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A Steinhaus type theorem

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Abstract. In this work a theorem of Steinhaus which states that the sum $A_1 + A_2$ of two measurable sets with positive Lebesgue measure contains an interval is generalized to the case

$$F(A_1, A_2, \dots, A_n)$$

where F is a continuously differentiable function mapping the product of Euclidean spaces with different dimensions into a Euclidean space.

A famous theorem of STEINHAUS [13] asserts that, for any measurable sets $A, B \subset \mathbb{R}$ with positive Lebesgue measure, $A+B$ has an interior point. This theorem allows various generalizations and modifications. A large part of these papers are based on Weil's idea [14] that the convolution of the characteristic functions χ_A and χ_B (in case A and B has finite measure) is a continuous function, hence the function

$$t \mapsto \mu(A \cap (t - B)) = \int \chi_B(t-y)\chi_A(y)\, d\mu(y)$$

is continuous and as follows from Fubini's theorem, not everywhere zero. This means that $A + B$ contains a nonvoid open set. This proof works directly when μ is a Haar measure on a locally compact Hausdorff group.

In the generalizations the following problem is treated: if we replace the addition by a binary operation $F(x, y)$, under what conditions on F can we prove that $F(A, B)$ contains a nonvoid open set? The first step was done by ERDŐS and OXTOBY [3] proving in the case $x, y \in \mathbb{R}$ that, if F is a continuously differentiable function with nonvanishing partial derivates, then $F(A, B)$ contains a nonvoid open set.

Further generalizations detail the case when x and y are from different topological measure spaces and F satisfies certain solvability conditions in

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x and y. See in this direction KUCZMA [7] and SANDER [11]. Sander has pointed out that one of the sets A, B may be nonmeasurable. These results apply to the case when $x, y \in \mathbb{R}^n$ and F is a continuously differentiable function of which the partial derivates are nonsingular.

A Steinhaus-type result for more than two sets is implicitly used in JÁRAI [5]. The most important feature of Weil's idea has been generalized by KRAUSZ in [9] proving that the function

$$t \mapsto \mu(\cap_{i=1}^{n} g_{i,t}^{-1}(A_i))$$

is continuous if the functions $g_{i,t}$ does not map sets with positive measure into zero sets and depend smoothly on the parameter t. We note that several variants of Steinhaus' theorem have applications in the theory of functional equations. More detailed references may be found in KUCZMA [7], KUCZMA and KUCZMA [8] and SANDER [12].

In this paper we first replace the theorem stating that the convolution

$$t \mapsto \int_Y f_1(t-y) f_2(y)\, d\mu(y)$$

is continuous with a theorem that the function

$$t \mapsto \int_Y h(f_1(g_1(t,y)), \dots , f_n(g_n(t,y)))\, d\mu(y)$$

is continuous. Second, we generalize Steinhaus' theorem to the case

$$F(A_1, A_2, \dots , A_n)$$

where F is a continuously differentiable function mapping the product of Euclidean spaces with different dimensions into a Euclidean space. The results of Weil and Krausz and the well-known case $F(A, B)$, $A, B \subset \mathbb{R}^n$ are corollaries.

We shall use the terminology of FEDERER [4] about measure theory. By a measure on X we mean a countable subadditive nonnegative function defined on 2^X; (an outer measure in another terminology). λ^k shall denote the k dimensional Lebesgue measure on $\mathbb{R}^k$. Other notions of analysis are used in the usual way. In case of doubt we refer the reader to DIEUDONNÉ [2].

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Theorem 1. *Let T, Y and X_i $(i = 1, 2, \dots, n)$ be locally compact Hausdorff spaces, and let Z, Z_i $(i = 1, 2, \dots, n)$ be separable Banach spaces. Let ν and μ_i be finite Radon measures over Y and X_i, respectively. Consider the functions $f_i : X_i \to Z_i$, $g_i : T \times Y \to X_i$, $h : Z_1 \times \ldots \times Z_n \to Z$. Suppose that, with the notations*

$$g_{i,t}(y) = g_i(t, y) \quad \text{whenever } (t, y) \in T \times Y,$$

the following conditions hold:

(1) *h maps bounded subsets into bounded subsets, and is uniformly continuous on every bounded subset;*

(2) $f_i \in \mathcal{L}^\infty(\mu_i) \qquad (i = 1, 2, \dots, n)$;

(3) *g is continuous and for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mu_i(g_{i,t}(B)) \geq \delta$ whenever $B \subset Y$, $\nu(B) \geq \varepsilon$, $t \in T$ and $1 \leq i \leq n$.*

Then the function

$$f(t) = \int_Y h(f_1(g_1(t, y)), \dots, f_n(g_n(t, y)))\, d\nu(y)$$

is continuous on T.

Proof. First we prove that the integral exists. We may replace each f_i by a bounded Borel function defined an all of X_i, which is almost equal to f_i. This switch does not change the integral, because, by (3), the set of points y for which the value of $f_i(g_i(t, y))$ are changed, has measure 0.

Hence we may assume that the functions f_i are bounded Borel functions. By (1) the function

$$y \mapsto h(f_1(g_1(t, y)), \dots, f_n(g_n(t, y))) \tag{4}$$

is a Borel function whenever $t \in T$ is fixed, and its image is a bounded subset of Z. Hence the integral exists whenever $t \in T$.

Now let $\varepsilon > 0$ and $t_0 \in T$. Let us choose a real number $M > 0$, for which the image of (4) is contained in the closed ball with center 0 and radius M. By (3), there exists a $\delta > 0$ such that $B \subset Y$, $\nu(B) \geq \varepsilon' = \varepsilon/(16Mn)$, $t \in T$ and $1 \leq i \leq n$ implies $\mu_i(g_{i,t}(B)) \geq \delta$. Let us choose a compact set $C \subset Y$ for which $\nu(Y \setminus C) < \varepsilon/(8M)$. Let V be a neighbourhood of t_0 with compact closure in T. By Luzin's theorem there exists a compact set C_i in X_i for which $\mu_i(X_i \setminus C_i) < \delta$ and $f_i|C_i$ is continuous. Let us choose uniformities on the spaces T, Y, $X_1, \dots, X_n$ compatible with its topology. By (1) there exists an $\alpha > 0$ such that

$$|h(z_1, \dots, z_n) - h(z_1', \dots, z_n')| < \frac{\varepsilon}{2\nu(Y)}$$

whenever $z_i, z_i' \in Z_i$, $|z_i - z_i'| < \alpha$ and $|z_i|, |z_i'| \le \|f_i\|_u$, where $\| \; \|_u$ is the uniform norm. Because of the uniform continuity of $f_i|C_i$ there exists a reflexive symmetric relation β_i in the uniformity of X_i such that

$$|f_i(x_i) - f_i(x_i')| < \alpha$$

whenever $x_i, x_i' \in C_i$ and x_i and x_i' are β_i-near, that is, $(x_i, x_i') \in \beta_i$. Because of the uniform continuity of g_i on the compact set $\bar{V} \times C$, there exists a reflexive symmetric relation γ in the uniformity of Y and a reflexive symmetric relation η in the uniformity of T such that $g_i(t_0, y)$ and $g_i(t, y')$ are β_i-near in X_i whenever t and t_0 are η-near in $\bar{V}$ and y and y' are γ-near in C. Now let t be an element of V which is η-near to t_0, and let

$$K = \bigcap_{i=1}^{n} g_{i,t_0}^{-1}(C_i) \cap \bigcap_{i=1}^{n} g_{i,t}^{-1}(C_i) \cap C.$$

Then

$$Y \setminus K = Y \setminus C \cup \left(\bigcup_{i=1}^{n} g_{i,t}^{-1}(X_i \setminus C_i) \right) \cup \left(\bigcup_{i=1}^{n} g_{i,t}^{-1}(X_i \setminus C_i) \right)$$

and hence (using (3) and that $\mu_i(X_i \setminus C_i) < \delta$)

$$\nu(Y \setminus K) < \frac{\varepsilon}{8M} + n\frac{\varepsilon}{16Mn} + n\frac{\varepsilon}{16Mn} = \frac{\varepsilon}{4M}. \tag{5}$$

Using this, we have

$$\begin{aligned}
|f(t) - f(t_0)| \le & \int_Y |h(f_1(g_1(t,y)), \ldots, f_n(g_n(t,y))) \\
& - h(f_1(g_1(t_0,y)) \ldots, f_n(g_n(t_0,y)))| \, d\nu(y) \\
= & \int_{Y \setminus K} |h(f_1(g_1(t,y)), \ldots, f_n(g_n(t,y))) \\
& - h(f_1(g_1(t_0,y)), \ldots, f_n(g_n(t_0,y)))| \, d\nu(y) \\
& + \int_K |h(f_1(g_1(t,y)), \ldots, f_n(g_n(t,y))) \\
& - h(f_1(g_1(t_0,y)), \ldots, f_n(g_n(t_0,y)))| \, d\nu(y).
\end{aligned}$$

By (5) the first term on the right side is not greater than $2M\varepsilon/(4M) = \varepsilon/2$. By the choice of K, α, $\beta_1, \ldots, \beta_n$, γ and η, the second term on the right side is not greater than $\nu(Y)\varepsilon/(2\nu(Y)) = \varepsilon/2$. $\square$

As an illustration how one can get earlier results from the general theorem above, we give two examples. The following result was first proved by KRAUSZ in [9] (under the condition that every A_i is measurable).

Corollary 1. *Let X_i, Y, T, μ_i, ν and g_i be the same as in Theorem 1. Suppose that condition (3) of Theorem 1 is satisfied, and let A_i be a subset of X_i. Suppose that A_i is μ_i measurable if $2 \le i \le n$. Then the function*

$$f(t) = \nu\left(\bigcap_{i=1}^{n} g_{i,t}^{-1}(A_i)\right) \quad \text{whenever } t \in T$$

is continuous on T.

PROOF. By condition (3) of Theorem 1 the set $g_{1,t}^{-1}(B_1)$ is a ν hull of $g_{1,t}^{-1}(A_1)$ whenever B_1 is a μ_1 hull of A_1. Hence

$$f(t) = \int_Y \chi_{B_1}(g_1(t,y)) \cdot \chi_{A_2}(g_2(t,y)) \cdot \ldots \cdot \chi_{A_n}(g_n(t,y))\, d\nu(y)$$

where χ_{A_i} is the characteristic function of A_i and χ_{B_1} is the characteristic function of B_1. $\square$

The following corollary is well-known.

Corollary 2. *Let G be a locally compact Hausdorff group and let μ be a left Haar measure on G. Let A_i $(i = 1, 2, \ldots, n)$ be a subset of G with finite measure. Suppose that A_i is μ measurable if $2 \le i \le n$. Then the mapping*

$$(t_1, \ldots, t_n) \mapsto \mu(t_1 A_1 \cap t_2 A_2 \cap \ldots \cap t_n A_n)$$

of G^n into $\mathbb{R}$ is continuous.

PROOF. Because the replacement of A_1 by a μ hull does not change this function, we may suppose that A_1 is μ measurable too. If $\mu(A_i)$ is equal to 0 for some i, then there is nothing to prove. If $\mu(A_i) > 0$ for each i, then let $\varepsilon > 0$ and T be an open subset of G^n with compact closure. Let us choose compact sets C_i such that $C_i \subset A_i$ and $\mu(A_i \setminus C_i) < \varepsilon$. Let

$$Y = \cup\{t_1 C_1 \cup t_2 C_2 \cup \ldots \cup t_n C_n : (t_1, \ldots, t_n) \in T\}.$$

Then Y is an open subset of G with compact closure as well as the sets

$$X_i = \{t_i^{-1} y : y \in Y, (t_1, \ldots, t_n) \in T\}.$$

Let

$$g_i(t, y) = t_i^{-1} y \quad \text{if } (t_1, \ldots, t_n) \in T \text{ and } y \in Y.$$

Applying the preceding corollary we get that the function

$$t \mapsto \mu(t_1 C_1 \cap t_2 C_2 \cap \ldots \cap t_n C_n)$$

is continuous on T. But

$$\begin{aligned} 0 &\le \mu(t_1 A_1 \cap t_2 A_2 \cap \ldots \cap (t_n A_n) - \mu(t_1 C_1 \cap t_2 C_2 \cap \ldots \cap t_n C_n) \\ &\le \sum_{i=1}^{n} \mu(t_i A_i \setminus t_i C_i) \le n\varepsilon. \end{aligned}$$

Hence

$$(t_1, \ldots , t_n) \mapsto \mu(t_1 A_1 \cap \ldots \cap t_n A_n)$$

is the uniform limit of continuous functions, and so itself is continuous. □

In the following Lemma which will be needed for the proof of our main result, we give sufficient conditions for the validity of condition (3) in Theorem 1.

Lemma 1. *Let Y be an open subset of $\mathbb{R}^k$, let T be a topological space, $c \in Y$ and $d \in T$. Let $g : T \times Y \to \mathbb{R}^r$ be a continuous function and suppose that $\dfrac{\partial g}{\partial y}$ is continuous and*

$$\operatorname{rank}\left(\frac{\partial g}{\partial y}(d, c)\right) = r.$$

Then there exist open neighbourhoods Y^* and T^* of c and d, respectively, and there exists a constant $0 < C < \infty$ such that $Y^* \subset Y$, $T^* \subset T$ and

$$\lambda^k(B) \le \lambda^r(g_t(B)) C (\operatorname{diam} B)^{k-r} \tag{1}$$

whenever $B \subset Y^*$ and $t \in T^*$. (Here $\operatorname{diam} B$ denotes the diameter of the set B.) Moreover,

(2) *$g_t^{-1}(A) \cap Y^*$ is λ^k measurable whenever A is a λ^r measurable subset of $\mathbb{R}^r$ and $t \in T^*$.*

Proof. Let $q = k - r$ and let us divide the coordinates of $y = (y_1, \ldots , y_k)$ into two groups $y' = (y'_1, \ldots , y'_q)$ and $y'' = (y''_1, \ldots , y''_r)$ so that the equation

$$\det\left(\frac{\partial g}{\partial y''}(d, c)\right) = \det\left(\frac{\partial g}{\partial y''}(d, c', c'')\right) \ne 0$$

be satisfied. Let

$$g_{t,y'}(y'') = g(t, y', y'') = g(t, y)$$

and let us introduce the notation

$$L(t,y') = \frac{\partial g}{\partial y''}(t,y',c'').$$

Using the proof of the inverse function theorem (see RUDIN [10], Theorem 9.24) we obtain that, if Y'' is an open ball with centre c'' in $\mathbb{R}^r$, $t \in T$, $(y',y'') \in Y$ and

$$\left\| \frac{\partial g}{\partial y''}(t,y',y'') - L(t,y') \right\| < \frac{1}{2\|L(t,y')^{-1}\|} \tag{3}$$

whenever $y'' \in Y''$, then $g_{t,y'}$ is a homeomorphic mapping of Y'' onto an open subset $U(t,y')$ of $\mathbb{R}^r$. (Here $\| \ \|$ is the operator norm.) Now let

$$0 < \beta < \frac{1}{2\|L(d,c')^{-1}\|}$$

and

$$0 < \gamma < \left| \det \frac{\partial g}{\partial y''}(d,c',c'') \right|. \tag{4}$$

Using the continuity of the expressions in (3) and (4) we can choose an open ball Y'' with centre c'' and open sets Y' and T^* such that $d \in T^*$, $c' \in Y'$, $Y^* = Y' \times Y'' \subset Y$, moreover $t \in T^*$, $y' \in Y'$, $y'' \in Y''$ implies that

$$\begin{aligned} &\left\| \frac{\partial g}{\partial y''}(t,y',y'') - L(t,y') \right\| < \beta; \\ &\beta < \frac{1}{2\|L(t,y')^{-1}\|}; \\ &\gamma < \left| \det \frac{\partial g}{\partial y''}(t,y',y'') \right|. \end{aligned}$$

Let $\alpha(q)$ denote the λ^q measure of the q dimensional unit ball ($\alpha(0) = 1$). We are going to prove that

$$\lambda^k(B) \le \lambda^r(g_t(B)) \frac{\alpha(q)}{\gamma} (\operatorname{diam} B)^{k-r}$$

whenever $B \subset Y^*$ and $t \in T^*$. Let $R = \operatorname{diam} B$. Then there exists a closed ball V with radius R in $\mathbb{R}^q$ such that

$$B \subset (V \cap Y') \times Y''.$$

Suppose to the contrary that there exists a $t \in T^*$ for which

$$\lambda^k(B) > \lambda^r(g_t(B)) C R^q$$

where $C = \alpha(q)/\gamma$. Then we can choose an open set U for which $g_t(B) \subset U$ and

$$\lambda^k(B) > \lambda^r(U)CR^q.$$

Let

$$B^* = g_t^{-1}(U) \cap ((V \cap Y') \times Y'').$$

Then $B \subset B^*$, B^* is a Borel set and $g_t(B^*) \subset U$, that is

$$\lambda^r(g_t(B^*))CR^q < \lambda^k(B) \le \lambda^k(B^*).$$

We are going to prove that this is impossible. Let

$$B^*_{y'} = \{y'' : (y', y'') \in B^*\} \quad \text{if } y' \in V \cap Y'.$$

Using the theorem concerning transformations of integrals we have that

$$\lambda^r(g_t(B^*)) \ge \lambda^r(g_{t,y'}(B^*_{y'})) = \int_{B^*_{y'}} \left|\det \frac{\partial g}{\partial y''}(t, y', y'')\right| d\lambda^r(y'') \ge \gamma\lambda^r(B^*_{y'})$$

whenever $y' \in V \cap Y'$. By Fubini's theorem

$$\begin{aligned}\lambda^k(B^*) &= \int_{V\cap Y'} \lambda^r(B^*_{y'})\, d\lambda^q(y') \\ &\le \frac{\lambda^r(g_t(B^*))}{\gamma}\lambda^q(V) = \lambda^r(g_t(B^*))CR^q\end{aligned}$$

which is a contradiction. Hence the proof of (1) is complete.

To prove (2) we may clearly suppose that A is bounded. Let C be a Borel set for which $A \subset C$ and $\lambda^r(C \setminus A) = 0$. Then

$$g_t^{-1}(A) \cap Y^* = (g_t^{-1}(C) \cap Y^*) \setminus (g_t^{-1}(C \setminus A) \cap Y^*).$$

Since $g_t^{-1}(C) \cap Y^*$ is a Borel set and $g_t^{-1}(C \setminus A) \cap Y^*$ has measure 0 by (1), the set $g_t^{-1}(A) \cap Y^*$ is λ^k measurable. □

Lemma 2. *Under the conditions of Lemma 1, if a subset D of $\mathbb{R}^r$ has density 1 in the point $g(d,c)$ then $g_d^{-1}(D) \cap Y^*$ has density 1 in the point c.*

Proof. By the continuity of $\dfrac{\partial g}{\partial y}(d, y)$ the function g_d satisfies the Lipschitz condition on a neighbourhood of c. Hence there exist a $\gamma > 0$ and an $0 < M < \infty$ such that $y \in Y^*$ and

$$|g(d, y) - g(d, c)| \le M|y - c|$$

whenever $|y-c|<\gamma$. Let $\alpha(k)$ and $\alpha(r)$ denote the λ^k and λ^r measures of the k and r dimensional unit balls, respectively. Let $\varepsilon>0$, and let

$$0<\delta<\frac{\varepsilon\alpha(k)}{M^r 2^{k-r}\alpha(r)C}.$$

Let us choose a $\beta>0$ such that, whenever V is a closed ball with centre $g(d,c)$ and radius less than β, then

$$\lambda^r(V\cap D)\geq(1-\delta)\lambda^r(V).$$

We prove that if W is a closed ball in Y^* with centre c and radius less than γ and β/M, then

$$\lambda^k(W\cap g_d^{-1}(D))\geq(1-\varepsilon)\lambda^k(W).$$

Suppose to the contrary that for a such W with radius R

$$\lambda^k(W\cap g_d^{-1}(D))<(1-\varepsilon)\lambda^k(W).$$

Then there exists a compact subset $B\subset W\setminus g_d^{-1}(D)$ for which

$$\lambda^k(B)>\varepsilon\lambda^k(W).$$

Hence by Lemma 1,

$$\varepsilon R^k\alpha(k)=\varepsilon\lambda^k(W)<\lambda^k(B)\leq C2^{k-r}R^{k-r}\lambda^r(g_d(B)).$$

But $g_d(B)$ is a compact subset of $V\setminus D$, where $V=g_d(W)$ is the closed ball in $\mathbb{R}^r$ with centre $g(d,c)$ and radius $MR<\beta$. Since $\lambda^r(V\setminus D)<\delta\lambda^r(V)$ we get

$$\varepsilon R^k\alpha(k)\leq C2^{k-r}R^{k-r}\delta\lambda^r(V)=C2^{k-r}R^{k-r}M^rR^r\alpha(r)\delta$$

which contradicts the choice of δ. $\square$

Theorem 2. *Let X be an r dimensional Euclidean space, and let $X_1,\dots,X_n$ be orthogonal subspaces of X with dimensions $r_1,\dots,r_n$. Suppose, that $r_i\geq 1$ $(1\leq i\leq n)$ and $\sum_{i=1}^n r_i=r$. Let U be an open subset of X and $F:U\to\mathbb{R}^m$ be a continuously differentiable function. For each $x\in U$ let N_x denote the nullspace of $F'(x)$. Let A_i be a subset of X_i $(i=1,\dots,n)$ and suppose that A_i is λ^{r_i} measurable for $2\leq i\leq n$. Let $a\in U$, $\dim N_a=r-m$. Let p_i denote the orthogonal projection of X onto X_i. Suppose that $p_i(N_a)=X_i$ and A_i has density 1 in the point $p_i(a)$ whenever $1\leq i\leq n$. Then*

$$F(A_1\times A_2\times\ldots\times A_n)$$

is a neighbourhood of $F(a)$*.*

Proof. Let $k = r - m$. Because of the fact that $x \mapsto \operatorname{rank} F'(x)$ is lower semicontinuous, and $\operatorname{rank}(F'(a)) = m$, we may suppose that $\operatorname{rank}(F'(x)) = m$ whenever $x \in U$. Similarly, choosing a smaller U if necessary, we may suppose that $p_i(N_x) = X_i$ whenever $x \in U$ and $1 \le i \le n$; to prove this, suppose to the contrary that there exists an i and for each natural number j there exists an $x_j \in U$ and there exist orthonormal vectors $e_1^{(j)}, \dots, e_{k-r_i+1}^{(j)}$ in N_{x_j} such that $x_j \to a$ and

$$p_i(e_s^{(j)}) = 0 \quad \text{whenever } j = 1, 2, \dots \text{ and } 1 \le s \le k - r_i + 1.$$

Using the compactness of the unit sphere we can pass over a subsequence and suppose that

$$e_s^{(j)} \to e_s \qquad \text{if } j \to \infty.$$

But this proves that the vectors e_s are orthonormal in N_a and

$$p_i(e_s) = 0 \qquad \text{whenever } 1 \le s \le k - r_i + 1,$$

which is a contradiction.

Now, choosing a smaller U if necessary and using the rank theorem (see Dieudonné [2], 10.3.1), we have that there exist mappings u, p and v and an open neighbourhood V of $b = F(a)$ in $\mathbb{R}^m$ with the following properties: u maps U onto the open cube I^r, where $I =]-1, 1[$, u is invertible and u and u^{-1} are continuously differentiable; v maps I^m onto V, v is invertible and v and v^{-1} are continuously differentiable; p is the projection

$$p : (x_1, \dots, x_r) \mapsto (x_1, \dots, x_m)$$

of I^r onto I^m; and finally $F = v \circ p \circ u$. We may write I^r as

$$I^r = T \times Y \qquad \text{where } T = I^m \text{ and } Y = I^k.$$

Let $u(a) = (d, c) \in T \times Y$. Now let us use some facts from differential geometry (see Dieudonné [2], mainly 16.8.8).

$U \cap F^{-1}(v(t))$ is a closed submanifold of U whenever $t \in T$. The tangent space of this submanifold in a point $x \in U \cap F^{-1}(v(t))$ is equal to the subspace N_x of X. Clearly u^{-1} is a diffeomorphism of the closed submanifold $\{t\} \times Y$ of $T \times Y$ onto $U \cap F^{-1}(v(t))$. Let

$$g_i = p_i \circ u^{-1} \qquad \text{if} \quad 1 \le i \le n.$$

By the choice of U, p_i is a submersion of $U \cap F^{-1}(v(t))$ into X_i. Hence the mapping

$$g_{i,t} : Y \to X$$

is a submersion, that is, it has rank r_i whenever $y \in Y$ and $t \in T$.

Now, by Lemma 1, there exist open sets T^* and Y^* and there exists a $0 < K < \infty$ such that $d \in T^* \subset T$, $c \in Y^* \subset Y$, and

$$\lambda^k(B) \leq K\lambda^{r_i}(g_{i,t}(B))$$

whenever $B \subset Y^*$, $t \in T^*$. Let $X_i^* = X_i$, $A_i^* = A_i$ and g_i^* the restriction of g_i onto $T^* \times Y^*$. Applying Corollary 1 to the sets and functions marked by stars we have that the function

$$f(t) = \lambda^k\left(\bigcap_{i=1}^n {g_{i,t}^*}^{-1}(A_i)\right) \quad \text{if} \quad t \in T^*$$

is continuous on T^*. By Lemma 2, ${g_{i,d}^*}^{-1}(A_i)$ has density 1 in the point c. Because $g_{i,d}^{-1}(A_i) \cap Y^*$ is measurable by Lemma 1 if $2 \leq i \leq n$, we have that

$$\bigcap_{i=1}^n {g_{i,d}^*}^{-1}(A_i)$$

has density 1 in the point c. Hence $f(d) > 0$ and we have that there exists a neighbourhood W of d for which

$$f(t) > 0 \quad \text{if} \quad t \in W.$$

Clearly $v(W)$ is a neighbourhood of b in $\mathbb{R}^m$. If $z \in v(W)$ then $t := v^{-1}(z) \in W$ and hence the set

$$\bigcap_{i=1}^n {g_{i,t}^*}^{-1}(A_i)$$

is nonvoid. If y is an element of this set then

$$F(u^{-1}(t,y)) = v(p(t,y)) = v(t) = z$$

and

$$x_i = p_i(u^{-1}(t,y)) = g_{i,t}^*(y) \in A_i \quad \text{if} \quad 1 \leq i \leq n.$$

This means that $F(x_1, \ldots, x_n) = z$. $\square$

Remark. Our theorem may be stated in the following form : If $\dim N_x = r - m$ and $p_i(N_x) = X_i$ for all $x \in U$ $(i = 1, 2, \ldots, n)$, $A_1 \times A_2 \times \ldots \times A_n \subset U$ moreover A_i is λ^{r_i} measurable for $2 \leq i \leq n$ and $\lambda^{r_i}(A_i) > 0$ $(i = 1, 2, \ldots, n)$, then $F(A_1 \times \ldots \times A_n)$ contains a nonvoid open set.

Corollary 3. *Let U be an open subset of $\mathbb{R}^r \times \mathbb{R}^r$ and $F : (x,y) \mapsto F(x,y)$ a continuously differentiable mapping of U into $\mathbb{R}^r$. Let $A, B \subset \mathbb{R}^r$ and suppose that B is λ^r measurable. If $(a,b) \in U$,*

$$\det \frac{\partial F}{\partial x}(a,b) \neq 0, \quad \det \frac{\partial F}{\partial y}(a,b) \neq 0,$$

A has density 1 in the point a and B has density 1 in the point b, then $F(A,B)$ is a neighbourhood of $F(a,b)$.

Proof. By Theorem 2 we have to prove only that $p_1(N_{a,b}) = \mathbb{R}^r$ and $p_2(N_{a,b}) = \mathbb{R}^r$ where $N_{a,b}$ is the nullspace of $F'(a,b)$. Let $(x,y) \in N_{a,b}$. If $p_1(x,y) = 0$ then $x = 0$. Hence

$$0 = F'(a,b)(x,y) = \frac{\partial F}{\partial y}(a,b)(y).$$

But

$$\det \frac{\partial F}{\partial y}(a,b) \neq 0,$$

hence $y = 0$. This proves that $p_1 : N_{a,b} \to \mathbb{R}^r$ is a one-to-one mapping, that is, $p_1(N_{a,b}) = \mathbb{R}^r$. Similarly $p_2(N_{a,b}) = \mathbb{R}^r$.

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