

Isosceles orthogonally additive mappings and inner product spaces

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Dedicated to Professor János Aczél on his 70th birthday

Abstract. In a normed vector space $(X, \|\cdot\|)$, consider James' *isosceles orthogonality*, i.e., $x \perp y \iff \|x + y\| = \|x - y\|$. It is known that any odd, orthogonally additive mapping from X into an Abelian group is unconditionally additive whenever $\dim X \geq 3$. In this paper a complementary result is presented: the existence of a nontrivial even orthogonally additive mapping characterizes inner product spaces for $\dim X \geq 2$. The proof uses some interesting connectivity theorems.

1. Introduction

Mappings from a vector space into an Abelian group that are additive on orthogonal pairs of vectors have been studied for long by several authors. For an early contribution, see [3]. Besides the usual inner product orthogonality, some other relation were considered such as orthogonalities defined by a bilinear/sesquilinear form (see e.g. [13, 10]), by a norm of the space (see e.g. [7, 8]) or just by some abstract properties (see e.g. [1, 4]).

Based on some weak assumptions and the homogeneity, the most crucial property of such a relation, a general abstract theory was developed in [5], resulting in an additive/quadratic representation of orthogonally additive mappings: odd solutions are additive while the even ones are quadratic. Moreover, the existence of a nontrivial, even solution is characteristic for (generalized) inner product orthogonalities (see e.g. [9, 11]).

In normed vector spaces, however, there are some natural generalizations of the usual inner product orthogonality, homogeneity of which characterizes inner product spaces. Such relations are, for instance, James'

isosceles orthogonality or the Pythagorean orthogonality (see [2]). Due to the lack of homogeneity, these relations are not covered by the general theory mentioned above.

Using connectivity theorems for intersection of spheres in normed spaces of dimension ≥ 3 , we have succeeded recently in proving the additivity of an odd, isosceles orthogonally additive mapping (see [12]). The main purpose of this paper is to present a complementary result: the existence of a nontrivial even isosceles orthogonally additive mapping characterizes inner product spaces even if dimension ≥ 2 .

More precisely, let $(X, \|\cdot\|)$ be a real normed vector space of dimension ≥ 2 and $(Y, +)$ be an Abelian group. Consider James' *isosceles orthogonality* $\perp$ in X defined by $x \perp y \iff \|x + y\| = \|x - y\|$ ($x, y \in X$). A mapping $f : X \rightarrow Y$ is said to be (*isosceles*) *orthogonally additive*, if it satisfies the conditional Cauchy equation

$$f(x + y) = f(x) + f(y), \quad \text{whenever } x \perp y.$$

Now we are ready to formulate our main result:

Theorem 1.1. *There exists a nontrivial, even, isosceles orthogonally additive mapping from X to Y if, and only if, X is an inner product space.*

The idea of the proof is as follows:

First we prove that an even solution f depends only on the norm of the argument, i.e., f has the same value on vectors of equal norm. Then using a characterization of inner product spaces due to Senechalle [6], we show that in a non-inner product space there are also vectors of different norm on which the solution f takes the same value. Finally, we use certain connectivity theorems to prove that f is constant in regions bounded by concentric spheres: since these regions cover the whole space (but zero), f is constant and so identically zero. For the detailed proof see Section 3 below.

2. Connectivity theorems

We start with a technical observation:

Lemma 2.1. *If $\|x\| = \|y\| = 1$ and there exists a scalar $0 < \lambda_0 < 1$ such that $\|\lambda_0 x + (1 - \lambda_0)y\| = 1$, then*

$$\|\lambda x + (1 - \lambda)y\| = 1$$

for all $0 < \lambda < 1$.

PROOF. For any $0 < \lambda < 1$, let $u(\lambda)$ denote the convex combination $\lambda x + (1 - \lambda)y$. Then obviously

$$\|u(\lambda)\| \leq \lambda\|x\| + (1 - \lambda)\|y\| = 1.$$

On the other hand, if $\lambda_0 < \lambda < 1$, then we have

$$u(\lambda_0) = \frac{\lambda_0}{\lambda}u(\lambda) + \left(1 - \frac{\lambda_0}{\lambda}\right)y$$

and so by $0 < \lambda_0/\lambda < 1$,

$$1 = \|u(\lambda_0)\| \leq \frac{\lambda_0}{\lambda}\|u(\lambda)\| + \left(1 - \frac{\lambda_0}{\lambda}\right),$$

whence $\|u(\lambda)\| \geq 1$ follows.

The case $0 < \lambda < \lambda_0$ can be treated in a similar way.

From now on, the sphere centered at $a \in X$ with radius $\delta > 0$ is denoted by

$$S_a(\delta) = \{x \in X \mid \|x - a\| = \delta\}.$$

For $S_0(1)$ we simply write S .

Now suppose that $0 < \delta \leq 1$ and $a \in X \setminus \{0\}$ with $1 - \delta \leq \|a\| \leq 1 + \delta$. Letting $u = a/\|a\| \in S$, for a vector $v \in S \setminus \text{lin}\{u\}$, consider the closed halfplane

$$P_v^+ = \{\sigma u + \tau v \mid \sigma, \tau \in \mathbb{R}, \tau \geq 0\}.$$

Then we are going to show that

Lemma 2.2. *The set*

$$K_v = S \cap S_a(\delta) \cap P_v^+$$

is non-empty, closed and convex. In fact, K_v is a closed interval.

PROOF. For $\delta = 1$, $0 < \|a\| \leq 2$, the statement has already been proved in [12], Lemma 2.3. Thus we may and do assume that $\delta < 1$.

The set K_v is obviously closed. On the other hand, the mapping $\varphi(x) = \|x - a\|$ is continuous on the connected semi-circle $S \cap P_v^+$ containing $-u$ and u . Since

$$\varphi(-u) = 1 + \|a\| > 1 > \delta \geq |1 - \|a\|| = \|u - a\| = \varphi(u),$$

there exists a vector $x \in S \cap P_v^+$ with $\varphi(x) = \delta$, i.e., $x \in K_v$.

Next we prove the convexity of K_v . To do this, take any vectors $x, y \in K_v$. Then all the vectors

$$x' = \frac{x-a}{\delta}, \quad y' = \frac{y-a}{\delta}, \quad x'' = -x, \quad y'' = -y, \quad x''' = -x', \quad y''' = -y'$$

are in S . We have to show that the line segment $[x, y]$ is contained in K_v . For $x = y$ there is nothing to prove, while for $b = y - x \neq 0$ the following cases may occur:

Case I. $b = \beta a$. Changing the role of x and y , if it is necessary, we may assume that $\beta > 0$. Then $x = \lambda x' + \mu y$ with

$$\lambda = \frac{\beta\delta}{\beta+1} > 0 \quad \text{and} \quad \mu = \frac{1}{\beta+1} > 0.$$

Thus

$$1 = \|x\| \leq \lambda\|x'\| + \mu\|y\| = \lambda + \mu = \frac{\beta\delta + 1}{\beta + 1},$$

whence the contradiction $\delta \geq 1$ follows.

Case II. Vectors a and b are linearly independent. Then x and y can be expressed as

$$x = \alpha a + \beta b \quad \text{and} \quad y = \alpha a + (\beta + 1)b.$$

Without loss of generality, we may assume that $\beta \geq 0$ ($x, y \in P_v^+$ implies that β and $\beta + 1$ are of the same sign, thus the role of x and y should be changed only, when $\beta < 0$).

First we claim that $\alpha > 0$ and $\alpha \neq 1$. The value $\alpha = 0$ is impossible since otherwise $1 = \|x\| = \beta\|b\| < (\beta + 1)\|b\| = \|y\| = 1$ would follow. Similarly, $\alpha = 1$ would imply $1 = \|x'\| = \frac{\beta}{\delta}\|b\| < \frac{\beta+1}{\delta}\|b\| = \|y'\| = 1$. Now, on contrary, assume that $\alpha < 0$. Then $x = \lambda x' + \mu y$ with

$$\lambda = \frac{-\alpha\delta}{\beta - \alpha + 1} > 0 \quad \text{and} \quad \mu = \frac{\beta}{\beta - \alpha + 1} \geq 0.$$

Thus

$$1 = \|x\| \leq \lambda\|x'\| + \mu\|y\| = \lambda + \mu = \frac{\beta - \alpha\delta}{\beta - \alpha + 1},$$

whence the contradiction $\alpha \geq \frac{1}{1-\delta} > 0$ follows.

Now there may occur two different cases:

(i) $0 < \alpha < 1$. Then for

$$\lambda = 1 - \frac{\alpha}{\beta + 1} > 0 \quad \text{and} \quad \mu = \frac{1 - \alpha}{\delta(\beta + 1)} > 0,$$

we have $x' = \lambda y' + \mu y''$, whence

$$1 = \|x'\| \leq \lambda \|y'\| + \mu \|y''\| = \lambda + \mu = 1 + \frac{1 - \alpha - \delta\alpha}{\delta(\beta + 1)}$$

and so $\alpha \leq \frac{1}{1+\delta}$. On the other hand, for

$$\lambda = \frac{\delta\alpha}{\beta + 1} > 0 \quad \text{and} \quad \mu = 1 + \frac{\alpha - 1}{\beta + 1} > 0,$$

we have $x'' = \lambda y' + \mu y''$, whence

$$1 = \|x''\| \leq \lambda \|y'\| + \mu \|y''\| = \lambda + \mu = 1 + \frac{\delta\alpha + \alpha - 1}{\beta + 1}$$

and so $\alpha \geq \frac{1}{1+\delta}$. Thus we have proved that

$$\alpha = \frac{1}{1 + \delta}.$$

This fact implies that $x' = (1 - \lambda')y' + \lambda'y''$ with

$$0 < \lambda' = \frac{1}{(1 + \delta)(\beta + 1)} < 1,$$

and $x'' = \lambda''y' + (1 - \lambda'')y''$ with

$$0 < \lambda'' = \frac{\delta}{(1 + \delta)(\beta + 1)} < 1.$$

Hence by Lemma 2.1,

$$[x', y'] \subset [y', y''] \subset S \quad \text{and} \quad [x'', y''] \subset [y', y''] \subset S.$$

From the second inclusion it follows that

$$[x, y] = -[x'', y''] \subset -S = S$$

while from the first one, we have for all $0 \leq \lambda \leq 1$ that

$$1 = \|\lambda x' + (1 - \lambda)y'\| = \left\| \lambda \frac{x - a}{\delta} + (1 - \lambda) \frac{y - a}{\delta} \right\| = \frac{1}{\delta} \|(\lambda x + [1 - \lambda]y) - a\|,$$

i.e., $\lambda x + (1 - \lambda)y \in S_a(\delta)$. Thus

$$[x, y] \subset S \cap S_a(\delta) \cap P_v^+ = K_v$$

what was to be proved.

(ii) $\alpha > 1$. Then for

$$\lambda = \frac{\alpha - 1}{\delta(\alpha + \beta)} > 0 \quad \text{and} \quad \mu = \frac{\delta\beta}{\delta(\alpha + \beta)} \geq 0,$$

we have $x' = \lambda x + \mu y'$, whence

$$1 = \|x'\| \leq \lambda\|x\| + \mu\|y'\| = \lambda + \mu = \frac{\alpha - 1 + \delta\beta}{\delta(\alpha + \beta)}$$

and so $\alpha \geq \frac{1}{1-\delta}$. On the other hand, for

$$\lambda = \frac{\beta + 1}{\alpha + \beta} > 0 \quad \text{and} \quad \mu = \frac{\delta\alpha}{\alpha + \beta} > 0,$$

we have $y = \lambda x + \mu y'$, whence

$$1 = \|y\| \leq \lambda\|x\| + \mu\|y'\| = \lambda + \mu = \frac{\beta + 1 + \delta\alpha}{\alpha + \beta}$$

and so $\alpha \leq \frac{1}{1-\delta}$. Thus we have proved that

$$\alpha = \frac{1}{1-\delta}.$$

This fact implies that $x' = \lambda'x + (1 - \lambda')y'$ with

$$0 < \lambda' = \frac{1}{1 + \beta(1 - \delta)} \leq 1$$

and $y = (1 - \lambda'')x + \lambda''y'$ with

$$0 < \lambda'' = \frac{\delta}{1 + \beta(1 - \delta)} < 1.$$

This latter means by Lemma 2.1 that $[x, y'] \subset S$. Also it follows that

$$[x', y'] \subset [x, y'] \subset S \quad \text{and} \quad [x, y] \subset [x, y'] \subset S.$$

From the first inclusion, we have for all $0 \leq \lambda \leq 1$ that

$$1 = \|\lambda x' + (1 - \lambda)y'\| = \left\| \lambda \frac{x - a}{\delta} + (1 - \lambda) \frac{y - a}{\delta} \right\| = \frac{1}{\delta} \|(\lambda x + [1 - \lambda]y) - a\|,$$

i.e., $\lambda x + (1 - \lambda)y \in S_a(\delta)$. Thus

$$[x, y] \subset S \cap S_a(\delta) \cap P_v^+ = K_v$$

what was to be proved.

Finally, note that a convex subset of a plane without interior point is necessarily an interval.

Theorem 2.3. *In a real normed vector space of dimension ≥ 3 , the intersection of two spheres is connected (or empty).*

PROOF. For $\sigma \geq \tau > 0$ and $s, t \in X$, $\sigma - \tau \leq \|s - t\| \leq \sigma + \tau$, consider the intersection

$$K' = S_s(\sigma) \cap S_t(\tau).$$

Then the continuous mapping $\varphi : X \rightarrow X$,

$$\varphi(x) = \sigma x + s$$

carries $S \cap S_a(\delta)$ onto K' , where $\delta = \tau/\sigma$ and $a = (t - s)/\sigma$. So it is sufficient to prove the connectivity of

$$K = S \cap S_a(\delta)$$

for $0 < \delta \leq 1$ and $1 - \delta \leq \|a\| \leq 1 + \delta$. If $a = 0$, then $\delta = 1$ and so there is nothing to prove. Thus we may and do assume that $a \neq 0$ whence $u = a/\|a\| \in S$.

Let now $b \in K$ be arbitrarily fixed and for any $x \in K$ choose vectors $v_{x1}, v_{x2} \in S$ such that $a, b, x \in \text{lin}\{u, v_{x1}, v_{x2}\} = M_x$ and $\dim M_x = 3$. Then $W_x = S \cap M_x$ is a closed, bounded sphere in the finite dimensional subspace M_x and so it is compact. Furthermore, in the two dimensional subspace $L_x = \text{lin}\{v_{x1}, v_{x2}\}$, the set $V_x = S \cap L_x$ is a connected circle. Now consider the relation $F_x \subset V_x \times W_x$ defined for any $v \in V_x$ by $F_x(v) = K_v = S \cap S_a(\delta) \cap P_v^+$, the non-empty, closed, connected interval introduced in the previous lemma. It is known from [12], Corollary 2.2, that a closed relation between a connected and a compact metric space is itself connected provided it is defined everywhere with connected values. Thus as soon as it will have been shown that F_x is closed, the connectivity of

$$F_x(V_x) = \bigcup_{v \in V_x} (K \cap P_v^+) = K \cap M_x \ni x, b$$

will be clear.

For that reason take a sequence $(v_n, w_n) \in F_x$ converging to some $(v_0, w_0) \in V_x \times W_x$. It is clear that $\varepsilon = \text{dist}(u, L_x) > 0$. Since $w_n \in F_x(v_n) \subset P_{v_n}^+$, it can be written as

$$w_n = \sigma_n u + \tau_n v_n \quad \text{with} \quad \tau_n \geq 0.$$

On the other hand, w_0 can be combined linearly from u and some $v \in V_x$:

$$w_0 = \sigma_0 u + \tau_0 v \quad \text{with} \quad \tau_0 \geq 0.$$

Then we have

$$\frac{\|w_n - w_0\|}{|\sigma_n - \sigma_0|} = \left\| u - \frac{\tau_0 v - \tau_n v_n}{\sigma_n - \sigma_0} \right\| \geq \varepsilon,$$

whence

$$|\sigma_n - \sigma_0| \leq \frac{\|w_n - w_0\|}{\varepsilon} \longrightarrow 0.$$

Thus

$$\tau_n v_n = w_n - \sigma_n u \longrightarrow w_0 - \sigma_0 u = \tau_0 v,$$

and by the continuity of the norm,

$$\tau_n = \|\tau_n v_n\| \longrightarrow \|\tau_0 v\| = \tau_0.$$

Hence by $v_n \longrightarrow v_0$,

$$\tau_n v_n \rightarrow \tau_0 v_0$$

holds true, as well and by the uniqueness of the limit, $\tau_0 v_0 = \tau_0 v$, i.e.,

$$w_0 = \sigma_0 u + \tau_0 v_0 \in P_{v_0}^+.$$

On the other hand, $w_n \in K_{v_n} \subset K$ and because of K is closed, $w_0 \in K$, i.e., $w_0 \in K \cap P_{v_0}^+ = K_{v_0} = F_x(v_0)$. Thus $(v_0, w_0) \in F_x$ and therefore F_x is closed.

Finally, $\bigcap_{x \in K} F_x(V_x)$ is non-empty (namely it contains b), whence

$$K = \bigcup_{x \in K} F_x(V_x)$$

is connected.

Theorem 2.4. *In a real normed vector space of dimension ≥ 3 , the set*

$$C_\delta = \{(u, v) \in S \times S \mid \|u - v\| = \delta\} \subset S \times S$$

is connected for any $0 \leq \delta \leq 2$.

PROOF. The set C_δ is a relation on the connected sphere S and by Theorem 2.3,

$$C_\delta(u) = \{v \in S \mid \|u - v\| = \delta\} = S \cap S_u(\delta)$$

is non-empty, connected for all $u \in S$. We shall use [12], Lemma 2.1 to prove the connectivity of relation C_δ .

If $\delta = 0$, then C_0 is the diagonal of $S \times S$, so it is connected.

If $\delta = 2$, then for any sequence $u_n \in S$ converging to $u_0 \in S$, we have $-u_n \in C_2(u_n)$ for all $n \in \mathbb{N}$. Since $-u_n \longrightarrow -u_0 \in C_2(u_0)$, by virtue of [12], Lemma 2.1, the proof is complete.

Now consider the case of $0 < \delta < 2$. For a sequence $u_n \in S$ converging to $u_0 \in S$, choose $v \in S \setminus \text{lin}\{u_0\}$. Then

$$C_\delta(u_n) \cap \text{lin}\{u_0, v\} \neq \emptyset$$

for sufficiently large n 's. Since any sequence $v_n \in C_\delta(u_n) \cap \text{lin}\{u_0, v\}$ is contained in the compact metric space $S \cap \text{lin}\{u_0, v\}$, the Bolzano-Weierstrass Theorem ensures the existence of a convergent selection $v_{n_k} \longrightarrow v_0 \in S$. Since $\|u_{n_k} - v_{n_k}\| = \delta$ and $u_{n_k} \longrightarrow u_0$, we have $\|u_0 - v_0\| = \delta$, i.e., $v_0 \in C_\delta(u_0)$. Therefore [12], Lemma 2.1 completes the proof.

Theorem 2.5. *Suppose that $s, t \in X$ is a normed base for the two dimensional space X : $\|s\| = \|t\| = 1$, $\text{lin}\{s, t\} = X$. For a couple of vectors $x = \sigma_x s + \tau_x t$ and $y = \sigma_y s + \tau_y t$ in X , let denote*

$$\det(x, y) = \sigma_x \tau_y - \tau_x \sigma_y.$$

Then for any $0 \leq \delta \leq 2$, the sets

$$C_\delta^\pm = \{(u, v) \in S \times S \mid \|u - v\| = \delta, \det(u, v) \in \mathbb{R}_\pm\}$$

are connected.

PROOF. We perform the proof for C_δ^+ which is a relation on the compact, connected circle S . For any $u \in S$, the set

$${}_u P^+ = \{x \in X \mid \det(u, x) \geq 0\}$$

is a closed halfplane, so

$$C_\delta^+(u) = S \cap S_u(\delta) \cap {}_u P^+$$

is a non-empty, connected interval.

Now we are going to show that C_δ^+ is closed. Indeed, if $(u_n, v_n) \in C_\delta^+$ converging to some $(u_0, v_0) \in S \times S$, then by the continuity of the norm, $\delta = \|u_n - v_n\| \longrightarrow \|u_0 - v_0\|$, whence $\|u_0 - v_0\| = \delta$. Furthermore, by the continuity of the determinant,

$$\det(u_n, v_n) \longrightarrow \det(u_0, v_0)$$

showing $\det(u_0, v_0) \geq 0$.

These mean that

$$v_0 \in S \cap S_{u_0}(\delta) \cap {}_{u_0} P^+ = C_\delta^+(u_0),$$

i.e., $(u_0, v_0) \in C_\delta^+$. Thus [12], Corollary 2.2 completes the proof.

3. Proof of the main theorem

PROOF. Assume that X is not an inner product space and $f : X \rightarrow Y$ is an even, orthogonally additive mapping. We are going to show that f is identically zero.

First we claim that f takes the same value on vectors of equal norm. Indeed, for $x, y \in X$, $\|x\| = \|y\|$, letting $p = (x + y)/2$, $q = (x - y)/2$, we have

$$\|p + q\| = \|x\| = \|y\| = \|p - q\|,$$

i.e., $p \perp \pm q$. Hence

$$f(x) = f(p + q) = f(p) + f(q) = f(p) + f(-q) = f(p - q) = f(y).$$

This means that f depends only on the norm of x , i.e., there exists a function on the non-negative reals into Y such that

$$f(x) = \varphi(\|x\|), \quad x \in X.$$

Also, observe that

$$f(2x) = f(2p + 2q) = f(2p) + f(2q) = f(x + y) + f(x - y),$$

or in another equivalent form

$$f(x + y) = f(2x) - f(x - y), \quad x, y \in X, \quad \|x\| = \|y\|.$$

By a theorem of Senechalle [6], if X fails to be an inner product space, then there exist couples of vectors $(u_1, v_1), (u_2, v_2) \in S \times S$ such that

$$\|u_1 - v_1\| = \|u_2 - v_2\| = \delta \quad \text{but} \quad \|u_1 + v_1\| = \rho_1 < \rho_2 = \|u_2 + v_2\|.$$

Now consider the scalar set

$$\Gamma_\delta = \{\|u + v\| \mid u, v \in S, \|u - v\| = \delta\}$$

which is image of the set

$$C_\delta = \{(u, v) \in S \times S \mid \|u - v\| = \delta\}$$

through the continuous sum-norm function $\gamma : S \times S \rightarrow \mathbb{R}$, $\gamma(u, v) = \|u + v\|$.

When $\dim X \geq 3$, by Theorem 2.4, C_δ is connected and so is Γ_δ . For $\dim X = 2$, by Theorem 2.5, $C_\delta = C_\delta^+ \cup C_\delta^-$ holds with connected C_δ^+ , C_δ^- and

$$(u, v) \in C_\delta^+ \iff (v, u) \in C_\delta^-,$$

whence

$$\gamma(C_\delta^+) = \gamma(C_\delta^-) = \gamma(C_\delta) = \Gamma_\delta$$

is connected. This means in both cases that Γ_δ is an interval containing $[\rho_1, \rho_2]$.

Now let $\mu > 0$ be fixed and take any $\rho_1 \leq \rho \leq \rho_2$. Then $\rho \in \Gamma_\delta$ and so $\rho = \|u + v\|$ for some $u, v \in S$ with $\|u - v\| = \delta$. Using the above observation for $x = \mu u$, $y = \mu v$, ($\mu > 0$) we have

$$\varphi(\mu\rho) = f(\mu u + \mu v) = f(2\mu u) - f(\mu u - \mu v) = \varphi(2\mu) - \varphi(\mu\delta),$$

i.e., $\varphi(\mu\rho)$ does not depend on ρ . Thus φ is constant on all the intervals $[\mu\rho_1, \mu\rho_2]$, $\mu > 0$, whence so is on the whole positive real line:

$$\varphi(\rho) = c, \quad \rho > 0.$$

Finally, choosing any $u, v \in S$, $u \perp v$, we have $u + v \neq 0$ and so

$$c = f(u + v) = f(u) + f(v) = c + c.$$

Thus $c = 0$ and so for all $x \in X \setminus \{0\}$

$$f(x) = \varphi(\|x\|) = 0,$$

which, together with the obvious equality $f(0) = 0$, means that f is identically zero.

This completes the proof.

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