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Twisted quadratic moments for Dirichlet L-functions at $s = 2$

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Abstract. Let c, n be given positive integers. Let $q > 2$ be coprime with c. Let X_q be the multiplicative group of order $\phi(q)$ of the Dirichlet characters modulo q. Set

$$M(q,c,n) := \frac{2}{\phi(q)} \sum_{\substack{\chi \in X_q \\ \chi(-1)=(-1)^n}} \chi(c)|L(n,\chi)|^2.$$

The goal of this paper is to explain how one can compute explicit formulas for $M(q,c,n)$ for given small integers n and c. As an example, we give explicit formulas for $M(q,c,2)$ for $c \in \{1,2,3,4,6\}$, and for $M(p,5,2)$ for p a prime integer. As a consequence, we show that a previously published formula for $M(p,3,2)$ is false.

1. Introduction

Let c, n be given positive integers. Let $q > 2$ be coprime with c. Let X_q be the multiplicative group of order $\phi(q)$ of the Dirichlet characters modulo q. Set

$$M(q,c,n) := \frac{2}{\phi(q)} \sum_{\substack{\chi \in X_q \\ \chi(-1)=(-1)^n}} \chi(c)|L(n,\chi)|^2.$$

In [4], we developed a method for obtaining formulas for $M(q,1,n)$. For example, by [4, Theorem 2], we have

$$M(q,1,2) = \frac{\pi^2}{90} \times \left\{ \phi_4(q) + \frac{10}{q^2}\phi_2(q) \right\}, \tag{1}$$

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where

$$\phi_k(q) = \prod_{p|q}\left(1 - \frac{1}{p^k}\right) \qquad (k \in \mathbb{Z}_{\geq 1}).$$

In [5], for a given $c > 1$, we developed a method for obtaining formulas for $M(q,c,1)$. For example, in the cases where the multiplicative group $(\mathbb{Z}/c\mathbb{Z})^*$ is trivial of order 1, i.e. for $c \in \{1,2\}$, by [5, (1) and (3)] and [6, Theorem 2], we have

$$M(q,c,1) = \frac{\pi^2}{6c} \times \left\{\phi_2(q) - \frac{3c\phi_1(q)}{q}\right\}. \tag{2}$$

In the cases where the multiplicative group $(\mathbb{Z}/c\mathbb{Z})^*$ is of order 2, i.e. for $c \in \{3,4,6\}$, by [5, (4) and (5)] and [6, Theorem 4], we have

$$M(q,c,1) = \frac{\pi^2}{6c} \times \left\{\phi_2(q) - \frac{3c\phi_1(q)}{q} - \frac{(c-1)(c-2)\chi_c(q)}{q}\prod_{p|q}\left(1 - \frac{\chi_c(p)}{p}\right)\right\},$$

where χ_c is the non-trivial character on the multiplicative group $(\mathbb{Z}/c\mathbb{Z})^*$, i.e. where $\chi_c(n) = 1$ if $n \equiv 1 \pmod{c}$, and $\chi_c(n) = -1$ if $n \equiv -1 \pmod{c}$. We point out that according to [6, Section 4], as c gets bigger, such explicit formula become very complicated, there is no known closed formula for $M(q,c,2)$ and we gave formulas for $c \in \{1,2,3,4,5,6,8,10\}$. Restricting himself to prime moduli, in [2, Corollary 1.1], H. Liu gives explicit formulas for $M(p,c,2)$ for $c = 1,2,3,4$, without explaining why he did not consider neither non-prime moduli nor values of $c > 4$. Here, focussing on the particular example of $M(q,c,2)$, we explain how to extend Liu's results to non-prime moduli and values of $c > 4$. We will prove the following result and show that the formula for $M(p,3,2,)$ in [2, Corollary 1.1] is false (compare with [6, Theorem 4]):

Theorem 1. *Let $q > 1$ be coprime with $c \in \{1,2,3,4,6\}$. Define*

c	u_c	v_c
1	10	0
2	70	0
3	210	80
4	490	360
6	1830	2240

Let X_q^+ be the group of order $\phi(q)/2$ of the even Dirichlet characters modulo q. For $c \in \{3,4,6\}$, let χ_c be the only non-trivial character on the multiplicative group $(\mathbb{Z}/c\mathbb{Z})^* = \{\pm 1\}$. Then

$$\begin{aligned} M(q,c,2) &:= \frac{2}{\phi(q)}\sum_{\chi\in X_q^+}\chi(c)|L(2,\chi)|^2 \\ &= \frac{\pi^4}{90c^2}\times\left\{\phi_4(q)+\frac{u_c\phi_2(q)}{q^2}+\frac{v_c\chi_c(q)}{q^3}\prod_{p\mid q}\left(1-\frac{\chi_c(p)}{p}\right)\right\}. \end{aligned}$$

As a Corollary, we recover the formulas in [2, Corollary 1.1], correct the formula for $M(p,3,2)$ and give a new formula, the one for $M(p,6,2)$:

Corollary 2. *Let $p>3$ be a prime integer. We have*

$$M(p,c,2)=\frac{\pi^4}{90c^2}\times\frac{(p^2-1)(p^2+u_c+1)}{p^4}\qquad (c\in\{1,2\}),$$

and

$$M(p,c,2)=\frac{\pi^4}{90c^2}\times\frac{p^4+u_cp^2+v_c\chi_c(p)p-(u_c+v_c+1)}{p^4}\qquad (c\in\{3,4,6\}).$$

We also have the explicit formulas

$$M(p,3,2)=\frac{\pi^4}{810}\times\frac{p^4+210p^2+80p\chi_3(p)-291}{p^4},$$

$$M(p,4,2)=\frac{\pi^4}{1440}\times\frac{p^4+490p^2+360\chi_4(p)-851}{p^4}$$

and

$$M(p,6,2)=\frac{\pi^4}{3240}\times\frac{p^4+1830p^2+2240p\chi_6(p)-4071}{p^4}.$$

Let us explain how we performed some numerical computation to check our formulas for $M(p,c,2)$, and let us justify that the formula for $M(p,3,2)$ in [2, Corollary 1.1] is not correct. Let $p>3$ be a prime integer. We refer to [7, Chapter 4] for the justification of what follows. For χ_0 the trivial character modulo p, we have $L(s,\chi_0)=(1-p^{-s})\zeta(s)$ and $L(2,\chi_0)=\frac{p^2-1}{p^2}\frac{\pi^2}{6}$. For $\chi_0\neq\chi\in X_p^+$, we have

$$L(2,\chi)=-W_\chi\frac{2\pi^2}{p^{3/2}}L(-1,\bar{\chi})=W_\chi\frac{\pi^2}{p^{3/2}}B_{2,\bar{\chi}}=W_\chi\frac{\pi^2}{p^{5/2}}\sum_{a=1}^{p-1}\bar{\chi}(a)a^2,$$

where the root number W_χ is a complex number of absolute value equal to 1.

Hence,

$$M(p,c,2)=\frac{2\pi^4}{p-1}\times\left\{\frac{(p^2-1)^2}{36p^4}+\sum_{\chi_0\neq\chi\in X_p^+}\chi(c)\frac{\left|\sum_{a=1}^{p-1}\chi(a)a^2\right|^2}{p^5}\right\}.$$

Taking $p=5$, for which the only $\chi_0\neq\chi\in X_5^+$ is the Legendre symbol $\left(\frac{\bullet}{5}\right)$, for which $\sum_{a=1}^{4}\chi(a)a^2=1-2^2-3^2+4^2=4$, we obtain

$$M(5,c,2)=\frac{8\pi^4}{5^5}\left(5+\left(\frac{c}{5}\right)\right).$$

For $c\in\{1,2,3,4\}$, this formula yields the same value as the ones in Corollary 2. In particular, they give $M(5,3,2)=32\pi^4/5^5$, whereas the formula in [2, Corollary 1.1] gives the wrong estimation $M(5,3,2)=238\pi^4/(9\cdot 5^5)$.

2. Proof of Theorem 1

Lemma 3. *Assume that* $\gcd(c,d)=1$. *Set*

$$S_{\cot^k}(d)=\sum_{a=1}^{d-1}\cot^k\left(\frac{\pi a}{d}\right),$$

with the convention that $S_{\cot^k}(1)=0$, *and*

$$S_2(c,d):=\sum_{a=1}^{d-1}\cot^2\left(\frac{\pi a}{d}\right)\cot^2\left(\frac{\pi ac}{d}\right),\tag{3}$$

with the convention that $S_2(c,1)=0$ *for* $c\geq 1$. *Then*

$$M(q,c,2)=\frac{\pi^4}{2q^4}\sum_{d\mid q}\mu(q/d)\Big((d-1)+2S_{\cot^2}(d)+S_2(c,d)\Big).\tag{4}$$

In particular,

$$M(p,c,2)=\frac{\pi^4}{2q^4}\Big((p-1)+2S_{\cot^2}(p)+S_2(c,p)\Big).\tag{5}$$

Proof. From [4, Proposition 3], we have

$$L(2,\chi) = -\frac{\pi^2}{2q^2}\sum_{a=1}^{q-1}\chi(a)\cot'(\pi a/q) \qquad (\chi\in X_q^+),$$

and as in the proof of [4, Proposition 3], we obtain

$$M(q,c,2) = \frac{\pi^4}{2q^4}\sum_{\substack{a=1\\ \gcd(a,q)=1}}^{q-1}\cot'(\pi a/q)\cot'(\pi ac/q),$$

with $\cot' = -1-\cot^2$ the derivative of cot. Using $\sum_{d|a \text{ and } q}\mu(d) = 0$ or 1 for either $\gcd(a,q)>1$ or $\gcd(a,q)=1$, respectively, we obtain

$$M(q,c,2) = \frac{\pi^4}{2q^4}\sum_{\substack{d>1\\ d|q}}\mu(q/d)\sum_{a=1}^{d-1}\left(1+\cot^2\left(\frac{\pi a}{d}\right)\right)\left(1+\cot^2\left(\frac{\pi ac}{d}\right)\right), \qquad (6)$$

and the desired result, thanks to the conventions, $S_{\cot^2}(1) = S_2(c,1) = 0$. □

Lemma 4. *Let $c>1$ be an integer. It holds that*

$$(\cot^2 x)(\cot^2(cx)) = \frac{1}{c^2}\cot^4 x - \frac{2(c^2-1)}{3c^2}\cot^2 x + \frac{c^4-1}{15c^2} - \frac{2}{c^2}\sum_{k=1}^{c-1}\frac{\frac{2\cot^3(k\pi/c)+\cot(k\pi/c)}{\sin^2(k\pi/c)}}{\cot(k\pi/c)-\cot x} + \frac{1}{c^2}\sum_{k=1}^{c-1}\frac{\frac{\cot^2(k\pi/c)}{\sin^4(k\pi/c)}}{(\cot(k\pi/c)-\cot x)^2}.$$

Proof. Adapt the proof of [5, Lemma 4]. □

Lemma 5. *Let $d>1$ be an integer and $\theta\in(0,\pi)\setminus\in\{\pi a/d;\ 1\le a\le d-1\}$. Set*

$$T_m(\theta,d) := \frac{1}{d}\sum_{a=1}^{d-1}\frac{1}{(\cot\theta-\cot(\pi a/d))^m}.$$

Then

$$T_1(\theta,d) = (\sin^2\theta)\Big(\cot\theta-\cot(d\theta)\Big),$$

$$T_2(\theta,d) = (\sin^4\theta)\Big(d\cot^2(d\theta) - 2(\cot\theta)(\cot(d\theta)) + \cot^2\theta + (d-1)\Big)$$

and

$$\begin{aligned}&-2(2\cot^3\theta+\cot\theta)\frac{T_1(\theta,d)}{\sin^2\theta} + (\cot^2\theta)\frac{T_2(\theta,d)}{\sin^4\theta)}\\ &\quad = (d-3)\cot^2\theta - 3\cot^4\theta + 2(\cot^3\theta+\cot\theta)\cot(d\theta) + d(\cot^2\theta)(\cot^2(d\theta)).\end{aligned}$$

Proof. For $T_1(\theta, d)$, see [5, Lemma 5] and take $\alpha = \cot\theta$. For $T_2(\theta, d)$, notice that $T_2(\theta, d) = (\sin^2\theta)\frac{dT_1(\theta,d)}{d\theta}$. $\square$

Now, noticing that $\sum_{d|q}\mu(q/d)d^k = q^k\phi_k(q)$ and $\phi_0(q) = 0$ for $q > 1$, Theorem 1 follows from (4) and the following Proposition:

Proposition 6. *Assume that* $\gcd(c, d) = 1$. *Set*

$$F_c(d) := \sum_{k=1}^{c-1}(\cot^3(k\pi/c) + \cot(k\pi/c))\cot(kd\pi/c),$$

with the convention that $F_1(d) = 0$ *for* $d \geq 1$. *Then*

$$(d-1) + 2S_{\cot^2}(d) + S_2(c, d) = \frac{Q_c(d)}{45c^2} + \frac{d^2}{c^2}S_2(d, c) + \frac{2d}{c^2}F_c(d),$$

where $Q_c(d) := d^4 + 5(7c^2 - 9c + 4)d^2 - (3c^4 + 5c^2 + 3)$. *Notice that* $S_2(d, c)$ *and* $F_c(d)$ *depend only on* d *modulo* c. *In particular, we have*

c	$S_2(d,c)$	$F_c(d)$	$45c^2\big((d-1) + 2S_{\cot^2}(d) + S_2(c,d)\big)$
1	0	0	$d^4 + 10d^2 - 11$
2	0	0	$d^4 + 70d^2 - 71$
3	$\frac{2}{9}$	$\frac{8}{9}\chi_3(d)$	$d^4 + 210d^2 + 80d\chi_3(d) - 291$
4	2	$4\chi_4(d)$	$d^4 + 490d^2 + 360d\chi_4(d) - 851$
6	$\frac{164}{9}$	$\frac{224}{9}\chi_6(d)$	$d^4 + 1830d^2 + 2240d\chi_6(d) - 4071$

Proof. Using (3) and Lemma 4, we obtain

$$\begin{aligned} c^2S_2(c,d) = S_{\cot^4}(d) &- \frac{2(c^2-1)}{3}S_{\cot^2}(d) + \frac{c^4-1}{15}(d-1) \\ &- 2d\sum_{k=1}^{c-1}(2\cot^3(k\pi/c) + \cot(k\pi/c))\frac{T_1(k\pi/c, d)}{\sin^2(k\pi/c)} \\ &+ d\sum_{k=1}^{c-1}(\cot^2(k\pi/c))\frac{T_2(k\pi/c, d)}{\sin^4(k\pi/c)}. \end{aligned}$$

Using the last assertion in Lemma 5 for $\theta = k\pi/c$, we obtain

$$\begin{aligned} c^2S_2(c,d) = S_{\cot^4}(d) &- \frac{2(c^2-1)}{3}S_{\cot^2}(d) + \frac{c^4-1}{15}(d-1) \\ &+ d\Big((d-3)S_{\cot^2}(c) - 3S_{\cot^4}(c)\Big) \end{aligned}$$

$$+ 2d\sum_{k=1}^{c-1}(\cot^3(k\pi/c)+\cot(k\pi/c))\cot(kd\pi/c)$$
$$+ d^2\sum_{k=1}^{c-1}(\cot^2(k\pi/c))(\cot^2(kd\pi/c)).$$

Now, for $n>1$, $\cot(\pi k/n)$, $1\le k\le n-1$, are the roots of

$$\frac{(X+i)^n-(X-i)^n}{2in}=X^{n-1}-\frac{(n-1)(n-2)}{6}X^{n-3}+\cdots\in\mathbb{Q}[X].$$

Hence, $S_{\cot^2}(n)=\frac{(n-1)(n-2)}{3}$ and $S_{\cot^4}(n)=\frac{(n-1)(n-2)(n^2+3n-13)}{45}$. The desired result follows. $\square$

Corollary 7. *For $p\neq 2,5$ a prime integer, we have*

$$M(q,5,2)=\frac{\pi^4}{2250p^4}\times\begin{cases}p^4+994p^2+1008p-2003 & \text{if } p\equiv 1 \pmod 5,\\ p^4+706p^2+144p-2003 & \text{if } p\equiv 2 \pmod 5,\\ p^4+706p^2-144p-2003 & \text{if } p\equiv 3 \pmod 5,\\ p^4+994p^2-1008p-2003 & \text{if } p\equiv 4 \pmod 5.\end{cases}$$

Proof. Follows from (5), Proposition 6 and the computation of $S_2(r,5)$ and $F_5(r)$ for the value r of d modulo 5 ranging in $\{1,2,3,4\}$. $\square$

Remark 8. As in [8], for q a positive integer, set

$$d(q;a_1,\dots,a_n)=(-1)^{n/2}\sum_{k=1}^{q-1}\cot\left(\frac{ka_1}{q}\right)\cdots\cot\left(\frac{ka_n}{q}\right).$$

In [8, (47)], Zagier gave a reciprocity law for these generalized Dedekind sums under the assumption that q and the a_k's be pairwise coprime. For $S_2(c,q)=d(q;1,1,c,c)$ this assumption is not fulfilled. Hence, Proposition 6, where $F_c(x):=d(c;1,1,1,x)+d(c;1,x)$, which can be viewed as a reciprocity law for the sums $S_2(c,d)$, does not follow from Zagier's reciprocity law.

3. Conclusion

[5, Lemma 4], which deals with $(\cot x)(\cot(cx))$, and the present Corollary 4, which deals with $(\cot^2 x)(\cot^2(cx))$, could easily be generalized to evaluate

$(\cot^m x)(\cot^n(cx))$ for small values of m and n. Lemma 5 can be very easily generalized to evaluate $T_m(\theta,d)$ for small values of m. Lemma 3 can be generalized to express $M(q,c,n)$ in terms of

$$S_{k,l}(c,d) := \sum_{a=1}^{d-1} \cot^k\left(\frac{\pi a}{d}\right)\cot^l\left(\frac{\pi ac}{d}\right),$$

see [4]. Hence, following the method developed here one could obtain explicit formulas for $M(q,c,n)$ and $M(p,c,n)$ for other small values of n and c. As explained here and in [6], the formulas for $M(q,c,n)$ would get more complicated as $\phi(c)$ increases. We would get $\phi(c)$ twisted quadratic moments formulas, one for each value of d modulo c (notice that $\phi(c)=1$ if and only if $c=1,2$, and $\phi(c)=2$ if and only if $c=3,4,6$). Only asymptotic estimates as in [1] and [3] could yield explicit formulas for all c's.

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