\mathbb{C}^3)$ can be identified with the complex projective plane $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight, and when $m = 2$, $G_2(\mathbb{C}^4)$ is isometric to the real Grassmannian manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspace in $\mathbb{R}^6$. In this paper, m is assumed to be $m \geq 3$.

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, with N and A a unit normal vector field and the shape operator, respectively. Let g and ∇ be the induced metric from the ambient space and the corresponding Levi-Civita connection,

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respectively. The vector field $\xi := -JN$ for the Kähler structure J and the normal vector field N is said to be the Reeb vector field. The almost contact metric 3-structure vector fields ξ_ν are defined by $\xi_\nu = -J_\nu N$ for $\nu \in \{1,2,3\}$. We denote by $\mathfrak{D}^\perp$ the distribution defined by $\mathfrak{D}^\perp = \operatorname{span}\{\xi_1,\xi_2,\xi_3\}$, and by $\mathfrak{D}$ its orthogonal complement distribution satisfying $T_pM = \mathfrak{D}_p \oplus \mathfrak{D}_p^\perp$ at each point $p \in M$. A real hypersurface in $G_2(\mathbb{C}^{m+2})$ is said to be Hopf if ξ is an eigenvector field of the shape operator, i.e., $A\xi = \alpha\xi$, and $\alpha = g(A\xi,\xi)$ is said to be the Hopf principal curvature.

One of the most known classification results for Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ was obtained by Berndt and Suh [3].

Theorem 1.1 ([3])**.** *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both* $\operatorname{span}\{\xi\}$ *and $\mathfrak{D}^\perp$ are invariant under the shape operator if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic quaternionic projective space $\mathbb{H}P^m$ in $G_2(\mathbb{C}^{m+2})$.*

Following Theorem 1.1, many characterizations and non-existence results for Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ were obtained. Among others, one of the most mentioned conditions in this framework is the so-called normal Jacobi operator $\overline{R}_N$ associated to the normal vector field N. Berndt [1] introduced the notion of normal Jacobi operator for real hypersurfaces in quaternionic projective space $\mathbb{H}P^m$ and quaternionic hyperbolic space $\mathbb{H}H^m$, respectively. Later, such notion was considered in [20] for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ which is defined by

$$\overline{R}_N = \overline{R}(\cdot\,,N)N \in \operatorname{End}(T_pM), \quad p \in M, \tag{1}$$

where $\overline{R}$ denotes the Riemannian curvature tensor of $G_2(\mathbb{C}^{m+2})$.

Jeong, Lee and Suh in [9] proved that there do not exist Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator, i.e., $\mathcal{L}_X\overline{R}_N = 0$ for any vector field X, if the integral curves of $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components of the Reeb vector field are totally geodesic. Actually, such result generalized those in [12] under a weaker condition, namely the normal Jacobi operator is Lie ξ-parallel, i.e., $\mathcal{L}_\xi\overline{R}_N = 0$. The normal Jacobi operator is said to be parallel if it satisfies $\nabla_X\overline{R}_N = 0$ for any vector field X. Under such condition and applying Theorem 1.1, Jeong, Kim and Suh proved that following

Theorem 1.2 ([7])**.** *There exist no Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ such that the normal Jacobi operator is parallel.*

Theorem 1.2 shows that parallelism of the normal Jacobi operator is a rather strong condition for Hopf hypersurfaces. Therefore, many authors tried to weaken such a condition and generalize Theorem 1.2. Machado, Pérez, Jeong and Suh in [18] proved that there does not exist any Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ whose normal Jacobi operator is of Codazzi type, i.e., $(\nabla_X \overline{R}_N)Y = (\nabla_Y \overline{R}_N)X$ for any vector fields X, Y, if the distribution $\mathfrak{D}$ or $\mathfrak{D}^\perp$ components of the Reeb vector field are invariant under the shape operator. Panagiotidou and Tripathi in [22] proved that there do not exist Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ such that the normal Jacobi operator is semi-parallel, i.e., $R(X,Y)\overline{R}_N Z = \overline{R}_N(R(X,Y)Z)$ for any vector fields X, Y, Z, and $\alpha \neq 0$ and $\mathfrak{D}$ or $\mathfrak{D}^\perp$ component of the Reeb vector field is invariant under the shape operator. Recently, such result was improved by Huang, Lee and Suh in [6] by removing the restriction $\alpha \neq 0$. Also, De and Loo in [5] proved that there does not exist any real hypersurface with pseudo-parallel normal Jacobi operator. Machado, Pérez, Jeong and Suh in [19] proved that there do not exist Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-parallel normal Jacobi operator, i.e., $\nabla_X \overline{R}_N = 0$ for any vector field X belonging to $\mathfrak{D}$, if the distribution $\mathfrak{D}$ or $\mathfrak{D}^\perp$ component of the Reeb vector field is invariant under the shape operator.

Throughout this paper, we denote by $\mathfrak{F}$ the distribution $\operatorname{span}\{\xi, \xi_1, \xi_2, \xi_3\} = \operatorname{span}\{\xi\} + \mathfrak{D}^\perp$. The normal Jacobi operator on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is said to be generalized $\mathfrak{F}$-recurrent if it satisfies

$$\nabla_X \overline{R}_N = \rho \otimes \xi + \omega(X)\overline{R}_N \tag{2}$$

for any vector field X belonging to $\mathfrak{F}$, where both ρ and ω are 1-forms. Obviously, when $\rho = \omega = 0$, then the generalized $\mathfrak{F}$-recurrence condition reduces to the $\mathfrak{F}$-parallelism (see [13]). When $\omega = 0$, structure Jacobi operator satisfying (2) for $X = \xi$ was considered in [25] for real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$. When $\rho = 0$, (2) becomes $\mathfrak{F}$-recurrent condition. In the present paper, we generalize Theorem 1.2 by considering (2), and prove

Theorem 1.3. *There are no Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ if the normal Jacobi operator is generalized $\mathfrak{F}$-recurrent.*

It follows from Theorem 1.3 directly that

Corollary 1.4. *There are no Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ if the normal Jacobi operator is $\mathfrak{F}$-recurrent.*

A real hypersurface in $G_2(\mathbb{C}^{m+2})$ is said to be with recurrent normal Jacobi operator if $\nabla_X \overline{R}_N = \omega(X)\overline{R}_N$ for any vector field X and certain one-form ω. It is clear that when $\rho = 0$, Theorem 1.3 extends the following

Corollary 1.5 ([11])**.** *There does not exist Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with recurrent normal Jacobi operator.*

Moreover, when $\rho = \omega = 0$, Theorem 1.3 reduces to the following

Corollary 1.6 ([13])**.** *There does not exist Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{F}$-parallel normal Jacobi operator.*

The normal Jacobi operator $\overline{R}_N$ on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is said to be $\mathfrak{D}^{\perp}$-recurrent if it satisfies

$$\nabla_X \overline{R}_N = \omega(X)\overline{R}_N \tag{3}$$

for any vector field X belonging to $\mathfrak{D}^{\perp}$, where ω is an 1-form. If considering some other restrictions and the above definition, we obtain the following generalization of Corollaries 1.4, 1.5 and 1.6.

Theorem 1.7. *There are no Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ if the normal Jacobi operator is $\mathfrak{D}^{\perp}$-recurrent and the Hopf principal curvature is invariant along the Reeb vector field.*

In particular, if $\omega = 0$, $\mathfrak{D}^{\perp}$-recurrence for normal Jacobi operator becomes $\mathfrak{D}^{\perp}$-parallelism, and in this case Theorem 1.7 reduces to

Corollary 1.8 ([24])**.** *There does not exist Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}^{\perp}$-parallel normal Jacobi operator if the distribution $\mathfrak{D}$ or $\mathfrak{D}^{\perp}$ component of the Reeb vector field is invariant under the shape operator.*

At the end of this paper, we also discuss Reeb parallelism of the normal Jacobi operator for hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

2. Preliminaries

In this section, we collect some fundamental formulas shown in [2], [3], [4]. Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with real codimension one, and N be a unit normal vector field. On M there exists an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure J of $G_2(\mathbb{C}^{m+2})$. In this paper we put

$$JX = \phi X + \eta(X)N, \qquad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{4}$$

for any vector field X. From the first term of (4), it follows that

$$\phi^2 = -\operatorname{id} + \eta \otimes \xi, \qquad \eta(\xi) = 1, \quad \phi\xi = 0, \qquad \eta(X) = g(X, \xi), \tag{5}$$

where the Reeb vector field ξ is determined by $\xi := -JN$. Let $\{J_1, J_2, J_3\}$ be a canonical local basis of quaternionic Kähler structure $\mathfrak{J}$ of $G_2(\mathbb{C}^{m+2})$. Then, from the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, we have an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ as the following:

$$\begin{aligned}
\phi_\nu^2 &= -\operatorname{id} + \eta_\nu \otimes \xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \phi_\nu \xi_\nu = 0,\\
\phi_\nu \xi_{\nu+1} &= \xi_{\nu+2}, \quad \phi_{\nu+1}\xi_\nu = -\xi_{\nu+2},\\
\phi_\nu \phi_{\nu+1} &= \phi_{\nu+2} + \eta_{\nu+1} \otimes \xi_\nu,\\
\phi_{\nu+1}\phi_\nu &= -\phi_{\nu+2} + \eta_\nu \otimes \xi_{\nu+1},
\end{aligned} \tag{6}$$

where the index is taken modulo three. According to condition $J_\nu J = JJ_\nu$, the relationships between two almost contact metric structures are given by

$$\phi\phi_\nu = \phi_\nu\phi + \eta_\nu \otimes \xi - \eta \otimes \xi_\nu, \ \phi\xi_\nu = \phi_\nu\xi, \qquad \eta_\nu(\phi\,\cdot) = \eta(\phi_\nu\cdot). \tag{7}$$

Since J is parallel with respect to the Riemannian connection of $G_2(\mathbb{C}^{m+2})$, we have

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \qquad \nabla_X\xi = \phi AX \tag{8}$$

for any vector fields X, Y, where we have used the Guass and Weingarten formulas. Similarly, since J_ν is a quaternionic Kähler structure of $G_2(\mathbb{C}^{m+2})$, we have

$$\begin{aligned}
\nabla_X\xi_\nu &= q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX,\\
\nabla_X\phi_\nu &= q_{\nu+2}(X)\phi_{\nu+1} - q_{\nu+1}(X)\phi_{\nu+2} + \eta_\nu \otimes AX - g(AX, \cdot)\xi_\nu
\end{aligned} \tag{9}$$

for any vector field X.

The Riemannian curvature tensor $\overline{R}$ of $G_2(\mathbb{C}^{m+2})$ is given by (see [3])

$$\begin{aligned}
\overline{R}(X,Y)Z = {}& g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ\\
&+ \sum_{\nu=1}^{3}\{g(J_\nu Y, Z)J_\nu X - g(J_\nu X, Z)J_\nu Y - 2g(J_\nu X, Y)J_\nu Z\}\\
&+ \sum_{\nu=1}^{3}\{g(J_\nu JY, Z)J_\nu JX - g(J_\nu JX, Z)J_\nu JY\}
\end{aligned} \tag{10}$$

for any vector fields X, Y, Z.

3. Proofs of the main results

According to (10), the normal Jacobi operator is given by

$$\begin{aligned}\overline{R}_N(X) = \overline{R}(X,N)N = X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)(\phi_\nu\phi X - \eta(X)\xi_\nu) - \eta_\nu(\phi X)\phi_\nu\xi\}\end{aligned} \tag{11}$$

for any vector field X. Taking the covariant derivative of (11) and making use of formulas in Section 2, we obtain

$$\begin{aligned}(\nabla_X\overline{R}_N)Y = 3g(\phi AX,Y)\xi + 3\eta(Y)\phi AX + 3\sum_{\nu=1}^{3}\{g(\phi_\nu AX,Y)\xi_\nu + \eta_\nu(Y)\phi_\nu AX\} \\ - \sum_{\nu=1}^{3}\{2\eta_\nu(\phi AX)(\phi_\nu\phi Y - \eta(Y)\xi_\nu) - g(\phi_\nu AX,\phi Y)\phi_\nu\xi \\ - \eta(Y)\eta_\nu(AX)\phi_\nu\xi - \eta_\nu(\phi Y)(\phi_\nu\phi AX - g(AX,\xi)\xi_\nu)\}\end{aligned} \tag{12}$$

for any vector fields X, Y.

Using the Codazzi equation and $A\xi = \alpha\xi$, we have (see [3])

Lemma 3.1. *If M is a orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, then*

$$\operatorname{grad}\alpha = \xi(\alpha)\xi + 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi, \tag{13}$$

where grad *is the gradient operator.*

Lemma 3.2. *If M is a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ such that the normal Jacobi operator is generalized $\mathfrak{F}$-recurrent, then either $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$.*

Proof. Let us suppose that $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ with X_0 a unit vector field orthogonal to $\mathfrak{D}^\perp$. If either $\eta(X_0) = 0$ or $\eta(\xi_1) = 0$, then the lemma is verified. In what follows, we suppose that $\eta(X_0)\eta(\xi_1) \neq 0$. Because the normal Jacobi operator is generalized $\mathfrak{F}$-recurrent, from (2) we have

$$(\nabla_\xi\overline{R}_N)\xi = \rho(\xi)\xi + \omega(\xi)\overline{R}_N(\xi). \tag{14}$$

With the aid of (11), (12) and $A\xi = \alpha\xi$, (14) can be written as the following:

$$4\alpha\eta(\xi_1)\phi_1\xi = \rho(\xi)\xi + 4\omega(\xi)(\xi + \eta(\xi_1)\xi_1). \tag{15}$$

Taking the inner product of (15) with $\phi_1\xi$ gives $4\alpha\eta(\xi_1)g(\phi_1\xi,\phi_1\xi)=0$, and hence $\alpha g(\phi_1\xi,\phi_1\xi)=0$ because of $\eta_1(\xi)\neq 0$. We assume that $\alpha\neq 0$ and hence we get $g(\phi_1\xi,\phi_1\xi)=0$, which is also equivalent to $\eta_1^2(\xi)-1=0$. Without loss of generality, let us consider $\eta_1(\xi)=1$, and this reduces to $\xi=\eta(X_0)X_0+\xi_1$. It follows directly that $1=g(\xi,\xi)=g(\eta(X_0)X_0+\xi_1,\eta(X_0)X_0+\xi_1)=\eta^2(X_0)+1$, and this implies a contradiction. Then, our assumption is wrong, and we must have $\alpha=0$. In this case, (13) becomes $\phi_1\xi=0$ because of $\eta_1(\xi)\neq 0$. Applying this, the action of ϕ_1 on $\xi=\eta(X_0)X_0+\eta(\xi_1)\xi_1$ implies $\phi_1X_0=0$ because of $\eta(X_0)\neq 0$. However, it is easy to check that $0=g(\phi_1X_0,\phi_1X_0)=g(X_0,X_0)-\eta_1^2(X_0)=1$, a contradiction. $\square$

Lemma 3.3. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ such that the normal Jacobi operator is generalized $\mathfrak{F}$-recurrent. If $\xi\in\mathfrak{D}^\perp$, then the normal Jacobi operator is $\mathfrak{F}$-parallel.*

Proof. If $\xi\in\mathfrak{D}^\perp$, it follows that $JN\in\mathfrak{J}N$. We assume that J_1 is the almost Hermitian structure of $\mathfrak{J}$ such that $JN=J_1N$. Then we have

$$\xi=\xi_1,\quad \phi\xi_2=-\xi_3,\quad \phi\xi_3=\xi_2,\quad \phi\mathfrak{D}\subset\mathfrak{D}. \tag{16}$$

Because M is Hopf, using (6), $\phi_2\xi=-\xi_3$ and $\phi_3\xi=\xi_2$ in (12), we obtain $(\nabla_\xi\overline{R}_N)(Y)=0$. Since the normal Jacobi operator is assumed to be generalized $\mathfrak{F}$-recurrent, applying the previous relation in (2), we obtain

$$\rho(Y)\xi+\omega(\xi)(Y+4\eta(Y)\xi+3\sum_{\nu=1}^{3}\eta_\nu(Y)\xi_\nu-\phi_1\phi Y+\sum_{\nu=1}^{3}\eta_\nu(\phi Y)\phi_\nu\xi)=0 \tag{17}$$

for any vector field Y, where we used relations (11). Replacing Y in (17) by ξ_2 and applying (16), we obtain $\rho(\xi_2)\xi+4\omega(\xi)\xi_2=0$, which is equivalent to

$$\rho(\xi_2)=\omega(\xi)=0. \tag{18}$$

In view of the second term of the above relations, (17) becomes

$$-4\alpha\sum_{\nu=1}^{3}\eta_\nu(\phi Y)\xi_\nu+3\alpha\sum_{\nu=1}^{3}\eta_\nu(Y)\phi_\nu\xi+\alpha\sum_{\nu=1}^{3}g(\phi_\nu\xi,\phi Y)\phi_\nu\xi-\rho(Y)\xi=0 \tag{19}$$

for any vector field Y.

Taking the inner product of (19) with ξ and applying (16), we obtain $\rho=0$. In this context, the generalized $\mathfrak{F}$-recurrence condition for the normal Jacobi operator becomes $\mathfrak{F}$-recurrence, i.e.,

$$(\nabla_X\overline{R}_N)Y=\omega(X)\overline{R}_N(Y) \tag{20}$$

for any vector field X belonging to $\mathfrak{F}$ and any vector field Y on M. Replacing X by ξ_2 in (20) and applying (11), (12), (16) and $A\xi = \alpha\xi$, we get

$$\begin{aligned}
&3g(\phi A\xi_2, Y)\xi + 3\eta(Y)\phi A\xi_2 + 3\sum_{\nu=1}^{3}\{g(\phi_\nu A\xi_2, Y)\xi_\nu + \eta_\nu(Y)\phi_\nu A\xi_2\}\\
&\quad - \sum_{\nu=1}^{3}\{2\eta_\nu(\phi A\xi_2)(\phi_\nu\phi Y - \eta(Y)\xi_\nu) - g(\phi_\nu A\xi_2, \phi Y)\phi_\nu\xi\\
&\quad - \eta(Y)\eta_\nu(A\xi_2)\phi_\nu\xi - \eta_\nu(\phi Y)\phi_\nu\phi A\xi_2\}\\
&= \omega(\xi_2)(Y + 3\eta(Y)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(Y)\xi_\nu\\
&\quad - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)(\phi_v\phi Y - \eta(Y)\xi_\nu) - \eta_\nu(\phi Y)\phi_\nu\xi\})
\end{aligned} \tag{21}$$

for any vector field Y. Substituting $Y = \xi$ into (21), with the aid of (16), it follows that

$$3\phi A\xi_2 + 3\phi_1 A\xi_2 + 6g(A\xi_2, \xi_3)\xi_2 - 6g(A\xi_2, \xi_2)\xi_3 = 8\omega(\xi_2)\xi. \tag{22}$$

In view of assumption $\xi = \xi_1$, taking the inner product of equation (22) with ξ, we obtain $\omega(\xi_2) = 0$.

Similarly, replacing X by ξ_3 in (20) and applying relations (11), (12) and (16), we get

$$\begin{aligned}
&3g(\phi A\xi_3, Y)\xi + 3\eta(Y)\phi A\xi_3 + 3\sum_{\nu=1}^{3}\{g(\phi_\nu A\xi_3, Y)\xi_\nu + \eta_\nu(Y)\phi_\nu A\xi_3\}\\
&\quad - \sum_{\nu=1}^{3}\{2\eta_\nu(\phi A\xi_3)(\phi_\nu\phi Y - \eta(Y)\xi_\nu) - g(\phi_\nu A\xi_3, \phi Y)\phi_\nu\xi\\
&\quad - \eta(Y)\eta_\nu(A\xi_3)\phi_\nu\xi - \eta_\nu(\phi Y)\phi_\nu\phi A\xi_3\}\\
&= \omega(\xi_3)(Y + 3\eta(Y)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(Y)\xi_\nu\\
&\quad - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)(\phi_v\phi Y - \eta(Y)\xi_\nu) - \eta_\nu(\phi Y)\phi_\nu\xi\})
\end{aligned} \tag{23}$$

for any vector field Y. Substituting $Y = \xi$ into (23), with the help of (16), we obtain

$$3\phi A\xi_3 + 3\phi_1 A\xi_3 - 6g(A\xi_2, \xi_3)\xi_3 + 6g(A\xi_3, \xi_3)\xi_2 = 8\omega(\xi_3)\xi. \tag{24}$$

In view of $\xi = \xi_1$, the inner product of (24) with ξ implies $\omega(\xi_3) = 0$.

Taking into account $\omega(\xi) = \omega(\xi_2) = \omega(\xi_3) = 0$ and $\rho = 0$, we observe from (20) that the normal Jacobi operator is $\mathfrak{F}$-parallel when $\xi \in \mathfrak{D}^\perp$. $\square$

Before giving proofs of our main results, we also need the following two results.

Lemma 3.4 ([17])**.** *Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$. Then $\xi \in \mathfrak{D}$ if and only if $A\mathfrak{D} \subset \mathfrak{D}$ and M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

Proposition 3.5 ([3])**.** *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $A\mathfrak{D} \subset \mathfrak{D}$ and $\xi \in \mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r),$$

with some $r \in (0, \frac{\pi}{4})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = m(\gamma) = 3, \quad m(\lambda) = m(\mu) = 4n - 4,$$

and the corresponding eigenspaces are

$$\begin{aligned} &T_\alpha = \mathbb{R}\xi = \operatorname{span}\{\xi\}, \\ &T_\beta = \mathfrak{J}J\xi = \operatorname{span}\{\xi_1, \xi_2, \xi_3\}, \\ &T_\gamma = \mathfrak{J}\xi = \operatorname{span}\{\phi_1\xi, \phi_2\xi, \phi_3\xi\}, \quad T_\lambda, \quad T_\mu, \end{aligned}$$

where $T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp$, $\mathfrak{J}T_\lambda = T_\lambda$, $\mathfrak{J}T_\mu = T_\mu$, $JT_\lambda = T_\mu$.

Proof of Theorem 1.3. According to Lemma 3.2, we consider only two cases. When $\xi \in \mathfrak{D}^\perp$, by Lemma 3.3, the normal Jacobi operator is $\mathfrak{F}$-parallel. Jeong and Suh in [13] proved that there are no Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{F}$-parallel normal Jacobi operator. On the other hand, when $\xi \in \mathfrak{D}$, according to Lemma 3.4, M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$. Next, we show that on such hypersurfaces the normal Jacobi operator cannot be generalized $\mathfrak{F}$-recurrent.

If the normal Jacobi operator is generalized $\mathfrak{F}$-recurrent, we have

$$(\nabla_X \overline{R}_N)Y = \rho(Y)\xi + \omega(X)\overline{R}_N(Y)$$

for any vector field $X \in \mathfrak{F}$ and any vector field Y. Replacing X and Y by ξ_1 and ξ, respectively, and applying (11), (12) and Proposition 3.5, we obtain

$$4\beta\phi_1\xi = \rho(\xi_1)\xi + 4\omega(\xi_1)\xi.$$

The inner product of the above relation with $\phi_1\xi$ gives $4\beta g(\phi_1\xi, \phi_1\xi) = 4\beta = 0$. However, by Proposition 3.5, β cannot be zero, and we arrive at a contradiction. This completes the proof. □

Lemma 3.6 ([10], [16])**.** *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$. Then the Hopf principal curvature is invariant along the Reeb vector field if and only if the $\mathfrak{D}$ and $\mathfrak{D}^\perp$-components of the Reeb vector field are invariant under the shape operator.*

The proof of our second result depends on the following lemma.

Lemma 3.7. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ such that the normal Jacobi operator is $\mathfrak{D}^\perp$-recurrent. If the Hopf principal curvature is invariant along the Reeb vector field, then either $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$.*

Proof. Following the proof of Lemma 3.2, let us suppose that $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$, with X_0 a unit vector field orthogonal to $\mathfrak{D}^\perp$ and $\eta(X_0)\eta(\xi_1) \neq 0$. Since the normal Jacobi operator is $\mathfrak{D}^\perp$-recurrent, we have

$$(\nabla_X \overline{R}_N)Y = \omega(X)\overline{R}_N(Y)$$

for any vector field X belonging to $\mathfrak{F}$ and any vector field Y. Replacing X and Y by ξ_1 and ξ, respectively, in the previous equation and recalling (11), (12) imply

$$\begin{aligned} &3\phi A\xi_1 + 3\sum_{\nu=1}^{3} g(\phi_\nu A\xi_1, \xi)\xi_\nu + 3\eta(\xi_1)\phi_1 A\xi_1 \\ &\quad + 2\sum_{\nu=1}^{3}\eta_\nu(\phi A\xi_1)\xi_\nu + \sum_{\nu=1}^{3}\eta_\nu(A\xi_1)\phi_\nu\xi = 4\omega(\xi_1)(\xi + \eta(\xi_1)\xi_1), \end{aligned} \tag{25}$$

where we applied $A\xi = \alpha\xi$. Moreover, applying Lemma 3.6, we know the distributions $\mathfrak{D}$ or $\mathfrak{D}^\perp$ component of the Reeb vector field are invariant under the shape operator. The action of A on $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ gives

$$AX_0 = \alpha X_0, \qquad A\xi_1 = \alpha\xi_1. \tag{26}$$

Substituting the second term of (26) into the previous relation implies

$$\alpha\phi\xi_1 = \omega(\xi_1)(\xi + \eta_1(\xi)\xi_1). \tag{27}$$

Taking the inner product of the above equation with $\phi\xi_1$ gives $\alpha g(\phi\xi_1, \phi\xi_1) = 0$. The remaining proof has been already shown in that of Lemma 3.2. □

Proof of Theorem 1.7. Following Lemma 3.7, next we consider only two cases. When $\xi \in \mathfrak{D}^{\perp}$, as shown in Lemma 3.3, we may put $\xi = \xi_1$, and then the inner product of (25) with ξ gives $\omega(\xi_1) = 0$. On the other hand, proceeding similar with the proof of Lemma 3.3, we obtain directly $\omega(\xi_2) = \omega(\xi_3) = 0$. In this context, we see that the $\mathfrak{D}^{\perp}$-recurrent normal Jacobi operator is in fact $\mathfrak{D}^{\perp}$-parallel. Therefore, the non-existence proof for such case follows immediately from Suh and Jeong [24, Theorem 1] and Lemma 3.6.

When $\xi \in \mathfrak{D}$, we omit the proof for this case, because it is very similar with that of Theorem 1.3. □

Comparing the first two main theorems in the Introduction, we observe that Theorem 1.3 does not require the additional condition $\xi(\alpha) = 0$, but its assumption (i.e., $\mathfrak{F}$-recurrence) is stronger than that of Theorem 1.7 (i.e., $\mathfrak{D}^{\perp}$-recurrence). On the other hand, Hopf hypersurfaces can be classified under the conditions of Reeb parallel structure Jacobi operator $\nabla_{\xi} R_{\xi} = 0$ and $\xi(\alpha)$=0 with $\alpha \neq 0$ (see [8]), or Reeb parallel shape operator $\nabla_{\xi} A = 0$ (see [15]), even Reeb parallel Ricci operator with non-vanishing geodesic Reeb flow, but not Reeb parallel normal Jacobi operator $\nabla_{\xi} \overline{R}_N = 0$, due to the following

Theorem 3.8. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ such that $\xi \in \mathfrak{D}^{\perp}$. Then the normal Jacobi operator is Reeb parallel.*

Proof. As seen in proof of Lemma 3.3, we may put $\xi = \xi_1$ because of $\xi \in \mathfrak{D}^{\perp}$. Substituting $X = \xi$ into (12) and using $A\xi = \alpha\xi$ implies

$$\begin{aligned}(\nabla_{\xi} \overline{R}_N)Y = 3\alpha \sum_{\nu=1}^{3} g(\phi_{\nu}\xi, Y)\xi_{\nu} + 3\alpha \sum_{\nu=1}^{3} \eta_{\nu}(Y)\phi_{\nu}\xi + \alpha \sum_{\nu=1}^{3} g(\phi_{\nu}\xi, \phi Y)\phi_{\nu}\xi \\ - \alpha\eta(Y) \sum_{\nu=1}^{3} \eta_{\nu}(\xi)\phi_{\nu}\xi - \alpha \sum_{\nu=1}^{3} \eta_{\nu}(\phi Y)\xi_{\nu} \end{aligned} \tag{28}$$

for any vector field Y. The application of (16) in (28) implies $(\nabla_{\xi} \overline{R}_N)Y = 0$ for any vector field Y. □

However, the above conclusion is not true for Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\xi \in \mathfrak{D}$, because by applying Lemma 3.4 and Proposition 3.5 in (27) for $Y = \xi_1$, we have $(\nabla_{\xi} \overline{R}_N)\xi_1 = 4\alpha\phi_1\xi \neq 0$.

Pak and Pérez in [21] proved that the normal Jacobi operator in $G_2(\mathbb{C}^{m+2})$ is Reeb parallel with respect to the generalized Tanaka–Webster connection.

In view of Theorem 3.8, one observes that the classification for Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ needs some additional restrictions. In fact, such situation has been considered by Jeong and Suh in [14], who classified Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with Reeb parallel normal Jacobi operator and $g(A\mathfrak{D},\mathfrak{D}^{\perp})=0$.

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