

On the structure of univoque numbers

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1. Introduction

We shall continue our investigation in [1] on univoque sequences generated by Θ -adic expansion of real numbers. A method for the computation of the Hausdorff dimension of the set of univoque numbers will be presented.

Let $\frac{1}{2} \leq \Theta < 1$, $L = L(\Theta) = \Theta + \Theta^2 + \dots = \frac{\Theta}{1 - \Theta}$, $\lambda = \Theta L$.

For $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots) \in \{0, 1\}^{\mathbb{N}}$ let

$$\langle \varepsilon, \Theta \rangle := \sum_{n=1}^{\infty} \varepsilon_n \Theta^n.$$

A sequence ε is said to be univoque with respect to Θ if $\langle \varepsilon, \Theta \rangle = \langle \delta, \Theta \rangle$, $\delta \in \{0, 1\}^{\mathbb{N}}$ implies that $\varepsilon = \delta$, i.e. that $\varepsilon_j = \delta_j$ ($j = 1, 2, \dots$).

It is known that for any $x \in [0, L(\Theta)]$ there exists an $\varepsilon \in \{0, 1\}^{\mathbb{N}}$ such that $x = \langle \varepsilon, \Theta \rangle$, namely this is true for $\varepsilon_n = \varepsilon_n(x)$, where $\varepsilon_n(x)$ is defined by induction on n , as follows:

$$(1.1) \quad \varepsilon_n(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n-1} \varepsilon_i(x) \Theta^i + \Theta^n \leq x, \\ 0 & \text{if } \sum_{i=1}^{n-1} \varepsilon_i(x) \Theta^i + \Theta^n > x. \end{cases}$$

The expansion $\langle \varepsilon(x), \Theta \rangle = x$, $\varepsilon(x) = (\varepsilon_1(x), \dots)$ is called the **regular** expansion of x .

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Every $x \in (0, L(\Theta))$ can be expanded by the digits, $\delta_n = \delta_n(x)$ ($n = 1, 2, \dots$) as well, where they are defined from

$$(1.2) \quad \delta_n(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n-1} \delta_i(x) \Theta^i + \Theta^n < x, \\ 0 & \text{if } \sum_{i=1}^{n-1} \delta_i(x) \Theta^i + \Theta^n \geq x. \end{cases}$$

The expansion $x = \langle \delta(x), \Theta \rangle$, $\delta(x) = (\delta_1(x), \dots)$ is called the **quasi-regular** expansion of x .

The expansions $\varepsilon(x), \delta(x)$ are the same except, if the regular expansion of x is finite (i.e. if $\varepsilon_n(x) = 0$ for all large n).

Let $R(\Theta) = \{\varepsilon(x) \mid x \in [0, L]\}$, $R_1(\Theta) = \{\varepsilon(x) \mid x \in [0, 1)\}$.

Let $l = l(\Theta) = (l_1, l_2, \dots) \in \{0, 1\}^{\mathbb{N}}$ be the quasi-regular expansion of 1, i.e. $\delta_j(1) = l_j$ ($j = 1, 2, \dots$). If 1 has a finite regular expansion in the base Θ , and $\langle \varepsilon(1), \Theta \rangle = s_1 \Theta + \dots + s_k \Theta^k$, $s_k = 1$, then $\delta(1) = (s_1, s_2, \dots, s_k - 1, 0, s_1, \dots, s_k - 1, 0, \dots)$, i.e. $\delta(1)$ is a periodic sequence with period k .

W. PARRY [2] gave a simple characterization of the sequences $a = \{a_1, a_2, \dots\} \in \{0, 1\}^{\mathbb{N}}$ of $R_1(\Theta)$: $a \in R_1(\Theta)$, if and only if

$$(1.3) \quad \{a_r, a_{r+1}, \dots\} < \{l_1, l_2, \dots\} \quad (r = 1, 2, \dots)$$

holds, in the sense of the lexicographic ordering.

He proved furthermore that $l \in \{0, 1\}^{\mathbb{N}}$ is the regular expansion of 1 for a suitable $\Theta \in [\frac{1}{2}, 1)$, if and only if

$$(1.4) \quad \begin{aligned} & l_1 = 1 \text{ and} \\ & \{l_{k+1}, l_{k+2}, \dots\} < \{l_1, l_2, \dots\} \\ & k = 1, 2, \dots \end{aligned}$$

holds.

One can prove simply that the periodic sequence $l \in \{0, 1\}^{\mathbb{N}}$ with $l_1 = 1$ is the quasi-regular expansion of 1 with a suitable $\Theta \in [\frac{1}{2}, 1)$ if and only if

$$(1.5) \quad \{l_k, l_{k+1}, \dots\} \leq \{l_1, l_2, \dots\} \quad (k = 1, 2, \dots)$$

holds. If (1.5) holds, then with the corresponding Θ as base, the regular expansion of 1 is finite. In [1] we proved the following assertions (Theorem 2.1 and 2.4 which are formulated now as Lemma 1 and 2).

Lemma 1. *The sequence $\varepsilon \in \{0, 1\}^{\mathbb{N}}$ is univoque with respect to Θ if and only if both of the sequences $\varepsilon, \underline{1} - \varepsilon \in R(\Theta)$, where $\underline{1} = \{1, 1, \dots\}$.*

Let $U(\Theta)$ be the set of univoque sequences.

Lemma 2. *If $\frac{1}{2} \leq \Theta' < \Theta < 1$, then $U(\Theta) \subseteq U(\Theta')$.*

Definition. The number $\Theta \in (\frac{1}{2}, 1)$ is said to be stable from below, if $U(\Theta) = U(\Theta')$ holds for some $\Theta' < \Theta$. Similarly, Θ is stable from above, if $U(\Theta'') = U(\Theta)$ holds for some $\Theta'' > \Theta$.

Remark. This definition is somewhat different from that was given in [1].

Let $H(= H_\Theta)$, $H^*(= H_\Theta^*)$ be the set of univoque numbers (with respect to Θ) on the intervals $[\Theta, 1)$, $[0, 1)$, respectively. It is clear that

$$(1.6) \quad H^* = \{0\} \cup \bigcup_{n=0}^{\infty} \Theta^n H.$$

The set of univoque numbers $x \in [1, L]$ can be given as $(L - H^*) \cap [1, L]$. Let

$$(1.6) \quad U_1(\Theta) := \{\varepsilon \in U(\Theta), \langle \varepsilon, \Theta \rangle \in H\},$$

i.e. $U_1(\Theta)$ is the set of those univoque sequences for which the represented number $\langle \varepsilon, \Theta \rangle$ falls into $[\Theta, 1)$.

2. A new notation for univoque sequences

First of all, let $\mathcal{K}_h$ denote the set of words of length h over $\mathbb{N}$, and $\mathcal{M}$ be the set of infinite words over $\mathbb{N}$, i.e. let

$$\begin{aligned} \mathcal{K}_h &:= \{m_1 m_2 \dots m_h; m_j \in \mathbb{N}\} \\ \mathcal{M} &:= \{\underline{m} = m_1 m_2 \dots; m_j \in \mathbb{N}\}. \end{aligned}$$

Let $F_h : \mathcal{M} \rightarrow \mathcal{K}_h$ be the mapping $F_h(\underline{m}) = m_1 \dots m_h$; let σ be the shift operator acting as $\sigma(m_1 m_2 \dots) = m_2 m_3 \dots$.

Let us define the ordering relations in $\mathcal{K}_h$ and in $\mathcal{M}$ by the following relations:

- (1) in $\mathcal{K}_1(= \mathbb{N})$: the common ordering
- (2) in $\mathcal{K}_2$: $n_1 n_2 < m_1 m_2$ holds if $n_1 < m_1$, or if

$$n_1 = m_1 \quad \text{and} \quad n_2 > m_2.$$

- (h) in $\mathcal{K}_h$: $n_1 n_2 \dots n_h < m_1 \dots m_h$, if

$$n_1 < m_1, \quad \text{or if} \quad n_1 = m_1 \quad \text{and} \quad n_2 \dots n_h > m_2 \dots m_h.$$

In other words, if $n_1 \dots n_h \neq m_1 \dots m_h$ and k is the smallest index for which $n_k \neq m_k$, then

for odd k : $n_1 \dots n_h < m_1 \dots m_h$, if $n_k < m_k$

for even k : $n_1 \dots n_h < m_1 \dots m_h$, if $n_k > m_k$.

Let $\underline{m}, \underline{n}$ be two distinct words in $\mathcal{M}$. We say that $\underline{m} < \underline{n}$, if $F_h(\underline{m}) \neq F_h(\underline{n})$ implies that $F_h(m) < F_h(n)$ in $\mathcal{K}_h$. It is clear that this definition is correct.

Let $E \subseteq \{0, 1\}^{\mathbb{N}}$ be the set of those sequences $\varepsilon = \{\varepsilon_1, \varepsilon_2, \dots\}$ in which both of 0 and 1 occurs infinitely often, and $\varepsilon_1 = 1$. Let $\varphi : E \rightarrow \mathcal{M}$ be the one to one mapping defined as follows: Let ε (considered as an infinite word over $\{0, 1\}$) of form $1^{a_1}0^{b_1}1^{a_2}0^{b_2} \dots$. Then $\varphi(\varepsilon) = a_1b_1a_2b_2 \dots$.

It is clear that, if $\varepsilon, \delta \in E$, then $\varepsilon < \delta$ holds in E (in the sense of the lexicographic ordering) if and only if $\varphi(\varepsilon) < \varphi(\delta)$ in $\mathcal{M}$.

We have $U_1(\Theta) \subseteq E$. Let $\mathcal{M}^{(0)} = \mathcal{M}_{\Theta}^{(0)} = \varphi(U_1(\Theta))$.
Let furthermore

$$\underline{t} = t_1t_2 \dots = \varphi(l(\Theta)),$$

where $l(\Theta)$ is the sequence getting as the quasiregular expansion of 1 in the base Θ .

From the Parry condition and Lemma 1 we have

Lemma 3. $\alpha \in \mathcal{M}$ belongs to $\mathcal{M}_{\Theta}^{(0)}$ if and only if

$$(2.1) \quad \sigma^l(\alpha) < \underline{t} \quad (l = 0, 1, 2, \dots).$$

PROOF. Clear.

Let $Y = y_1y_2 \dots \in \mathcal{M}$, $Y_h := F_h(Y) = y_1 \dots y_h$;

$$(2.2) \quad S(Y) := \{\alpha \in \mathcal{M} : \sigma^l(\alpha) < Y, l = 0, 1, 2, \dots\},$$

$$(2.3) \quad U_k(Y) := \{\alpha \in \mathcal{M} : F_k(\sigma^l(\alpha)) < Y_k, l = 0, 1, 2, \dots\},$$

$$(2.4) \quad V_k(Y) := \{\alpha \in \mathcal{M} : F_k(\sigma^l(\alpha)) \leq Y_k, l = 0, 1, 2, \dots\}.$$

It is clear that $U_1(Y) \subseteq U_2(Y) \subseteq \dots$ and $V_1(Y) \supseteq V_2(Y) \supseteq \dots$.

Lemma 4. For each $k, l \in \mathbb{N}$ we have

$$(2.5) \quad U_k(Y) \subseteq S(Y) \subseteq V_l(Y).$$

PROOF. Clear.

Lemma 5. *Let p be the smallest integer, if any, for which there exist $u, r \geq 1$, $u + r = p$ such that*

$$(2.6) \quad y_{u+1} \cdots y_{u+r} > Y_r$$

in the sense of ordering introduced in $\mathcal{K}_r$. Then

$$(2.7) \quad S(Y) = U_p(Y).$$

PROOF. If there is an $\alpha \in V_{u+r}(Y) \setminus U_{u+r}(Y)$, then $F_p(\sigma^j(\alpha)) = Y_p$ holds for some j . Then

$$F_r(\sigma^{j+r}(\alpha)) > Y_r,$$

i.e. $\alpha \notin V_r(Y)$. Hence $V_{u+r}(Y) = U_{u+r}(Y)$, and (2.7) follows from (2.5).

Lemma 6. *If $y_2 > y_1$, then*

$$(2.8) \quad S(Y) = \{\alpha = a_1 a_2 \dots \mid 1 \leq a_j \leq y_1 - 1\}.$$

Let $y_2 = y_1$ and denote $z = y_1 y_1 y_1 \dots$. If $z \geq Y$, then $S(Y)$ as in (2.8). If $z < Y$, then

$$(2.9) \quad S(Y) = \{\alpha = a_1 a_2 \dots \mid 1 \leq a_j \leq y_1 - 1, j = 1, 2, \dots\} \cup \\ \cup \{\alpha = \beta z \mid \beta = b_1 \dots b_h, 1 \leq b_j \leq y_1 - 1, h = 0, 1, 2, \dots\}$$

($h = 0$ is for the empty word!)

PROOF. The first assertion comes from Lemma 5 immediately. Assume that $y_2 = y_1$. If $\alpha \in S(Y)$, and the first occurrence of y_1 in the sequence is $a_{h+1} = y_1$, then $\alpha = a_1 a_2 \dots a_h z$, thus $\sigma^h(\alpha) = z$, $\sigma^h(\alpha) < Y$, this may occur only if $z < Y$. The further part of the lemma is clear.

Lemma 7. *Let $\underline{t} = t_1 t_2 \cdots = \varphi(l(\Theta))$, and assume that $t_2 \geq t_1$. Then*

$$\mathcal{M}_{\Theta}^{(0)} = \{\alpha = a_1 a_2 \dots \mid 1 \leq a_j \leq t_1 - 1, j = 1, 2, \dots\}.$$

PROOF. The assertion immediately follows from Lemma 4 and 6. The only critical element is $z = t_1 t_1 \dots$ in the case $t_2 = t_1$. Since $\underline{t}$ comes from a quasi regular expansion of 1, therefore $t_{2j+1} \leq t_1$ and in the case $t_{2j+1} = 1$ $t_{2j+2} > t_2 = t_1$, since $\sigma^{2j}(\underline{t}) \leq \underline{t}$. If $t_k = t_1$ for each k , then $z = \underline{t}$, and $z < \underline{t}$ does not hold. Let k be the smallest index for which $t_k \neq t_1$. If k odd, then $t_k < t_1$, but then $z > \underline{t}$. If k even, then $t_k > t_1$, and similarly we have $z > \underline{t}$. Thus $z \notin \mathcal{M}_{\Theta}^{(0)}$.

3. The structure of $M_{\Theta}^{(0)}$ in the case $t_2 \geq t_1$

Theorem 1. Assume that the condition stated in Lemma 7 holds. Then H is self-similar, it is the attractor of the iterated function system

$$(3.1) \quad H = \bigcup_{a_1=1}^{t_1-1} \bigcup_{a_2=1}^{t_1-1} f_{a_1, a_2}(H),$$

where $f_{a_1, a_2}(x) = \Theta^{a_1} + \Theta^{a_1+a_2}x$. The components on the right hand side of (3.1) are disjoint sets.

Let ξ denote the positive root of the polynomial $x^{t_1-1} + \dots + x - 1$; let

$$(3.2) \quad s := \frac{\log 1/\xi}{\log 1/\Theta} (< 1).$$

Then the Hausdorff dimension of H equals to its similarity dimension, $= s$.

PROOF. (3.1) is a consequence of Lemma 7. From the definition follows that the components are disjoint. H is closed and bounded. The further assertion follows from a theorem of Hutchinson (see G. EDGAR [3]).

4. On the set $\mathcal{F}$

Let $\mathcal{F}$ denote the set of those Θ for which 1 is univoque with respect to Θ . If the regular expansion of 1 is finite, then clearly $\Theta \notin \mathcal{F}$, since then 1 has another expansion. If $\Theta \in \mathcal{F}$, $\underline{t} = \varphi(l(\Theta))$, then $\langle \varphi^{-1}(\sigma^j(\underline{t})), \Theta \rangle \in H_{\Theta}$ for each $j \geq 1$, therefore

$$(4.1)_j \quad \sigma^j(\underline{t}) < \underline{t} \quad (j = 1, 2, \dots)$$

holds.

Let now $\underline{t} \in \mathcal{M}$ be an arbitrary sequence for which $(4.1)_j$ ($j = 1, 2, \dots$) holds. The fulfilment of the conditions $(4.1)_{2l}$ ($l = 1, 2, \dots$) guarantee the existence of a Θ for which $\varphi^{-1}(\underline{t}) = l(\Theta)$, $\Theta \in (\frac{1}{2}, 1)$. Then $(4.1)_j$ implies that $\langle \varphi^{-1}(\sigma^j(\underline{t})), \Theta \rangle \in H$ for $j \geq 1$ (see Lemma 3), thus 1 is univoque with respect to Θ . We have proved

Theorem 2. $\underline{t} \in \mathcal{M}$ is the image of the regular (quasi-regular) expansion of 1 in the base of a suitable $\Theta \in \mathcal{F}$ if and only if $(4.1)_j$ ($j = 1, 2, \dots$) holds.

Theorem 3. *The Lebesgue measure of $\mathcal{F}$ is zero, its Hausdorff dimension is 1.*

PROOF. **I.** In [1] we proved that $\Theta \in \mathcal{F}$ implies that $\Theta \leq \frac{\sqrt{5}-1}{2}$.

Let Θ and Θ' be such numbers for which $l(\Theta) = \{l_1, \dots, l_k, l_{k+1}, \dots\}$, $l(\Theta') = \{l_1, \dots, l_k, l'_{k+1}, l'_{k+2}, \dots\}$, $l_{k+1} = 0, l'_{k+1} = 1$.

Let $P_1(z) = \sum_{j=1}^{\infty} l_j z^j - 1$, $P_2(z) = \sum_{j=1}^k l_j z^j + \sum_{j=k+1}^{\infty} l'_j z^j - 1$. Since

$P_1(\Theta) = 0, P_2(\Theta') = 0, P_2(\Theta) = |P_2(\Theta) - P_1(\Theta)| \leq c\Theta^k, P_2(\Theta) - P_2(\Theta') = (\Theta - \Theta')P_2''(\xi), \xi \in (\Theta, \Theta')$, and $(0 <) c_1 < P_2''(\xi) < c_2$ with numerical constants c_1, c_2 , therefore

$$(4.2) \quad 0 < \Theta - \Theta' < c_3 \Theta^k.$$

Let $\mathcal{F}_K = \{\Theta \mid \Theta \in \mathcal{F}, t_1 = K\}$. If $\Theta \in \mathcal{F}_K$, then

$1 = \Theta + \dots + \Theta^{t_1} + \Theta^{t_1+t_2+1} + \dots$, $t_1 = K$, $t_2 \leq K$, consequently $\Theta_K = \max_{\Theta \in \mathcal{F}_K} \Theta$ satisfies

$$(4.3) \quad \sum_{j=1}^K \Theta_K^j \leq 1 - \Theta_K^{2K+1}.$$

Let R be an arbitrary large integer. Let us classify the elements of $\mathcal{F}_K$ according to the sequence $t_1, t_2, \dots, t_R$. The distance of two numbers, $\Theta_1, \Theta_2 \in \mathcal{F}_K$ with $F_R(\varphi(\Theta_1)) = F_R(\varphi(\Theta_2)) = t_1 t_2 \dots t_R$ is less than $c_3 \Theta_K^{t_1 + \dots + t_R}$, due to (4.2). Thus $\mathcal{F}_K$ can be covered by finitely many intervals the total length of which is less than

$$c_3 \left(\sum_{j=1}^K \Theta_K^j \right)^R \leq c_3 (1 - \Theta_K^{2K+1})^R.$$

The right hand side tends to zero as $R \rightarrow \infty$. Thus $\text{meas}(\mathcal{F}_K) = 0$, whence $\text{meas}(\mathcal{F}) = \sum \text{meas}(\mathcal{F}_K) = 0$. The first part of the theorem is proved.

II. Let $\mathcal{F}_K^{(0)}$ be the subset of $\mathcal{F}_K$ defined by the conditions $\mathcal{F}_K^{(0)} = \{\Theta \mid \varphi(l(\Theta)) = t_1 t_2 \dots; t_1 = K; 1 \leq t_j \leq K-1, j \geq 2\}$. We shall show that for any given $\sigma < 1$ there is a K such that the Hausdorff dimension of $\mathcal{F}_K^{(0)}$ is larger than σ .

Let $\Theta_{\min}, \Theta_{\max}$ denote the smallest and the largest elements of $\mathcal{F}_K^{(0)}$, respectively. Assume that $K \geq 3$. Then $\varphi(l(\Theta_{\min})) = K 1 (K-1) 1 (K-1) \dots$. Let Ψ_K be the positive root of the polynomial $1 - (z + \dots + z^{K-1})$. Then

$\Theta_{\min} < \Theta_{\max} < \Psi_K$, furthermore $\Psi_K - \Theta_{\min} < c \Psi_K^K$ holds with a suitable numerical constant c . The last inequality follows from (4.2).

Let $\Theta' < \Theta$, $\Theta', \Theta \in \mathcal{F}_K^{(0)}$ with $l(\Theta) = l_1 l_2 \dots$, $l(\Theta') = l'_1 l'_2 \dots$ such that $l_s = 0$, $l_{s+1} = 1$ and $l_j = l'_j$ for $1 \leq j \leq s$. Assume that $s > K$. Then $l_1 \Theta + \dots + l^{s-1} \Theta^{s-1} + \Theta^s > 1$, $l_1 \Theta' + \dots + l_{s-1} \Theta'^{s-1} + \Theta'^s < 1 - \Theta'^{s+K}$. The polynomial $h(z) := l_1 z + \dots + l_{s-1} z^{s-1} + z^s$ satisfies $(1 \leq) h'(z) \leq 9$ for $z \leq 0, 9$, say, whence

$$\Theta'^{s+K} < h(\Theta) - h(\Theta') = (\Theta - \Theta')h'(\xi), \quad \xi \in (\Theta', \Theta)$$

thus

$$(4.4) \quad \Theta - \Theta' \geq \frac{1}{9} \Theta'^{s+K} \geq \frac{1}{9} \cdot \Theta_{\min}^{s+K}.$$

Let $f(V)$ be the number of the sequences $l_1, \dots, l_V$ which occur as the first V elements of $l(\Theta) = \{l_1, l_2, \dots\}$ for some $\Theta \in \mathcal{F}_K^{(0)}$. For $V > U$ and given $l_1^*, \dots, l_U^*$ let $g(V \mid l_1^*, \dots, l_U^*)$ be the number of the distinct $l_1, \dots, l_V$ occuring in the beginning of $l(\Theta) = \{l_1, l_2, \dots\}$ for which the first U elements are fixed, $l_1 = l_1^*, \dots, l_U = l_U^*$.

Lemma 8. *With suitable positive constants $c(K), c_1(K), c_2(K)$ we have*

$$(4.5) \quad f(V) = c(K) \Psi_K^{-V} (1 + o(1)) \quad \text{as } V \rightarrow \infty,$$

$$(4.6) \quad c_1(K) < g(V \mid l_1^*, \dots, l_U^*) \Psi_K^{V-U} < c_2(K) \quad \text{if } U < V.$$

First we continue the proof of the theorem assuming the validity of Lemma 8, then we prove it.

Assume in contrary that dimension $(\mathcal{F}_K^{(0)}) < \sigma$. Then, for arbitrary choice of $\varepsilon, \delta > 0$, there is a covering $\mathcal{F}_K^{(0)} \subseteq \bigcup_{j=1}^{\infty} E_j$, such that $\text{diam } E_j < \delta$ and

$$\sum_{j=1}^{\infty} (\text{diam } E_j)^{\sigma} < \varepsilon.$$

Then there is such a covering with open intervals I_j , and even we may assume that the set of lengths of I_j belongs to the set $\{\Theta_{\min}^r \mid r = 1, 2, \dots\}$.

$\mathcal{F}_K^{(0)}$ is a closed set. Let $\mathcal{F}_K^{(0)} \subseteq \bigcup I_j$,

$$(4.7) \quad \sum_{j=1}^{\infty} (\text{diam } I_j)^{\sigma} < \varepsilon, \quad \text{diam } I_j < \delta.$$

From the Heine-Borel theorem we obtain that there is a finite subcover,
 $\mathcal{F}_K^{(0)} \subseteq \bigcup_{j=1}^p I_j$. Let M_r be the number of intervals I_j with length $\Theta_{\min}^r$.

Let $r_o = \left\lceil \frac{\log 1/\delta}{\log 1/\Theta_{\min}} \right\rceil$, and r_1 be the largest j for which $M_j \neq 0$. Then $M_j = 0$ for $j < r_o$. We have

$$(4.8) \quad \sum_{j=1}^p (\text{diam } I_j)^\sigma \leq \sum_{r_o \leq j \leq r_1} M_j \Theta_{\min}^{j\sigma} < \varepsilon.$$

Let $V > r_1$ and $\mathcal{F}_K^{(0)}(l_1, \dots, l_V)$ be the set of those $\Theta \in \mathcal{F}_K^{(0)}$, for which the first V elements of $l(\Theta)$ is the given sequence $l_1, \dots, l_V$.

If $\Theta_1, \Theta_2 \in \mathcal{F}_K^{(0)}$ are covered with the same interval I_j of length $\Theta_{\min}^r$, then $\Theta_{\min}^r > |\Theta_1 - \Theta_2|$ and by (4.4) we obtain that the first $r - K - 4$ digits of $l(\Theta_1)$ and of $l(\Theta_2)$ coincide.

Due to (4.6), the number of that sets among $\mathcal{F}_K^{(0)}(l_1, \dots, l_V)$ which have nonempty intersection with I_j , is less than

$$c_2(K) \Psi_K^{r-K-4} \cdot \Psi_K^{-V}.$$

Since any of $\mathcal{F}_K^{(0)}(l_1, \dots, l_V)$ has a nonempty intersection with at least one I_j , therefore

$$f(V) \leq c_2(K) \Psi_K^{-V} \sum_{r_o \leq r \leq r_1} M_r \Psi_K^{r-K-4}.$$

Then, from (4.5), taking the limit $V \rightarrow \infty$, we have

$$(A :=) \frac{1}{\Psi_K^{K+4}} \frac{c(K)}{c_2(K)} \leq \sum_{r_o \leq r \leq r_1} M_r \cdot \Psi_K^r.$$

From (4.8) it follows that $M_r < \varepsilon \Theta_{\min}^{-r\sigma}$, thus

$$(4.9) \quad A \leq \varepsilon \sum_{r_o \leq r < r_1} \left(\frac{\Psi_K}{\Theta_{\min}^\sigma} \right)^r.$$

If K is large enough, then $\Psi_K < \Theta_{\min}^\sigma$. For such choice of K the inequality (4.9) cannot be held if ε is small enough. This finishes the proof of the theorem.

PROOF of Lemma 8. Let $f_1(n)$ be the number of that sequences in $\{0, 1\}^n$, which do not contain K consecutive 1's and 0's.

Then $f_1(n) = f_1(n-1) + \dots + f_1(n-(K-1))$ for $n \geq K$. The characteristic polynomial $x^{K-1} - (x^{K-2} + \dots + x + 1)$ of this recursion

has only one root, namely Ψ_K^{-1} in the domain $|z| \geq 1$, therefore $f_1(n) = C\Psi_K^{-n}(1 + \sigma(1))$ ($n \rightarrow \infty$), $C > 0$ holds since $f(V) = f_1(V - k)$, (4.5) holds.

Since $g(V \mid l_1^*, \dots, l_U^*) \geq f_1(V - U)$, $g(V \mid l_1^*, \dots, l_U^*) \geq f_1(V - U - K)$ clearly hold, (4.6) is true.

5. On stable numbers

Theorem 4. $\Theta \in (\frac{1}{2}, 1)$ is stable from both sides if and only if

$$(5.1) \quad \sigma^j(\underline{t}) > \underline{t} \quad , \quad \underline{t} = \varphi(l(\Theta))$$

holds for at least one j . If (5.1) fails, then Θ is instable from below.

PROOF. Assume that (5.1) holds with $j = u$. Then for a suitable $r \geq 1$ we have

$$(5.2) \quad t_{u+1} \dots t_{u+r} > T_r ,$$

where in general $T_s := F_s(\underline{t}) = t_1 \dots t_s$. Then u is an even number due to (1.4), (1.5).

Then, from Lemma 5., applying it with $Y = \underline{t}$,

$$(5.3) \quad \varphi(U_1(\Theta)) = \mathcal{M}_{\Theta}^{(0)} = \{\alpha : F_p(\sigma^j(\alpha)) < T_p\} ,$$

where $p = u + r$. (5.3) remains true for all those $\tilde{\Theta}$ for which in the notation $\tilde{t} = \varphi(l(\tilde{\Theta}))$ the relation $F_p(\tilde{t}) = T_p$ holds: Hence

$$(5.4) \quad \mathcal{M}_{\tilde{\Theta}}^{(0)} = \mathcal{M}_{\Theta}^{(0)} , \quad U_1(\Theta) = U_1(\tilde{\Theta})$$

is valid in an open interval J around Θ . We may assume that $\Theta \leq \frac{\sqrt{5} - 1}{2}$, since for bigger Θ , $U(\Theta) = \{0, 1\}$. Then $L(\Theta) < 2$. The whole set of the univoque sequences, written as infinite words over $\{0, 1\}$ can be given by the relation

$$(5.5) \quad U(\Theta) = \{0\} \cup \{1\} \cup \bigcup_{\substack{k=0 \\ l=0}}^{\infty} (1^k 0^l U_1(\Theta)) .$$

(5.5) it follows immediately from Lemma 1 and from (1.6). (5.4) and (5.5) implies $U(\Theta) = U(\Theta')$. The first part of the theorem is proved.

Assume that

$$(5.6) \quad \sigma^j(\underline{t}) \leq \underline{t} \quad (j = 1, 2, \dots)$$

holds.

If $\underline{t}$ is periodic, then the regular expansion of 1 in the base Θ is finite, therefore 1 is not univoque with respect to Θ , $\underline{t} \notin \mathcal{M}(U(\Theta))$. For an arbitrary $\Theta' < \Theta$, the sequence $\underline{t}' = \varphi(l(\Theta'))$ is larger than $\underline{t}$, thus by (5.6),

$$\sigma^j(\underline{t}) < \underline{t}' \quad (j = 0, 1, 2, \dots).$$

Consequently $\underline{t} \in \mathcal{M}(U(\Theta'))$.

Assume that $\underline{t}$ is not periodic. Then $\sigma_j(\underline{t}) < \underline{t}$ ($j = 1, 2, \dots$). We may assume that $t_1 \geq 2$. Let $\alpha = 11\underline{t}$. It is not the image of the quasi regular expansion of any number with respect to Θ , therefore $\alpha \notin \mathcal{M}(U(\Theta))$.

The theorem is completely proved.

Let $\mathcal{F}_0$ be the set of those Θ for which $\underline{t} = \varphi(l(\Theta))$ is periodic and (5.6) holds. Thus Θ is unstable from below if $\Theta \in \mathcal{F} \cup \mathcal{F}_0$.

Let $\Theta \in \mathcal{F} \cup \mathcal{F}_0$ and $w = \sup_{\eta \in H_\Theta^0} \eta$. If $\eta \in H_\Theta^{(0)}$, then $\varepsilon(\eta) \leq \delta(w)$,

where $\varepsilon(\eta)$ is the regular expansion of η , and $\delta(w)$ is the quasi regular expansion of w (≤ 1) in the base Θ . If $\eta = \sum_{k=1}^{\infty} \varepsilon_k(\eta)\Theta^k$ is univoque,

then so is $\eta_l = \sum_{k=1}^{\infty} \varepsilon_{k+l}(\eta)\Theta^k$, and thus $\eta_l = w$. Furthermore, in the case $\varepsilon_l(\eta) = 1$ we have $L(\Theta) - \eta_l \leq w$. Since w can be approximated by η , hence we have

$$\sigma^j(\varphi(\varepsilon(\eta))) < \kappa = \varphi(\delta(w)),$$

and even

$$\sigma^j(\kappa) \leq \kappa \quad (j = 0, 1, 2, \dots)$$

holds.

Let us assume first that $w = 1$. Let $\eta_\nu \in H_\Theta$ $\eta_\nu \uparrow 1$, $\kappa^{(\nu)} = \varphi(\varepsilon(\eta_\nu))$. The sequence $11\kappa^{(\nu)} \in \mathcal{M}_\Theta^{(0)}$ due to the fact that $\sigma^j(11\kappa^{(\nu)}) < \kappa = \bar{t}$. Furthermore, for an arbitrary $\Theta' > \Theta$, if $t' = \varphi(l(\Theta'))$, then $t' < t$, and $\sigma^2(11\kappa^{(\nu)}) < t'$ does not hold for at least one ν . Thus $11\kappa^{(\nu)}$ is not the image of a univoque number with respect to Θ' . Consequently Θ is unstable from above.

Let now $w < 1$. Let $\Theta'(> \Theta)$ be close to Θ so that for $t' = \varphi(l(\Theta'))$, $\kappa < \underline{t}' < \underline{t}$. If $Y = y_1 y_2 \dots \in \mathcal{M}_\Theta^{(0)}$, then $\sigma^j(Y) \leq \kappa < t'$, consequently $Y \in \mathcal{M}_{\Theta'}^{(0)}$. Thus one get immediately that $U(\Theta') = U(\Theta)$, i.e. Θ is stable from above.

6. On the structure of $\mathcal{M}_{\Theta}^{(0)}$ for stable numbers

6.1 Due to Lemma 5 and Theorem 4, if Θ is stable (from both sides), then

$$(6.1) \quad \mathcal{M}_{\Theta}^{(0)} = \{\alpha : F_k(\sigma^j(\alpha)) < T_k, j = 0, 1, 2, \dots\},$$

where $T_k = t_1 \dots t_k$, k is the least index for which $t_{u+1} \dots t_k > T_{k-u}$ holds for some $u \leq k$. Starting from (6.1), we can compute such an $Y = y_1 \dots y_s$ for which

$$(6.2) \quad \mathcal{M}_{\Theta}^{(0)} = \{\alpha : F_s(\sigma^j(\alpha)) \leq Y_s, j = 0, 1, 2, \dots\}.$$

We may assume furthermore that Y_s cannot be substituted by a smaller sequence $Y'_s (< Y_s)$, and with a shorter one. If Y_s is so chosen then there is an element $\alpha (\in \mathcal{M}_{\Theta}^{(0)})$ with prefix Y_s .

Let $Y_v = y_1 \dots y_v$ ($v = 1, \dots, s$). If Y_s is such a sequence, then

$$(6.3) \quad \begin{cases} y_{u+1} \dots y_{u+r} \leq Y_r \\ 0 \leq u < u+r \leq s \end{cases}$$

We assume the fulfilment of (6.3) for the whole section 6.

Notations: If $w \in \mathcal{K}_h$, $Z \subseteq \mathcal{M}$, then $wZ = \{wz : z \in Z\}$.

The union of the sets $B_r (\subseteq \mathcal{M})$ is denoted as $\sum B_r$. Let $(\mathcal{M}_{\Theta}^{(0)} =) X$ be the set of sequences α determined by the inequalities in the right hand side of (6.2).

$\lambda(w)$ denotes the length of w . Thus $\lambda(w) = h$ for $w \in \mathcal{K}_h$.

Let $w^k = w \dots \overset{k}{w}$, and $w^{\mathbb{N}} = ww \dots$, $w^0 =$ empty word.

Let $X_w := \{\alpha : \alpha \in X, F_{\lambda(w)}(\alpha) = w\}$.

Lemma 9. Let $w = r_1 \dots r_h$, $1 \leq h < s$ be such a sequence for which

$$(6.4) \quad r_{u+1} \dots r_h \leq Y_{h-u} \quad (u = 0, \dots, h-1)$$

holds. If u^* is the smallest integer u for which $r_{u^*+1} \dots r_h = Y_{h-u^*}$, then

$$(6.5) \quad X_w = r_1 \dots r_{u^*} X_{Y_{h-u^*}}.$$

If (6.4) holds with the strict inequality for every u , then

$$(6.6) \quad X_w = w.$$

If (6.4) fails to hold for some u , then X_w is empty.

PROOF. Clear. $w\alpha \in X$ if $X \ni \alpha = a_1 a_2 \dots$, and

$$(6.7)_u \quad r_{u+1} \dots r_h a_1 \dots a_{s-(h-u)} \leq Y_s, \quad u = 0, \dots, h-1$$

holds. If $r_{u+1} \dots r_h < Y_{h-u}$, then u is not a critical value, $(6.7)_u$ is valid for each α . The least critical value is $u = u^*$. It means that $w\alpha \in X$ if and only if $Y_{h-u^*}\alpha \in X$. Thus (6.5), (6.6) holds. The last assertion is obvious.

Lemma 10. *Let k be an odd integer.*

1. *If $y_k = 1$ and $Y_{2k-1} = Y_k Y_{k-1}$, $s \geq 2k - 1$, then*

$$(6.8) \quad X_{Y_{k-1}} = X_{Y_k} = Y_k X_{Y_k} = \{Y_k^{\mathbb{N}} = Y_k Y_k \dots\}.$$

2. *If $y_k > 1$ and $s \geq 2k + h(k+1)$, $h \geq 0$ and*

$$(6.9) \quad Y_{2k+h(k+1)} = Y_k (Y_{k-1}(y_k - 1)1)^h Y_k,$$

then

$$(6.10) \quad X_{Y_k} = \sum_{j=0}^h Y_k (Y_{k-1}(y_k - 1)1)^j X_{Y_k}.$$

Here Y_0 is thought to be the empty word.

PROOF. 1. Let $\alpha \in X$, $\alpha = a_1 a_2 \dots$, $F_{k-1}(\alpha) = Y_{k-1}$. Then $F_{2k-1}(\alpha) \leq Y_{2k-1}$, whence $a_k a_{k+1} \dots a_{2k-1} \leq 1 Y_{k-1}$. Thus $a_k = 1$ and $(Y_{k-1} \geq) a_{k+1} \dots a_{2k-1} \geq Y_{k-1}$, i.e. $a_{k+1} \dots a_{2k-1} = Y_{k-1}$, and (6.8) holds.

2. Assume the fulfilment of (6.9). Let $\alpha = a_1 a_2 \dots \in X$, $F_k(\alpha) = Y_k$. Then $a_{k+1} \dots a_{2k} a_{2k+1} \geq Y_{k-1}(y_k - 1)1$. Hence $a_{k+1} \dots a_{2k} = Y_{k-1}$ and $a_{2k} > y_k - 1$. Then either $a_{2k} = y_k$, or $a_{2k} = y_k - 1$ and $a_{2k+1} = 1$. In the first case $\alpha = Y_k Y_k \dots$, in the second $\alpha = Y_k Y_{k-1}(y_k - 1)1 \alpha_1$ and

$$F_{h(k+1)-1}(\alpha_1) \geq (Y_{k-1}(y_k - 1)1)^{h-1} Y_k.$$

Similarly as above we obtain that either $\alpha_1 = Y_k \alpha_2$, or $\alpha_1 = Y_{k-1}(y_k - 1)1 \alpha_2$, and in the latter case

$$F_{(h-1)(k+1)-1}(\alpha_2) \geq (Y_{k-2}(y_k - 1)1)^{h-2} Y_k.$$

Iterating this argument at most h times we obtain (6.10).

Lemma 11. *Let k be even:*

1. *If $y_k = y_1$, $s \geq 2k - 2$, and*

$$(6.11) \quad Y_{2k-2} = Y_{k-1} Y_{k-1},$$

then

$$(6.12) \quad X_{Y_{k-1}} = X_{Y_k} = Y_{k-1} X_{Y_{k-1}} = \{Y_{k-1}^{\mathbb{N}}\}.$$

2. *Let $y_k < y_1$, $s \geq 2k + (k+1)h$, $h \geq 0$, and*

$$(6.13) \quad Y_{2k+(k+1)h} = Y_k (1 Y_{k-1}(y_k + 1))^h Y_k,$$

then

$$(6.14) \quad X_{Y_k} = \sum_{j=0}^h Y_k (1Y_{k-1}(y_k + 1))^j X_{Y_k}.$$

PROOF. It is very similar to that of Lemma 10. We leave it for the reader.

Remark. If (6.8) or (6.11) holds, then Y_s can be reduced to Y_k . We can exclude these cases.

6.2. Our purpose is to find an appropriate partition of X , the components of which are characterized by the prefixes of their elements, such that the relations among them allow to define a strongly connected Mauldin Williams multigraph.

Definition. **1.** We say that Y_s is of type A if for each $k \leq s$ there exists a suitable finite word w for which

$$(6.15) \quad wX \subseteq X_{Y_k}$$

holds.

2. We say that Y_s is of type B_l if l is the smallest integer for which no finite word w exists with the property $wX \subseteq X_{Y_l}$.

Theorem 5. Let Y_s be of type B_k . Then, for odd k (6.9), for even k (6.13) holds.

PROOF. *1.* Let $k = 1$. If $s = 1$, then $X_{Y_1} = Y_1X$, i.e. (6.15) holds. Let $s \geq 2$. $y_2 \leq y_1$ follows from (6.3). If $y_2 = y_1$, then Y_s can be reduced to Y_1 in (6.2), but we assumed that Y_s is the shortest which gives (6.2). If $y_2 \leq y_1 - 2$, then $X_{Y_1} \supseteq X_{Y_1(y_2+1)} = y_1(y_2 + 1)X$. It remains the case $y_2 = y_1 - 1$. If $s = 2$, then $X_{Y_1} \supseteq X_{Y_2} = Y_2X$. Let $s \geq 3$. If $y_3 > 1$, then

$$Y_21X = X_{Y_21} \subseteq X_{Y_2} \subseteq X_{Y_1}.$$

Let $y_3 = 1$. If $s = 3$, then $X_{Y_1} \supseteq X_{Y_3} = Y_3X$. Let $s \geq 4$. If $y_4 < y_2$ then $X_{Y_1} \supseteq X_{Y_3y_2} = Y_3y_2X$. If $y_4 = y_1$, then Y_s is of type (6.9). It remains the case $y_4 = y_2$. Continuing this argument, since s is finite, we conclude that Y_s is of form (6.9).

2. Let $k > 1$, k odd, Y_s be of type B_k .

Due to the minimality of k $X_{Y_{k-1}} \neq X_{Y_k}$, thus there is an l , $l \neq y_k$ such that $X_{Y_{k-1}l} \neq \emptyset$. Since k odd, therefore $l < y_k$, and so $y_k > 1$.

From (6.3) we obtain that $y_{u+1} \dots y_{k-1} y_k \leq Y_{k-u}$, $y_{u+1} \dots y_{k-1} l \leq Y_{k-u}$, which for odd u implies that

$$y_{u+1} \dots y_{k-1} < Y_{k-u-1} \quad (u \text{ odd}).$$

Let

$$\alpha = (Y_{k-1}(y_k - 1)1)^{\mathbb{N}}.$$

Let us observe first that

$$(6.17) \quad F_K(\sigma^j(\alpha)) < Y_k \quad (j = 0, 1, 2, \dots)$$

holds. The sequence $\beta := Y_k \alpha$ cannot go through all the tests

$$(6.18) \quad F_s(\sigma^j(\beta)) \leq Y_s$$

Assume in contrary that (6.18) holds.

Let $\Delta = \beta(Y_{k-1}(y_k - 1)1)^r$, r large. Then X_Δ is non empty, $\beta\alpha \in X_\Delta$. Hence, similarly as in the proof of Lemma 9, we get that $\Delta = \Delta_1 Y_j$ with an appropriate $j \in \{0, 1, \dots, k-1\}$ such that $X_\Delta = \Delta_1 X_{Y_j}$. ($Y_0 =$ empty word, $X_{Y_0} = X$). Since $j < k$, therefore $w^*X \subseteq X_{Y_j}$ would imply that $X_{Y_k} \supseteq X_\Delta \supseteq \Delta_1 w^*X$, thus Y_s cannot be of type B_k .

Let v be the smallest j for which (6.18) fails to hold. Since $v \leq k$, taking into account (6.16), we get that v is an even number. Let $s_0 + 1 (\leq s)$ be the smallest number for which $F_{s_0+1}(\sigma^V(\beta)) > Y_{s_0+1}$. Then $F_{s_0}(\sigma^V(\beta)) = Y_{s_0}$. Furthermore $s_0 \geq k$. We prove that $v = 0$. If $v \neq 0$, then $y_{v+1} \dots y_k y_1 \dots y_v = Y_k$, i.e. $Y_k = Y_{k-v} Y_v = Y_v Y_{k-v}$. Consequently, if in a sequence $X \ni \gamma = c_1 c_2 \dots$, $c_1 \dots c_{k-v} = Y_{k-v}$, $k-v$ is odd, then $F_k(\gamma) \leq Y_k$ implies that $c_{k-v-1} \dots c_k = Y_v$. Hence we obtain that $X_{Y_{k-v}} = X_{Y_k}$; $k-v < k$, which contradicts to the minimality of k . We obtained that $v = 0$,

$$F_{s_0}(\beta) = Y_{s_0}, \quad F_{s_0+1}(\beta) > Y_{s_0+1}.$$

Hence we obtain that $F_{s_0-k}(\alpha) = y_{k+1} \dots y_{s_0}$ and $F_{s_0+1-k}(\beta) < y_{k+1} \dots y_{s_0+1}$. Let r be the largest integer for which $r(k+1) \leq s_0 - k$. Since $\sigma^{k+1}(\alpha) = \alpha$, we obtain that $F_{s_0-k-r(k+1)}(\alpha) = y_{k+r(k+1)+1} \dots y_{s_0}$ and that

$$F_{s_0-l}(\alpha) = y_{l+1} \dots y_{s_0}, \quad F_{s_0+1-l}(\alpha) < y_{l+1} \dots y_{s_0+1}, \quad l = k + r(k+1).$$

This can occur only if $y_{l+1} \dots y_{s_0+1} = Y_k$. This proves the theorem for odd k .

3. The case $k = \text{even}$ can be proved similarly. We omit the details.

6.3. Assume that Y_s is of type A . Let W be the set of the following finite words:

- (1) $i \in W$, if $i \in \{1, \dots, y_1 - 1\}$.
- (2) for every k , $1 \leq k < s$, $w = y_1 \dots y_k i \in W$, if $y_{u+1} \dots y_k i \leq Y_{k+1-u}$ ($u = 0, \dots, k-1$) and $i \neq y_{k+1}$, $i \leq y_1$ hold;
- (3) $Y_s \in W$.

Then X_w ($w \in W$) are mutually disjoint sets, $\sum X_w = X$.

Assume now that Y_s is of type B_k . Then W ($= W^{(k)}$) is defined as follows:

- (1) $i \in W^{(k)}$, if $i \in \{1, \dots, y_1 - 1\}$
- (2) for every j , $1 \leq j < k$, $w = y_1 \dots y_j l$ for which $y_{u+1} \dots y_j l \leq Y_{j+1-u}$ ($u = 0, \dots, j-1$) $l \neq y_{j+1}$, $l \leq y_1$ hold, let $w \in W^{(k)}$
- (3) $Y_k \in W^{(k)}$.

Then $\{X_w, w \in W^{(k)}\}$ is a subdivision of X into the mutually disjoint sets X_w .

Now we define the directed multigraph $G(W)$ (resp. $G(W^{(k)})$) over the set W (resp. $W^{(k)}$) as the set of nodes by the following relation.

Let first Y_s is of type A .

For $1 \leq i < y_1$ we have $X_i = iX$, thus

$$(6.19) \quad X_i = \sum_{w \in W} i X_w.$$

Let $z = Y_r i \in W$, $1 \leq r < s$. If h is the largest number ($h = 0$ is included with $X_{Y_0} = X$) for which $z = Y_{r-h} Y_h$, then

$$(6.20) \quad X_z = Y_{r-h} X_{Y_h} = \sum_{\substack{w \in W \\ \lambda(w) > h}} Y_{r-h} X_w,$$

(see Lemma 9). Especially in the case $h = 0$ we have

$$(6.21) \quad X_z = \sum_{w \in W} z X_w.$$

Finally we give a formula for X_{Y_s} . We have $X_{Y_s} = \sum_{1 \leq l \leq y_1}^* X_{Y_s l}$, where

the asterisk means that we sum only for those l for which additionally $y_{u+1} \dots y_s l \leq Y_{s+l-u}$ ($u = 1, 2, \dots, s-1$) holds. Let u_l^* be the smallest value, if any, for which $y_{u_l^*+1} \dots y_s l = Y_{s+l-u_l^*}$. For such an l we have $X_{Y_s l} = Y_{u_l^*} X_{Y_{s+1-u_l^*}}$. Such an l will be called of first kind. If such a

u does not exist (we say l is of second kind), then clearly $X_{Y_s l} = Y_s l X$. Consequently

$$(6.22) \quad X_{Y_s} = \sum_l' Y_{u_l^*} X_{Y_{s+1-u_l^*}} + \sum_l'' Y_s l X,$$

where in $\sum_l'$ we sum over the l of first kind, and in $\sum_l''$ over the others. At least one of the sums on the right hand side is non-empty.

Since $y_2 < y_1$ ($y_2 = y_1$ leads to the reducible case $s = 2$, $Y_2 = y_1 y_1$), therefore $s + 1 - u_l^* \leq s - 1$. Thus, by Lemma 9 we obtain that

$$(6.23) \quad X_{Y_s} = \sum_l' \sum_{\substack{w \in W \\ \lambda(w) > s+1-u_l^*}} Y_{u_l^*} X_w + \sum_l'' \sum_{w \in W} Y_s l X_w.$$

Construction of $G(W)$:

Let $z \in W$, $z \neq Y_s$. Then direct edges to that $w \in W$ which occur in the formula (6.19), (6.20), (6.21) respectively. The edge is labeled by the corresponding "coefficient" standing before X_w . For example, if z is subjected to (6.20), then we direct one edge to a $w \in W$ if $\lambda(w) > h$, and label this with Y_{r-h} . For $z = Y_s$ and $w \in W$ we direct as many edges from z to w as many times X_w occurs in the right hand side of (6.23), and label them with the corresponding coefficients $Y_{u_l^*}$ or $Y_s l$.

Theorem 6. *If Y_s is of type A then $G(W)$ is strongly connected.*

PROOF. The assertion is an immediate consequence of Lemma 9 and (6.22), whence we obtain that for each $w \in W$, $X_w \supseteq zX$ with an appropriate finite word z holds.

Assume now that Y_s is of type $B^{(k)}$. The construction of $G(W^{(k)})$ is similar as earlier. The relations (6.19), (6.20), (6.21) are valid. Instead of (6.22) we use the relation, (6.10), (6.14). Thus for odd k , from the point Y_k $h + 1$ loops are going out which are labelled by $Y_k(Y_{k-1}(y_k - 1)1)^j$ ($j = 0, \dots, h$). Thus from Y_k we cannot reach any element of $W^{(k)} \setminus \{Y_k\}$. Furthermore the graph $G(W^{(k)}) \setminus \{Y_k\}$ is strongly connected, due to the minimality condition in the definition B_k .

Theorem 7. *If Y_s is of type $B^{(k)}$ then $G(W^{(k)} \setminus \{Y_k\})$ is strongly connected.*

Example 1. Let $Y = Y_4 = 4213$. Then $W = \{1, 2, 3, 44, 43, 4214, 4213\}$
 We have: $X_i = \sum_{w \in W} iX_w$,

$$X_4 = 4 X_4 = 4 X_{44} + 4 X_{43} + 4 X_{4214} + 4 X_{4213}$$

$$X_{43} = 43 X = \sum_{w \in W} 43 X_w$$

$$X_{4214} = 421 X_4 = 421 X_{44} + 421 X_{43} + 421 X_{4214} + 421 X_{4223}$$

$$X_{4213} = 4213 X = \sum_{w \in W} 4213 X_w .$$

We draw $G(W)$ in a simplified form. If $z \in W$ is such a node which is subjected to the formula $X_z = \sum_{w \in W} zX^w$, then the corresponding edges

are not drawn and the nodes are marked with asterisk.

Then $G(W)$

Example 2. Let $Y_{21} = 322(3211)^3 322333$. Computing X we can substitute Y_{21} by $Y_{18} = 322(3211)^3 322$. Then Y is of type $B^{(3)}$. $W = W^{(3)} = \{1, 2, 33, 321, 322\}$

We have: $X_z = \sum_{w \in W} X_w$ if $w = 1; 2; 321$. We denote them with z^* . Fur-

thermore $X_{33} = 3 X_{33} + 3 X_{321} + 3 X_{322}$, $X_{322} = \sum_{j=0}^3 322(3211)^j X_{322}$.

Thus the simplified form of $G(W^{(3)})$ is the following

6.4. Let $\Theta + \Theta^2 < 1$. If V is an arbitrary subset of $H(= H_\Theta)$, then $L - V = \{L - x : x \in V\} \subseteq H^*$.
For some $\alpha \in X$ let $\Psi(\alpha) := \langle \varphi^{-1}(\alpha), \Theta \rangle$. If $x = \Psi(\alpha)$, $y = \Psi(\sigma(\alpha))$, then

$$x = \Theta + \cdots + \Theta^{a_1} + \Theta^{a_1}(L - y).$$

The assumption $\Theta + \Theta^2 < 1$ guarantees that if $x \in [\Theta, 1)$ then $y \in [\Theta, 1)$. Thus, if

$$K = iQ, \quad K, Q \subseteq X,$$

then

$$\Psi(K) = \Theta + \cdots + \Theta^i + \Theta^i(L - \Psi(Q)).$$

For an arbitrary finite word z we define the similarity $f_z : \mathbb{R} \rightarrow \mathbb{R}$ recursively, by the following rules:

- (1) For $z = i \in \mathbb{N}$ let $f_i(x) = \Theta + \cdots + \Theta^i + \Theta^i(L - x)$.

- (2) If $f_{i_1 \dots i_j}$ are defined for every $j \leq r$ and every $i_1 \dots i_r \in \mathcal{K}_r$, then $f_{i_1 \dots i_r i_{r+1}}(x) = f_{i_1}(f_{i_2 \dots i_{r+1}}(x))$. It is clear that $f_{i_1 \dots i_r}$ is a linear function with contraction factor

$$r(i_1 \dots i_r) = \Theta^{i_1 + \dots + i_r}.$$

Let $H(w) := \Psi(X_w)$ defined for finite words w . Then $\{H(w) \mid w \in W\}$ is a partition of $H(= \Psi(X))$ into disjoint non-empty compact sets.

The multigraph $G(W)$ ($G(W^{(k)})$) generates the following relation among them:

$$(6.24) \quad H(z) = \bigcup f_e(H(w)) \quad (z \in W)$$

where in the right hand side we sum over all edges leaving z . e denotes the label of the edge and w the endpoint.

Assume that Y_s is of type A . Then $G(W)$ is a Mauldin-Williams graph. The open set condition (due to Moran) clearly satisfied, therefore the similarity dimension equals to the Hausdorff dimension of the components $H(z)$. All of the components have positive finite measures (with respect to the σ -dimensional Hausdorff-measure μ_σ). σ can be computed as the only nonnegative real number for which the equation system

$$(6.25) \quad q_z^\sigma = \sum r(e)^\sigma q_w^\sigma \quad (z \in W)$$

has positive $q_z(z \in W)$ solution.

Let us consider now the case when Y_s is of type $B^{(k)}$. Assume that $t_1 \geq 3$. Let $m = t_1 + \dots + t_k$. The set $H(Y_k)$ is self-similar, it is the attractor of the iterated function system

$$H(Y_k) = \bigcup f_e(H(Y_k)),$$

where in the right hand side we sum over the loops coinciding Y_k . Thus its similarity dimension = Hausdorff dimension = λ can be computed from

$$1 = \sum_e r(e)^\lambda.$$

Since $r(e)$ run over the values $\Theta^m, \Theta^{2m}, \dots, \Theta^{hm}$, for odd k , and over the values $\Theta^{m+j(m+2)}$ ($j = 0, \dots, h$) for even k , we have

$$(6.25)_{k \text{ odd}} \quad 1 = \Theta^{m\lambda} + \Theta^{2m\lambda} + \dots + \Theta^{(h+1)m\lambda}$$

$$(6.25)_{k \text{ even}} \quad 1 = \Theta^{m\lambda} + \Theta^{m\lambda+(m+2)\lambda} + \dots + \Theta^{m\lambda+h(m+2)\lambda}$$

Let $X^{(1)} := \{\alpha : F_k(\alpha) < Y_k\}$, $\Psi(X^{(1)}) = \tilde{H}$

It is clear that $\tilde{H}^* \subseteq H$.

Let $X^{(2)} = X \setminus X^{(1)}$. Then $X^{(2)}$ can be represented as the union of countable many sets of form $z X_{Y_k}$. Since the $\lambda + \varepsilon$ dimensional measure of $\Psi(z X_{Y_k})$ is zero for all of these subsets, therefore

$$\mu_{\lambda+\varepsilon}(X^{(2)}) = 0 \quad \text{for every } \varepsilon > 0.$$

Let $X_w^{(1)} = X_w \setminus X^{(2)}$ defined for $w \in W^k \setminus Y_k$.

Then $X^{(1)}$ is the union of the disjoint sets $X_w^{(1)}$ the relation among them are defined by the strongly connected multigraph $G(W \setminus \{Y_k\})$. Thus the Hausdorff-dimension σ of the sets $\Psi(X^{(1)})$, $\Psi(X_w^{(1)})$ can be computed, $\mu_\sigma(\Psi(X_w^{(1)})) > 0$.

If we can prove that $\sigma > \lambda$, then we conclude that

$$\infty > \mu_\sigma(X_w) > 0 \quad \text{if } w \neq Y_k, 0 < \mu_\sigma(X) < \infty.$$

Let $Z = \{\alpha : \alpha = a_1 a_2 \dots, 0 \leq a_i < y_1\}$. Then $Z \subseteq X^{(1)}$, $D := \Psi(Z) \subseteq H^*$. Furthermore D is a self-similar set,

$$D = \sum_{i=1}^{y_1-1} f_i(D),$$

its Hausdorff dimension is that η for which

$$1 = \sum_{i=1}^{y_1-1} \Theta^{i\eta}$$

holds. Since $\eta \leq \sigma$, if is enough to prove that $\lambda < \eta$.

But this is clear, if $y_1 \geq 3$. λ as a function of h in (6.25) is monotonically increasing. Thus $\lambda \leq \lambda_0$, where

$$1 = \frac{\Theta^{m\lambda_0}}{1 - \Theta^{m\lambda_0}}, \quad \text{i.e. } \Theta^{\lambda_0} = \left(\frac{1}{2}\right)^{1/m}.$$

Since $m \geq 3$, therefore $\Theta^{\lambda_0} > 3/4$, $\frac{3}{4} + \left(\frac{3}{4}\right)^2 > 1$, consequently $\eta > \lambda_0$.

Finally we observe that the above method is applicable even in the case $y_1 = 2$. If $s = 1$, then this is clear. If $s \geq 2$ and $y_1 = y_2$, then Y_3 can be reduced to $Y_2 = 22$, and we get that $X_1 = 1X$, $X_2 = 2X_2$, which implies that $H(\Theta)$ is a countable set, therefore its Hausdorff dimension equals to zero. We should consider only the cases when Y_s is of type B . Let $y_2 = 1$. Assume that Y_s is of type B_k . If $k = 1$, then Y_s has the prefix $2(11)^h 2$ with some integer $h \geq 0$. Hence we obtain that

$$X_1 = 1X, \quad X_{2(11)^j 2} = 2(11)^j X_2 \quad (j = 0, \dots, h)$$

whence

$$\Psi(X_2) = \sum_{j=0}^h f_{2(11)^j}(\Psi(X_2))$$

follows. Then $\Psi(X_2)$ is a self-similar set, its Hausdorff dimension λ can be computed as the solution of the equation

$$1 = \sum_{j=0}^h \Theta^{\lambda(2+2j)}.$$

We have $0 < \lambda < 1$.

Furthermore $0 < \mu_\lambda(\Psi(X_2)) < \infty$. Since

$$X_1 = \{1^{\mathbb{N}}\} + \sum_{l=1}^{\infty} 1^l X_2,$$

$$\text{therefore } \mu_\lambda(\Psi(X_1)) = \mu_\lambda(\Psi(1^{\mathbb{N}})) + \sum_{l=1}^{\infty} \mu_\lambda(\Psi(1^l X_2)) = 0 + \sum_{l=1}^{\infty} \Theta^{l\lambda} \mu_\lambda(\Psi(X_2)),$$

thus $0 < \mu_\lambda(\Psi(X_1)) < \infty$.

Assume that $k \geq 2$. If k is odd and $y_k = 1$, or if k is even and $y_k = y_1 (= 2)$, then Y_s is of form (6.8) or (6.11) respectively, thus it is reducible. These cases can be excluded.

Let $k (\geq 3)$ be odd. Then k is at least so large then the index of the second occurrence of 2 in $y_1 y_2 \dots$. Thus $Y_s = 2 1^r 2 \dots$ and $k \geq r + 2$. Since (6.9) holds, therefore $m := y_1 + \dots + y_k \geq r + 4$. The Hausdorff dimension λ of $\Psi(X_{Y_k})$ can be computed from the equation

$$(6.26) \quad \Theta^{m\lambda} + \dots + \Theta^{m\lambda(h+1)} = 1.$$

Let $X' = \{\alpha : F_{r+2}(\sigma^j(\alpha)) < 21^r 2\}$. If we prove that the Hausdorff dimension of $\Psi(X')$ is larger than λ , then we can compute it from the Mauldin-Williams graph omitting the node Y_k .

Let X'' be the attractor of

$$(6.27) \quad X'' = \sum_{\substack{l=0 \\ 2l < r}} 2 1^{2l} X''.$$

Then $X'' \subseteq X'$. The dimension σ of $\Psi(X'')$ is obtained from

$$(6.28) \quad 1 = \sum_{\substack{l=0 \\ 2l < r}} \Theta^{\sigma(2l+2)}.$$

$\sigma \leq \lambda$ would imply that $(\xi =) \Theta^\sigma \geq \Theta^\lambda (= \eta)$. From (6.26), (6.28) we can get immediately that it is impossible if $m \geq 6$, i.e. if $r \geq 2$. It remains the

case $r = 1$.

Let $k = \text{even}$ of form (6.13). Then $y_k = 1$. If $2\ 1^r 2$ is a prefix in Y_s , then $k \geq r - 1$, and so $m = y_1 + \dots + y_k \geq r$. Now the Hausdorff dimension λ of $\Psi(X_{Y_k})$ is computed from the equation,

$$(6.29) \quad \Theta^{m\lambda} + \Theta^{m\lambda+(m+2)\lambda} + \dots + \Theta^{m\lambda+h(m+2)\lambda} = 1.$$

If r is even, then Y_s is of type $B^{(1)}$ which was considered earlier. Let r be odd. Let us consider the set X'' defined by (6.27). The Hausdorff dimension of $\Psi(X'')$ is given as that σ for which (6.28) holds. Let $\xi = \Theta^\sigma$, $\eta = \Theta^\lambda$. Let $r > 1$. The smallest value of η is getting by for $h \rightarrow \infty$, i.e. for $1 = \eta^m + \eta^{2m+2}$. Furthermore, from (6.28), $1 = \xi^2 + \xi^4 + \dots + \xi^{r+1}$, and this implies that $\xi < \eta$ for $m \geq 3$.

Finally we consider the case when $r = 1$, $k = \text{even}$, Y_s is of form $B^{(k)}$. If $k = 2$, then Y_s is of form (6.13), i.e.

$$Y_{3h+6} = 21(121)^h 22, \text{ and } F_3(Y_s) \neq 212.$$

Then $k \geq 4$. Consequently either $k = 4$ and $Y_{8+5h} = 2121(12122)^h 2121$ for some $h \geq 0$ or $k \geq 6$.

Let $k = 4$, $W^{(0)} = \{1, 22, 211\}$, $X' = X'_1 + X'_{22} + X'_{211}$ defined by $X'_1 = 1X'$, $X''_{22} = 2X''_2$, $X'_{211} = 211X'$.

The Hausdorff-dimension σ of $\Psi(X')$ can be computed from: $q_1^\sigma = \Theta^\sigma R$, $q_{22}^\sigma = \Theta^{2\sigma}(R - q_1^\sigma)$, $q_{211}^\sigma = \Theta^{4\sigma} R$, $R = q_1^\sigma + q_{22}^\sigma + q_{211}^\sigma$ (> 0), i.e. it is the solution of the equation $1 = \Theta^\sigma + \Theta^{2\sigma} - \Theta^{3\sigma} + \Theta^{4\sigma}$. Since $m = 6$, similarly as above we deduce that $\sigma > \lambda$. The case $k \geq 6$ is similar, the proof is left to the reader.

6.5. Now we summarize our result for the computation of the Hausdorff dimension of H .

Assume that Y_s defining (6.2) cannot be further reduced. Then we have:

1. If Y_s is of type A , then the Hausdorff dimension of H_Θ equals to the similarity dimension of the Mauldin-Williams graph $G(W)$, $G(W)$ is strongly connected.
2. Assume that Y_s is of type $B^{(k)}$ and that $Y_{2j+2} \neq 2\ 1^{2j} 2$ ($j = 0, 1, \dots$).

Then the Hausdorff-dimension σ of $\Psi(X)$ is the same as the similarity dimension of (the strongly connected) graph $G(W \mid \{Y_k\})$.

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