Publ. Math. Debrecen
95/3-4 (2019), 321–334
DOI: 10.5486/PMD.2019.8410

Rings in which every element is the sum of a left zero-divisor and an idempotent

By EBRAHIM GHASHGHAEI (Ahvaz) and MUHAMMET TAMER KOŞAN (Ankara)

Abstract. A ring R is called *left zero-clean* if every element is the sum of a left zero-divisor and an idempotent. This class of rings is a natural generalization of O-rings and nil-clean rings. We determine when a skew polynomial ring is a left zero-clean ring. It is proved that a ring R is left zero-clean if and only if the upper triangular matrix ring $\mathbb{T}_n(R)$ is left zero-clean. It is shown that a commutative ring R is zero-clean if and only if the matrix ring $\mathbb{M}_n(R)$ is zero-clean for every positive integer $n \geq 1$. We characterize the zero-clean matrix rings over fields. We also determine when a 2×2 matrix A over a field is left zero-clean. A ring is called *uniquely left zero-clean* if every element is uniquely the sum of a left zero-divisor and an idempotent. We completely determine when a ring is uniquely left zero-clean.

1. Introduction

Throughout this paper, R will be an associative ring with identity, $\mathrm{U}(R)$ its group of units, $\mathrm{J}(R)$ its Jacobson radical, $\mathrm{Idem}(R)$ its set of idempotents, and $\mathrm{Nil}(R)$ is the set of nilpotent elements of R. For $x \in R$, $\mathrm{Ann}_l(x)=\{a \in R : ax=0\}$ and $\mathrm{Ann}_r(x) = \{a \in R : xa = 0\}$ denote the left annihilator and the right annihilator ideals of x in R, respectively. When $\mathrm{Ann}_r(x) \neq 0$, we say x is a left zero-divisor; otherwise it is a non-left zero-divisor. Let $\mathrm{Z}_l(R)$ (resp., $\mathrm{Z}_l^*(R)$) denote the set of left zero-divisors (resp., non-left zero-divisors) of R. Similarly, let $\mathrm{Z}_r(R)$ (resp., $\mathrm{Z}_r^*(R)$) denote the set of right zero-divisors (resp., non-right

---

Mathematics Subject Classification: 16U99, 15B33, 16W99.

Key words and phrases: idempotent, nil-clean, zero-clean, zero-divisor.
The first author would like to thank the National Elite Foundation of Iran for supporting this work. The second author is supported by TUBITAK (117F070).

zero-divisors) of R. A non-zero-divisor element is also known as a *regular element*, see [2]. Thus, for a commutative ring R, we will write $\mathrm{Z}_l(R) = \mathrm{Z}_r(R) = \mathrm{Z}(R)$ and $\mathrm{Z}_l^*(R) = \mathrm{Z}_r^*(R) = \mathrm{reg}(R)$, where $\mathrm{reg}(R)$ is the set of regular elements (i.e., non-zero-divisors) of R. We write $\mathbb{M}_n(R)$ and $\mathbb{T}_n(R)$ for the $n \times n$ matrix ring and the $n \times n$ upper triangular matrix ring over R, respectively. Also, let A be a matrix, we write $\mathrm{tr}(A)$ to denote the trace of A, and $\det(A)$ for the determinant of A. Moreover, $\mathbb{I}_n$ denotes the $n \times n$ identity matrix.

In [6], Diesl defined a ring element $a \in R$ to be *nil-clean* if it can be written in the form $t + e$ where $t \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$. If every $a \in R$ is nil-clean, R is said to be a nil-clean ring. In this paper, we introduce a class of rings which is a generalization of nil-clean rings. These rings, which will be called *left zero-clean* rings, are defined as rings R in which for every $a \in R$ there exist a left zero-divisor z and an idempotent e such that $a = z + e$. Examples of such rings include nil-clean rings and O-rings. We recall that Cohn [5] introduced the term *O-ring* for commutative rings with 1, in which every element different from 1 is a zero-divisor. Let us say a few words. Examples of O-rings are Boolean rings. In fact, for a commutative ring R, $R = \mathrm{Idem}(R) \bigcup \mathrm{Z}(R)$ if and only if R is an O-ring. In [5], Cohn showed that there exist O-rings which are not Boolean. It was conjectured by Anderson and Badawi that $R = \mathrm{Idem}(R) \bigcup \mathrm{Z}(R)$ if and only if R is a Boolean ring, see [1, p. 1022]. Indeed, Cohn's example indicates that this conjecture is false.

This article consists of 5 sections. Section 1 is the introduction. In Section 2, we investigate some fundamental properties of this class. We show that an abelian ring R is left zero-clean if and only if $a - 1 \in \mathrm{Z}_l(R)$ for each $a \in \mathrm{Z}_l^*(R)$. In this section, we show that a commutative ring R is a zero-clean ring if and only if $\mathrm{reg}(R) \subseteq 1 - \mathrm{Z}(R)$. In Section 3, the behavior of zero-clean rings under some classical ring constructions is studied. We determine when a skew polynomial ring is a left zero-clean ring. Also, it is shown that a ring R is left zero-clean if and only if $\mathbb{T}_n(R)$ is left zero-clean. The aim of Section 4 is to give some results of matrix rings over commutative zero-clean rings. It is shown that a commutative ring R is zero-clean if and only if $\mathbb{M}_n(R)$ is left zero-clean. We also determine when a 2×2 matrix A over a field is left zero-clean. In the last section, we define and give some characterizations of uniquely left zero-clean rings. Let us recall that a commutative ring R is *weakly présimplifiable* if and only if $\mathrm{Z}(R) \subseteq 1 - \mathrm{reg}(R)$, see [2]. It is shown that a commutative ring R is uniquely zero-clean if and only if $\mathrm{Z}(R) = 1 - \mathrm{reg}(R)$.

2. Zero-clean rings

We begin with a formal definition of the central concept of the article.

Definition 2.1. An element a of a ring R is called *left zero-clean* if it can be expressed as the sum of a left zero-divisor and an idempotent in R. Any equation of the form $a = z + e$ will be called a *left zero-clean decomposition* of $a \in R$, where $z \in \mathrm{Z}_l(R)$ and $e \in \mathrm{Idem}(R)$. A ring R is called *left zero-clean* if every element of R is left zero-clean. Right zero-clean rings are defined similarly. A ring which is both right and left zero-clean is called *zero-clean*.

Remark 2.2. It is clear that a left zero-divisor element and 1 are trivially left zero-clean.

The following example shows that there exist left zero-clean rings which are not right zero-clean.

Example 2.3. Let R be a $\mathbb{Z}_2$-algebra generated by x_i, $i \in \mathbb{N}$, with the relations $x_i x_j = 0$, for all $i < j$. It is easy to verify that R is a left zero-clean ring. On the other hand, x_1 has not a right zero-clean decomposition, and hence R is not right zero-clean.

The following example shows that there exists a left zero-clean ring which is neither an O-ring nor nil-clean.

Example 2.4. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \cdots$. It is straightforward to check that R is a zero-clean ring which is neither an O-ring nor a nil-clean ring.

The following example shows that the class of left zero-clean rings is not hereditary on subrings.

Example 2.5. Let R be a $\mathbb{Z}_2$-algebra generated by x_i, $i \in \mathbb{Z}$, with the relations $x_i x_j = 0$, for all $i < j$. Now, let S be the subring of R generated by x_k where $k \leq -1$. One can easily see S is not a left zero-clean ring.

A ring is called *abelian* if all its idempotents are central. Examples of abelian rings are reduced rings (e.g., strongly regular rings), one-sided duo rings (e.g., commutative rings), and of course, all rings with only trivial idempotents $\{0, 1\}$.

Theorem 2.6. *Let R be an abelian ring. The following statements are equivalent:*

(1) *R is a left zero-clean ring.*

(2) *If $a \in \mathrm{Z}_l^*(R)$, then $a - 1 \in \mathrm{Z}_l(R)$.*

Proof. (1) $\Rightarrow$ (2) Assume that R is a left zero-clean ring and $a \in \mathrm{Z}_l^*(R)$. Therefore, a has a left zero-clean decomposition, as $a = z + e$, where $z \in \mathrm{Z}_l(R)$ and $0 \neq e \in \mathrm{Idem}(R)$. If $e = 1$, then $a - 1 = z \in \mathrm{Z}_l(R)$. Now, suppose that $e \neq 1$ and consider $0 \neq z' \in \mathrm{Ann}_r(z)$. We have $a(z'(e-1)) = (z+e)(z'(e-1)) = 0$. Since a is a non-left zero-divisor, $z'(e-1) = 0$. Hence $z'e = z' \neq 0$, and we infer that $a - 1 = z + e - 1$. Now, we have $(a-1)(z'e) = (z+e-1)(z'e) = 0$, which implies that $a - 1$ is a left zero-divisor.

(2) $\Rightarrow$ (1) If $a \in \mathrm{Z}_l^*(R)$, then $(a-1)+1$ is a left zero-clean decomposition of a. $\square$

Corollary 2.7. *If R is an abelian left zero-clean ring, then $\mathrm{J}(R) \subseteq \mathrm{Z}_l(R)$.*

Proof. Suppose that $a \in \mathrm{J}(R) \setminus \mathrm{Z}_l(R)$. Then $1 - a$ is a unit which has to be a left-zero divisor by Theorem 2.6, a contradiction. $\square$

The following is an easy consequence of Theorem 2.6.

Corollary 2.8. *The only zero-clean domain is $\mathbb{Z}_2$.*

Recall that a ring R is *local* if the sum of any two non-units is a non-unit, or equivalently, if the ring has a unique maximal left ideal.

Proposition 2.9. *Let R be a left zero-clean ring. Then $\mathrm{Z}_l(R) \subseteq \mathrm{J}(R)$ if and only if R is local.*

Proof. Suppose that $\mathrm{Z}_l(R) \subseteq \mathrm{J}(R)$. Since $\mathrm{J}(R)$ contains no non-trivial idempotent, R has only trivial idempotents. Hence R is an abelian ring. Corollary 2.7 shows that $\mathrm{J}(R) \subseteq \mathrm{Z}_l(R)$. Thus, $\mathrm{Z}_l(R) = \mathrm{J}(R)$ and this means that $\mathrm{Z}_l(R)$ is an ideal. Now, we claim that $R \setminus \mathrm{J}(R) = \mathrm{U}(R)$. Suppose that $r \notin \mathrm{J}(R)$. By hypothesis, we infer that r is a non-left zero-divisor in R. Thus, there exists $j \in \mathrm{Z}_l(R) = \mathrm{J}(R)$ where $r = j + 1$. This implies that r is a unit. Thus, we conclude that every non-left zero-divisor is a unit. Note that $\mathrm{Z}_l(R)$ is an ideal, and so the sum of any two non-units in R is a non-unit. This means that R is local. The converse is clear. $\square$

Remark 2.10. It is clear that $\mathbb{Z}_6$ is a zero-clean ring, but $\mathbb{Z}_6/3\mathbb{Z}_6$ is not a zero-clean ring. Hence a homomorphic image of a left zero-clean ring need not be left zero-clean. Also, a homomorphic image of a non-left zero-clean ring may be left zero-clean. For example, $\mathbb{Z}$ is not a zero-clean ring, but $\mathbb{Z}_4 \cong \mathbb{Z}/4\mathbb{Z}$ is a zero-clean ring.

This next result needs no proof.

Lemma 2.11. *Let $\{R_i\}_{i\in I}$ be a family of rings. Then* $\mathrm{Z}_l^*(\prod_I R_i)=\prod_I \mathrm{Z}_l^*(R_i)$ *and* $\mathrm{Z}_r^*(\prod_I R_i) = \prod_I \mathrm{Z}_r^*(R_i)$. *Also,* $\mathrm{Idem}(\prod_I R_i) = \prod_I \mathrm{Idem}(R_i)$.

The following seems interesting.

Proposition 2.12. *A direct product of rings is left zero-clean if and only if at least one factor is left zero-clean.*

Proof. Let $\{R_i\}_{i\in I}$ be a family of rings and $R = \prod_I R_i$. First assume that, for each $i \in I$, R_i is not a left zero-clean ring. Thus, for each $i \in I$, there exists $a_i \in R_i$ which is not a left zero-clean element. Take $(a_i)_{i\in I} \in R$. We claim that $(a_i)_{i\in I}$ is not a left zero-clean element of R. Suppose that $(a_i)_{i\in I} = (z_i)_{i\in I} + (e_i)_{i\in I}$, where $(z_i)_{i\in I} \in \mathrm{Z}_l(R)$ and $(e_i)_{i\in I} \in \mathrm{Idem}(R)$. Lemma 2.11 shows that there exists $i_k \in I$ such that $z_{i_k} \in \mathrm{Z}_l(R_{i_k})$, and also, for each $i \in I$, $e_i \in \mathrm{Idem}(R_i)$. Therefore, we infer that $a_{i_k} = z_{i_k} + e_{i_k}$, where $z_{i_k} \in \mathrm{Z}_l(R_{i_k})$ and $e_{i_k} \in \mathrm{Idem}(R_{i_k})$, is a left zero-clean decomposition for $a_{i_k} \in R_{i_k}$. This is a contradiction.

Conversely, assume that $R = \prod_I R_i$ is not a zero-clean ring. Thus, there exists $(a_i)_{i\in I} \in R$ which is not a left zero-clean element. Suppose that there exists $i_k \in I$ such that R_{i_k} is a left zero-clean ring and we seek a contradiction. This implies that for $a_{i_k} \in R_{i_k}$, there exist $z_{i_k} \in \mathrm{Z}_l(R_{i_k})$ and $e_{i_k} \in \mathrm{Idem}(R_{i_k})$ such that $a_{i_k} = z_{i_k} + e_{i_k}$. Define $(p_i)_{i\in I}$ and $(q_i)_{i\in I}$ as follows:

$$p_{i_k} := \begin{cases} z_{i_k} & \text{if } i = i_k, \\ a_{ij} & \text{otherwise;} \end{cases} \qquad q_{i_k} := \begin{cases} e_{i_k} & \text{if } i = i_k, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.11 shows that $(p_i)_{i\in I} \in \mathrm{Z}_l(R)$, $(q_i)_{i\in I} \in \mathrm{Idem}(R)$ and $(a_i)_{i\in I} = (p_i)_{i\in I} + (q_i)_{i\in I}$. Hence $(a_i)_{i\in I} \in R$ is a left zero-clean element, which is a contradiction. □

Lemma 2.13. *Let R be a left zero-clean ring. Then 2 is a left zero-divisor.*

Proof. Suppose that 2 is a non-left zero-divisor. Then there exist $z \in \mathrm{Z}_l(R)$ and $0, 1 \neq e \in \mathrm{Idem}(R)$ such that $2 = z + e$. Therefore $1 - e = z - 1$, and so $z - 1$ is an idempotent. Thus $(z - 1)^2 = (z - 1)$, which implies $(3 - z)z = 2$, a contradiction. □

Corollary 2.14. *$\mathbb{Z}_n$ is a zero-clean ring if and only if $2|n$.*

Proof. It follows from Lemma 2.13 and Proposition 2.12. □

3. Zero-clean property under algebraic constructions

Let R be a ring, and σ be a ring endomorphism of R. Let $R[x;\sigma]$ denote the skew polynomial ring consisting of the polynomials in x with coefficients in R written on the left, with multiplication defined by $xr = \sigma(r)x$ for all $r \in R$. $R[[x;\sigma]]$ denotes the skew formal power series ring. It is clear that if we put $\sigma = 1_R$, then we have $R[x] = R[x;\sigma]$ and $R[[x]] = R[[x;\sigma]]$. Next, we will characterize when a skew polynomial ring is left zero-clean.

Theorem 3.1. *Let R be a ring, and σ be a ring endomorphism of R.*

(1) *If σ is a ring automorphism of R, then $R[x;\sigma]$, $R[[x;\sigma]]$ are never left zero-clean rings.*

(2) *If R is a domain, then $R[x;\sigma]$ is never a left zero-clean ring.*

(3) *If R is not a domain, then the following statements are equivalent:*

  (a) *$R[x;\sigma]$ is a left zero-clean ring.*

  (b) (i) *R is left zero-clean.*

    (ii) *σ is not injective.*

    (iii) *Every $r \in R$ has a left zero-clean decomposition $r = z + e$ where $z \in \mathrm{Z}_l(R)$, $e \in \mathrm{Idem}(R)$, and $\mathrm{Ann}_r(z) \bigcap \mathrm{Ker}(\sigma) \neq 0$.*

Proof. (1) Suppose that $x = z + e$ where $z \in \mathrm{Z}_l(R[x;\sigma])$ and $e \in \mathrm{Idem}(R[x;\sigma]) = \mathrm{Idem}(R)$. This implies that $-e + x = z$. Since σ is a ring automorphism, $-e + x$ is not a left zero-divisor, a contradiction. The proof is similar for $R[[x;\sigma]]$.

(2) It is clear.

(3) $(a) \Rightarrow (b)$ It is clear that R is a left zero-clean ring, since $\mathrm{Idem}(R[x;\sigma]) = \mathrm{Idem}(R)$. If x is a non-left zero divisor, then there exist $z \in \mathrm{Z}_l(R[x;\sigma])$ and $e \in \mathrm{Idem}(R[x;\sigma]) = \mathrm{Idem}(R)$ such that $x - z = e \in \mathrm{Idem}(R)$, which is a contradiction. Hence x is a left zero-divisor and this means that σ is not injective. Now suppose that $r \in R$, and consider the element $r + x$ in $R[x;\sigma]$. Therefore, there exist $z \in \mathrm{Z}_l(R[x;\sigma])$ and $e \in \mathrm{Idem}(R)$ such that $r + x = z + e$. Hence $z = (r - e) + x$. Since $z = (r - e) + x$ is a left-zero-divisor in $R[x;\sigma]$, there exists $b \in \mathrm{Ann}_r(r - e) \bigcap \mathrm{Ker}(\sigma)$. This means that r has a left zero-clean decomposition $r = (r - e) + e$ such that $\mathrm{Ann}_r(r - e) \bigcap \mathrm{Ker}(\sigma) \neq 0$.

$(b) \Rightarrow (a)$ Suppose that $a_0 + a_1x + a_2x^2 + ... + a_nx^n$ is an element of $R[x;\sigma]$. By our assumption, a_0 has a left zero-clean decomposition $a_0 = z + e$ where $z \in \mathrm{Z}\,1(R)$, $e \in \mathrm{Idem}(R)$, and $\mathrm{Ann}_r(z) \bigcap \mathrm{Ker}(\sigma) \neq 0$. Hence $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ has the left zero-clean decomposition $(z + a_1x + a_2x^2 + \cdots + a_nx^n) + e$. Thus $R[x;\sigma]$ is a left zero-clean ring. $\square$

We now turn our attention to formal triangular matrix rings. Before stating the next result, we recall the following lemma that is suitable for our purpose.

Lemma 3.2 ([7, Theorem 3.3])**.** *Let A, B be rings, and $M = {}_AM_B$ be a bimodule. Suppose $m \neq 0$ and $D = \begin{bmatrix} a & m \\ 0 & b \end{bmatrix}$ is an element of the formal triangular matrix $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$. Then the following statements are equivalent:*

(1) *D is a left zero-divisor.*

(2) *At least one of the following occurs:*

 (a) *$a \in \mathrm{Z}_l(A)$.*

 (b) *$a \in \mathrm{Z}_l^*(A)$, $b \in \mathrm{Z}_l(B)$, and there exists $0 \neq b'' \in \mathrm{Ann}_r(b)$ such that $mb'' = 0$.*

 (c) *$a \in \mathrm{Z}_l^*(A)$, $b \in \mathrm{Z}_l^*(B)$, and there exists $0 \neq m'' \in M$ such that $am'' = 0$.*

Theorem 3.3. *Let A, B be rings, $M = {}_AM_B$ be a bimodule, and M be a torsion-free A-module. The formal triangular matrix ring $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ is left zero-clean if and only if either A or B is left zero-clean.*

Proof. First assume that A is a left zero-clean ring, and $C = \begin{bmatrix} a & m \\ 0 & b \end{bmatrix}$ is an arbitrary element of $\mathrm{Z}_l^*(R)$. By Lemma 3.2, we infer that $a \in \mathrm{Z}_l^*(A)$, $b \in \mathrm{Z}_l^*(B)$, and $am' \neq 0$ for all $0 \neq m' \in M$. Since A is a left zero-clean ring, there exist $z \in \mathrm{Z}_l(A)$ and $e \in \mathrm{Idem}(A)$ such that $a = z + e$. Take $D = \begin{bmatrix} z & m \\ 0 & b \end{bmatrix}$ and $E = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$. It is easy to see that $D \in \mathrm{Z}_l(R)$, $E \in \mathrm{Idem}(R)$ and $C = D + E$. The proof for B is similar.

Conversely, assume that R is a left zero-clean ring, and A,B are not left zero-clean rings simultaneously. There exist $a \in A$ and $b \in B$ such that they are not left zero-clean elements. Now, consider the element $C = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ in R. Since R is a zero-clean ring, there exist $D = \begin{bmatrix} c & m \\ 0 & d \end{bmatrix} \in \mathrm{Z}_l(R)$ and $E = \begin{bmatrix} f & -m \\ 0 & g \end{bmatrix} \in \mathrm{Idem}(R)$ such that $C = D + E$. Since E is an idempotent, we have $f \in \mathrm{Idem}(A)$ and $g \in \mathrm{Idem}(B)$. Also, since D is a left zero-divisor element, we infer that either $c \in \mathrm{Z}_l(A)$ or $d \in \mathrm{Z}_l(B)$ by Theorem 3.2 (note that M is a torsion-free A-module). Thus, we conclude that either a or b is a left zero-clean element, a contradiction. $\square$

Using Theorem 3.3, an inductive argument gives immediately the following observation.

Corollary 3.4. *Let R be a ring and $n \geq 1$. Then R is left zero-clean if and only if $\mathbb{T}_n(R)$ is left zero-clean.*

4. Matrix rings over commutative rings

In this section, we will focus our attention on matrix rings. We begin with the following theorem.

Theorem 4.1. *Let R be a commutative ring. If $\mathbb{M}_n(R)$ is left zero-clean, then R is zero-clean.*

Proof. Suppose that $\mathbb{M}_n(R)$ is left zero-clean. Consider an element $a \in R$. By hypothesis, there is an idempotent matrix E such that $a\,\mathbb{I}_n - E$ is a left zero-divisor. That means that there is a nonzero vector v in R^n such that $(a\,\mathbb{I}_n - E)v = 0$. In other words, $Ev = av$. Since E is idempotent, this means that $E^2v = Ev$, implying that $a^2v = av$. Since v is not zero, it has at least one nonzero entry; call it "y". This means that $(a^2 - a)y = 0$. But then this means that $a^2 - a$ is a zero-divisor, which means that either a or $a - 1$ is a zero-divisor. Thus, a is left zero-clean. $\square$

Theorem 4.2. *Let R be an abelian ring. If R is left zero-clean, then $\mathbb{M}_n(R)$ is left zero-clean.*

Proof. Let $0 \neq A = [a_{ij}]_{n\times n}$ be an arbitrary element in $\mathbb{M}_n(R)$. Consider two different cases as follows.

Case 1. Suppose that $a_{11} \in \mathrm{Z}_l^*(R)$. Define $B := [b_{ij}]_{n\times n}$ such that

$$b_{ij} := \begin{cases} a_{11} - 1 & \text{if } i = 1 \text{ and } j = 1, \\ 0 & \text{if } i \neq 1 \text{ and } j = 1, \\ a_{ij} & \text{if } 1 \leq i \leq n \text{ and } j \neq 1. \end{cases}$$

Note that $a_{11} - 1 \in \mathrm{Z}_l(R)$ by Theorem 2.6. It is easy to see that $B \in \mathrm{Z}_l(\mathbb{M}_n(R))$ and $A - B \in \mathrm{Idem}(\mathbb{M}_n(R))$. Hence A is a left zero-clean element.

Case 2. Suppose that $a_{11} \in \mathrm{Z}_l(R)$. Define $C := [c_{ij}]_{n\times n}$ such that

$$c_{ij} := \begin{cases} 1 & \text{if } 2 \leq i = j \leq n, \\ a_{ij} & \text{if } i \neq 1 \text{ and } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, C is an idempotent and $A - C$ is a left zero-divisor. Thus A is a left zero-clean element. □

The following combines Theorem 4.1 and the commutative case of Theorem 4.2.

Corollary 4.3. *Let R be a commutative ring and $n \geq 1$. Then R is zero-clean if and only if $\mathbb{M}_n(R)$ is zero-clean.*

Breaz *et al.* [3, Theorem 3] have shown that $\mathbb{M}_n(F)$, where F is a field, is a nil-clean ring if and only if $F \cong \mathbb{Z}_2$. Now we are in a position to give the complete characterization of zero-clean matrix rings over fields. The following is a consequence of Corollaries 2.8 and 4.3.

Corollary 4.4. *Let F be a field. The following statements are equivalent:*

(1) $F \cong \mathbb{Z}_2$.

(2) *For every positive integer n, the matrix ring $\mathbb{M}_n(F)$ is zero-clean.*

(3) *There exists a positive integer n such that the matrix ring $\mathbb{M}_n(F)$ is zero-clean.*

The above result motivates us to ask when a 2×2 matrix over a field $F(\neq \mathbb{Z}_2)$ is a left zero-clean element. Before stating our result, we recall the following lemmas, which are standard facts from linear algebra.

Lemma 4.5 ([4, Theorem 9.1])**.** *Let R be a commutative ring and $A \in \mathbb{M}_n(R)$. Then A is a zero-divisor if and only if $\det(A) \in \mathrm{Z}(R)$.*

Lemma 4.6. *Let F be a field. Then $A \in \mathbb{M}_2(F)$ is a non-trivial idempotent if and only if $\det(A) = 0$ and $\mathrm{tr}(A) = 1$.*

First, we determine when a 2×2 diagonal matrix over a field $F(\neq \mathbb{Z}_2)$ is a left zero-clean element.

Theorem 4.7. *Let $F(\neq \mathbb{Z}_2)$ be a field and $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in \mathbb{M}_2(F)$. Then the following statements are equivalent:*

(1) $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ *is left zero-clean in* $\mathbb{M}_2(F)$.

(2) *Either* $a = d = 0, 1$ *or* $a \neq d$.

Proof. (1) $\Rightarrow$ (2) Suppose that $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ is a non-left zero-divisor. If $a = d = 1$, then $A = I$, and hence A is a left zero-clean element. Now, suppose that $a \neq d$. Let

$$D = \begin{bmatrix} a - (1 - (a-1)d(a-d)^{-1}) & -1 + (a-1)d(a-d)^{-1} \\ -(a-1)d(a-d)^{-1} & d - (a-1)d(a-d)^{-1} \end{bmatrix}.$$

Clearly, $\det(A) = 0$, and hence $D \in \mathrm{Z}_l(\mathbb{M}_2(F))$. Also, let

$$E = \begin{bmatrix} 1 - (a-1)d(a-d)^{-1} & 1 - (a-1)d(a-d)^{-1} \\ (a-1)d(a-d)^{-1} & (a-1)d(a-d)^{-1} \end{bmatrix}.$$

Lemma 4.6 shows that $E \in \mathrm{Idem}(\mathbb{M}_2(F))$. One can easily see that $A = D + E$, and so A is a left zero-clean element.

(2) $\Rightarrow$ (1) Suppose that $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ is left zero-clean. If $a = 0, 1$, then A is a trivial idempotent, and we are done. Otherwise, suppose that $a \neq 0, 1$. Then A is a non-left zero-divisor, and it has a left zero-clean decomposition. Suppose that $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} b & -c \\ -d & e \end{bmatrix} + \begin{bmatrix} f & c \\ d & k \end{bmatrix}$ where $M = \begin{bmatrix} b & -c \\ -d & e \end{bmatrix} \in \mathrm{Z}_l(\mathbb{M}_2(F))$ and $N = \begin{bmatrix} f & c \\ d & k \end{bmatrix} \in \mathrm{Idem}(\mathbb{M}_2(F))$. If N is a trivial idempotent, then either A or $A - I$ is a left zero divisor, a contradiction. Hence N is not a trivial idempotent. Thus $\mathrm{tr}(N) = 1$ and $\det(N) = 0$ by Lemma 4.6. Hence $f = 1 - k$ and $(1-k)k - cd = 0$. Now, we conclude that $M = \begin{bmatrix} a - (1-k) & -c \\ -d & a - k \end{bmatrix}$. Therefore, we have $\det(M) = a^2 - a + (1-k)k - cd = 0$, and hence $\det(M) = a^2 - a = 0$, a contradiction. $\square$

Next, we prove that every 2×2 non-diagonal matrix over a field $F(\neq \mathbb{Z}_2)$ is left zero-clean.

Theorem 4.8. *Let* F *be a field and* $F(\neq \mathbb{Z}_2)$.

(1) *If* $c \neq 0$, *then* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is left zero-clean in* $\mathbb{M}_2(F)$.

(2) *If* $b \neq 0$, *then* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is left zero-clean in* $\mathbb{M}_2(F)$.

Proof. (1) Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a non-left zero-divisor in $\mathbb{M}_2(F)$ and $c \neq 0$. Consider three different cases as follows:

Case 1. If $(a - d + c - b) \neq 0$, then take $E = \begin{bmatrix} 1-f & 1-f \\ f & f \end{bmatrix}$ where $f = ((a-1) + (1-b)c)(a-d+c-b)^{-1}$.

Case 2. If $(a-d+c-b) = 0$ and $(a-d+b-c) \neq 0$, then take $E = \begin{bmatrix} 1-f & f \\ 1-f & f \end{bmatrix}$ where $f = ((a-1) + (1-c)b)(a-d+b-c)^{-1}$.

Case 3. If $(a-d+c-b) = (a-d+b-c) = 0$, then we conclude that $a = d$ and $c = b$. Therefore, take $E = \begin{bmatrix} 0 & 0 \\ f & 1 \end{bmatrix}$ where $f = (c^2 - a^2 + a)c^{-1}$.
In each of the above cases, E is an idempotent by Lemma 4.6. Also, $\det(A–E)=0$, and so $A - E$ is a left zero-divisor in $\mathbb{M}_2(F)$. Hence $A = (A - E) + E$ is a left zero-clean decomposition of A.

(2) The proof is similar to part (1). □

With Theorems 4.7 and 4.8 at our disposal, we are now ready to determine when a 2×2 matrix over a field is left zero-clean.

Corollary 4.9. *Let* $F(\neq \mathbb{Z}_2)$ *be a field and* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_2(F)$. *Then the following statements are equivalent:*

(1) *A is not a left zero-clean element.*

(2) $a = d \neq 1, 0$ *and* $c = b = 0$.

5. Uniquely zero-clean rings

Definition 5.1. An element a in a ring R is called *uniquely left zero-clean* if there is a unique idempotent e such that $a - e$ is a left zero-divisor. We will say that a ring is uniquely left zero-clean if each of its elements is uniquely left zero-clean. Uniquely right zero-clean rings are defined similarly. A ring which is both uniquely right and uniquely left zero-clean is called uniquely zero-clean.

Lemma 5.2. (1) *A uniquely left zero-clean ring has only trivial idempotents.*

(2) *If R is a uniquely left zero-clean ring, then* $\mathrm{J}(R) \subseteq \mathrm{Z}_l(R)$.

PROOF. (1) Let R be a uniquely left zero-clean ring and $e \in \mathrm{Idem}(R)$. Since $1 = (1-e)+e$ and $1 = 0+1$ are two zero-clean decompositions of 1, we conclude that $e = 0$ or $e = 1$. Therefore, a uniquely left zero-clean ring has only trivial idempotents.

(2) It follows from Corollary 2.7. □

Corollary 5.3. *Let R be a uniquely left zero-clean ring. Then*

(1) $z = z+0$ *is the uniquely left zero-clean decomposition of z for each* $z \in \mathrm{Z}_l(R)$.

(2) $r = (r-1)+1$ *is the uniquely left zero-clean decomposition of r for each* $r \in \mathrm{Z}_l^*(R)$.

Corollary 5.4. $\mathbb{Z}_n$ *is uniquely zero-clean if and only if n is a power of* 2.

PROOF. It follows from Corollary 2.14 and Lemma 5.2 (1). □

Proposition 5.5. *Let R be a left zero-clean ring and* $\mathrm{J}(R) = \mathrm{Z}_l(R)$. *Then R is a uniquely left zero-clean ring.*

PROOF. The assumption $\mathrm{J}(R) = \mathrm{Z}_l(R)$ implies that R has only trivial idempotents. Since R is a left zero-clean ring, $r = (r-1)+1$ is the unique left zero-clean decomposition of r for each $r \in \mathrm{Z}_l^*(R)$. Now suppose that $z \in \mathrm{Z}_l(R) = \mathrm{J}(R)$. We claim that $z = z+0$ is the unique left zero-clean decomposition of z. Otherwise, $z = (z-1)+1$ is a left zero-clean decomposition of z where $z-1 \in \mathrm{Z}_l(R) = \mathrm{J}(R)$. Thus $1 = z-(z-1) \in \mathrm{J}(R)$, which is a contradiction. Hence R is a uniquely left zero-clean ring. □

In the preceding proposition, the condition $\mathrm{J}(R) = \mathrm{Z}_l(R)$ is not superfluous as follows.

Example 5.6. It is clear that, for $n \geq 2$, $\mathbb{T}_n(\mathbb{Z}_2)$ is a left zero-clean ring by Corollary 3.4. Note that $\mathrm{J}(\mathbb{T}_n(\mathbb{Z}_2)) \subset \mathrm{Z}_l(\mathbb{T}_n(\mathbb{Z}_2))$, but $\mathbb{T}_n(\mathbb{Z}_2)$ is not a uniquely left zero-clean ring by Lemma 5.2 (1).

Remark 5.7. Following [6], a ring R is called *uniquely nil-clean* if, for any $a \in R$, there exists a unique idempotent $e \in R$ such that $a-e \in R$ is nilpotent. It is clear that every Boolean ring is uniquely nil-clean. Note that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a uniquely nil-clean ring which is not uniquely zero-clean. On the other hand, suppose that R is a $\mathbb{Z}_2$-algebra generated by x_i, $i \in \mathbb{N}$, with the relations $x_i x_j = 0$, for all $i < j$. Then R is a uniquely left zero-clean ring, but it is not uniquely nil-clean.

We get the following characterizations for uniquely left zero-clean rings.

Theorem 5.8. *Let R be a ring. The following statements are equivalent:*

(1) *R is a uniquely left zero-clean ring.*

(2) (a) *If $z \in \mathrm{Z}_l(R)$, then $z-1 \in \mathrm{Z}_l^*(R)$.*

(b) *If $r \in \mathrm{Z}_l^*(R)$, then $r-1 \in \mathrm{Z}_l(R)$.*

(3) $\mathrm{Z}_l(R) = 1 - \mathrm{Z}_l^*(R)$.

Proof. (1) $\Rightarrow$ (2) (a) If $z \in \mathrm{Z}_l(R)$, then $z = (z-1)+1$ is not a left zero-clean decomposition z by Corollary 5.3. This means that $z-1 \in \mathrm{Z}_l^*(R)$.

(b) If $r \in \mathrm{Z}_l^*(R)$, then $r = (r-1)+1$ is the unique left zero-clean decomposition of r by Corollary 5.3. This means that $r - 1 = z \in \mathrm{Z}_l(R)$.

(2) $\Rightarrow$ (1) First, we claim that R has only trivial idempotents. Suppose that $0, 1 \neq e \in \mathrm{Idem}(R)$. Since $e \in \mathrm{Z}_l(R)$, we get $e-1 \in \mathrm{Z}_l^*(R)$, a contradiction. Thus R has only trivial idempotents. Now, suppose that $z \in \mathrm{Z}_l(R)$. Then $z = (z-1)+1$ is not a left zero-clean decomposition, because $z-1 \in \mathrm{Z}_l^*(R)$. This means that $z = z+0$ is the uniquely left zero-clean decomposition of z. It is also easy to see that $r = (r-1)+1$ is the uniquely left zero-clean decomposition of r for each $r \in \mathrm{Z}_l^*(R)$. Hence R is a uniquely left zero-clean ring.

(2) $\Leftrightarrow$ (3) It is clear. □

Following [2], a commutative ring R is called *weakly présimplifiable* if, for $x, y \in R$, $x = xy$ implies $x = 0$ or y is regular (i.e., non-zero-divisor). Anderson and Chun [2, Theorem 6] have shown that a commutative ring R is weakly présimplifiable if and only if $\mathrm{Z}(R) \subseteq 1 - \mathrm{reg}(R)$. It is natural to ask: When does the inclusion $\mathrm{reg}(R) \subseteq 1 - \mathrm{Z}(R)$ hold? With the help of Theorem 2.6, we make the following simplifying observation.

Corollary 5.9. *Let R be a commutative ring. The following statements are equivalent:*

(1) *R is a zero-clean ring.*

(2) $\mathrm{reg}(R) \subseteq 1 - \mathrm{Z}(R)$.

The following is an immediate consequence of Theorem 5.8.

Corollary 5.10. *Let R be a commutative ring. The following statements are equivalent:*

(1) *R is a uniquely zero-clean ring.*

(2) $\mathrm{Z}(R) = 1 - \mathrm{reg}(R)$.

With Corollary 5.10, this proves that every commutative uniquely zero-clean ring is weakly présimplifiable. However, the converse is false, by the following example.

Example 5.11. Clearly, $\mathbb{Z}_2$ is a weakly présimplifiable ring. By [2, Theorem 18(2)], the polynomial ring $\mathbb{Z}_2[x]$ is weakly présimplifiable, while $\mathbb{Z}_2[x]$ is not a uniquely zero-clean ring by Theorem 3.1(1).

ACKNOWLEDGEMENTS. The authors would like to express their thanks to ALEXANDER J. DIESL and SIMION BREAZ for their encouragement and guidance throughout this project, and to the referee for his/her comments and suggestions toward improving the content and style of the paper.

References

[1] D. F. ANDERSON and A. BADAWI, Von Neumann regular and related elements in commutative rings, *Algebra Colloq.* **19** (2012), 1017–1040.

[2] D. D. ANDERSON and S. CHUN, Associate elements in commutative rings, *Rocky Mountain J. Math.* **44** (2014), 717–731.

[3] S. BREAZ, G. CĂLUGĂREANU, P. DANCHEV and T. MICU, Nil-clean matrix rings, *Linear Algebra Appl.* **439** (2013), 3115–3119.

[4] W. C. BROWN, Matrices Over Commutative Rings, *Marcel Dekker*, *New York*, 1993.

[5] P. M. COHN, Rings of zero divisors, *Proc. Amer. Math. Soc.* **9** (1958), 909–914.

[6] A. J. DIESL, Nil clean rings, *J. Algebra* **383** (2013), 197–211.

[7] E. GHASHGHAEI, M. T. KOŞAN, M. NAMDARI and T. YILDIRIM, Rings in which every left zero-divisor is also a right zero-divisor and conversely, *J. Algebra Appl.* **18** (2019), 1950096, 14 pp.

EBRAHIM GHASHGHAEI
DEPARTMENT OF MATHEMATICS
SHAHID CHAMRAN UNIVERSITY OF AHVAZ
AHVAZ
IRAN

E-mail: e.ghashghaei@yahoo.com

MUHAMMET TAMER KOŞAN
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES
GAZI UNIVERSITY
ANKARA
TURKEY

E-mail: mtamerkosan@gazi.edu.tr
tkosan@gmail.com

(Received August 6, 2018; revised May 27, 2019)