

Fermat-Pell equation and the numbers of the form $w^2 + (w + 1)^2$

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1. Introduction

In the present paper we obtain recursive formulae for the determination of all non-negative (that is $X \geq 0$ and $Y \geq 0$) integral solutions of

$$(F) \quad X^2 - dY^2 = C \quad (d \neq \square, C \neq 0),$$

where $d \neq \square$ (non-square) is a natural number and C is an integer $\neq 0$ (Theorem 2.3 for $C > 0$ and Theorem 2.4 for $C < 0$ below). Also, we obtain same recursive formulae (Theorem 2.6 below).

The special case $d = 2$ and $C = 2k^2 - 1$, $k = 0, 1, 2, \dots$, of (F) constitute the connecting link with the numbers of the form

$$N(w) \equiv w^2 + (w + 1)^2$$

(for $w = (X - 1)/2$, we have $N(w) = Y^2 + k^2$).

In a forthcoming paper these recursive formulae will be used in the special case $d = 2$ and $C = 2k^2 - 1$ for the complete determination of all composite numbers of the form $w^2 + (w + 1)^2$.

The expression $x_1 + y_1\sqrt{d}$ will always denote the fundamental solution of

$$(P) \quad x^2 - dy^2 = 1 \quad (d \neq \square).$$

Also, $x_n + y_n\sqrt{d}$ $n = 0, 1, \dots$, will denote the sequence of all non-negative integral solutions of (P). These solutions are given in [3, p. 439] by the

following recursive formulae:

$$(1.1) \quad x_{n+1} = 2x_1x_n - x_{n-1}, \quad \text{where } x_0 = 1 \quad \text{and} \quad x_1 = x_1,$$

$$(1.2) \quad y_{n+1} = 2x_1y_n - y_{n-1}, \quad \text{where } y_0 = 0 \quad \text{and} \quad y_1 = y_1,$$

Let G be the group of all integral solutions of (P) . Let $Z \equiv X + Y\sqrt{d}$ be an integral solution of (F) . Consider the class

$$A \equiv \{Zz \mid z \in G\}$$

of solutions of (F) represented by Z . Define

$$\bar{A} \equiv \{-\bar{Z}z \mid z \in G\}.$$

Then $\bar{A}$ constitutes a class of solutions of (F) represented by $-\bar{Z}$. This class $\bar{A}$ is called *conjugate class* of A . If $A \neq \bar{A}$ then A is called *genuine* or *not ambiguous* class. If $A = \bar{A}$, then A is called *ambiguous* [cf. 2, p. 205].

Let $Z^* = X^* + Y^*\sqrt{d}$ be the *fundamental solution* (as defined in NAGELL in [2, p. 205]) of (F) belonging to the class A , then

$$A = \{Z^*z \mid z \in G\} \quad \text{and} \quad \bar{A} = \{-\bar{Z}^*z \mid z \in G\}.$$

Theorem 1.1. *The Diophantine equation (F) has a finite number of classes of solutions. The fundamental solutions of all such classes are determined by the following (equivalent) inequalities in case $C > 0$*

$$(1.3) \quad 0 < |X^*| \leq \sqrt{(x_1 + 1)C/2},$$

$$(1.4) \quad 0 \leq Y^* \leq (y_1/\sqrt{2(x_1 + 1)})\sqrt{C}$$

and by the following (equivalent) inequalities in case $C < 0$

$$(1.5) \quad 0 \leq |X^*| \leq \sqrt{(x_1 - 1)(-C)/2},$$

$$(1.6) \quad 0 < Y^* \leq (y_1/\sqrt{2(x_1 - 1)})\sqrt{(-C)}.$$

Moreover, A consists of all elements of the form

$$X + Y\sqrt{d} = (X^* + Y^*\sqrt{d})(x + y\sqrt{d}),$$

where $x + y\sqrt{d}$ ranges over the set of all integral solutions of (P) .

The Diophantine equation (F) has no solution at all when it has no solution satisfying the inequalities (1.3) and (1.4) or (1.5) and (1.6) respectively.

PROOF. See Theorem 109, in [2] (cf. also [1], [4] and [5]).

In case $C > 0$ the recursive description of all non-negative integral solutions of (F) belonging to a class of solutions A , is given by Theorem 2.1.

Its proof is based on Proposition 1.2. In the sequel A will always denote an arbitrarily chosen fixious class of solutions of (F) and $X^* + Y^*\sqrt{d}$ its fundamental solution.

Proposition 1.2. *Consider the Diophantine equation (F) , $C > 0$. Let A be a class of solutions with $X^* > 0$. Let*

$$X_n + Y_n\sqrt{d} \equiv (X^* + Y^*\sqrt{d})(x_n + y_n\sqrt{d}) \quad \text{for all } n = 0, 1, \dots,$$

$$X'_n + Y'_n\sqrt{d} \equiv (X^* - Y^*\sqrt{d})(x_n + y_n\sqrt{d}) \quad \text{for all } n = 1, 2, \dots$$

Then the set of all non-negative integral solutions of (F) belonging to A consists of all pairs (X_n, Y_n) , while the set of all non-negative (positive) integral solutions of (F) belonging to $\bar{A}$ consists of all pairs (X'_n, Y'_n) .

PROOF. By Theorem 1.1 the class A consists of all elements having one of the following typical forms:

$$\begin{aligned} (X^* + Y^*\sqrt{d})(x_n + y_n\sqrt{d}) &= (x_nX^* + dy_nY^*) + (y_nX^* + x_nY^*)\sqrt{d} \\ &\equiv X_n + Y_n\sqrt{d}, \end{aligned}$$

$$(X^* + Y^*\sqrt{d})(-x_n - y_n\sqrt{d}) = -X_n - Y_n\sqrt{d},$$

$$\begin{aligned} (X^* + Y^*\sqrt{d})(-x_n + y_n\sqrt{d}) &= -(x_nX^* - dy_nY^*) + (y_nX^* - x_nY^*)\sqrt{d} \\ &\equiv -X'_n + Y'_n\sqrt{d}, \end{aligned}$$

$$(X^* + Y^*\sqrt{d})(x_n - y_n\sqrt{d}) = X'_n - Y'_n\sqrt{d}.$$

Also, $\bar{A}$ consists of all elements having one of the following typical forms:

$$(-X^* + Y^*\sqrt{d})(x_n + y_n\sqrt{d}) = -X'_n - Y'_n\sqrt{d},$$

$$(-X^* + Y^*\sqrt{d})(-x_n - y_n\sqrt{d}) = X'_n + Y'_n\sqrt{d},$$

$$(-X^* + Y^*\sqrt{d})(-x_n + y_n\sqrt{d}) = X_n - Y_n\sqrt{d},$$

$$(-X^* + Y^*\sqrt{d})(x_n - y_n\sqrt{d}) = -X_n + Y_n\sqrt{d}.$$

The following hold true:

$$(1.7) \quad X_n = x_nX^* + dy_nY^* > 0,$$

$$(1.8) \quad Y_n = y_nX^* + x_nY^* \geq 0,$$

$$(1.9) \quad X'_n = x_nX^* - dy_nY^* > 0.$$

The last equality holds true because $x_n > y_n\sqrt{d}$ and $X^* > Y^*\sqrt{d}$.

It will be proved that:

$$(1.10) \quad Y'_n = y_nX^* - x_nY^* > 0 \quad \text{for every } n = 1, 2, \dots$$

In fact, by (1.4) we deduce that

$$Y^{*2} \leq (y_1^2 C)/(2(x_1 + 1)) < y_1^2 C \leq y_n^2 (X^{*2} - dY^{*2}) \quad \text{for every } n \geq 1,$$

that is

$$(y_n X^*)^2 - (x_n Y^*)^2 > 0.$$

Hence

$$y_n X^* - x_n Y^* > 0, \quad \text{that is} \quad Y'_n > 0.$$

From (1.7), (1.8), (1.9) and (1.10) follows the desired conclusion.

In the sequel X_n, X'_n, Y_n, Y'_n will have the same meaning as in Proposition 1.2.

2. Study of the generalized Fermat equation

Theorem 2.1. *Consider the Diophantine equation (F), $C > 0$. Let A be a class of solutions with $X^* > 0$. Then the sequence of all non-negative integral solutions of (F) belonging to A is determined by the following recursive formulae:*

$$(2.1) \quad X_{n+1} = 2x_1 X_n - X_{n-1}, \text{ where } X_0 = X^* \text{ and } X_1 = x_1 X^* + dy_1 Y^*,$$

$$(2.2) \quad Y_{n+1} = 2x_1 Y_n - Y_{n-1}, \text{ where } Y_0 = Y^* \text{ and } Y_1 = y_1 X^* + x_1 Y^*.$$

Also, the sequence of all non-negative (positive) integral solutions of (F) belonging to $\bar{A}$ is determined by the following recursive formulae:

$$(2.3) \quad X'_{n+1} = 2x_1 X'_n - X'_{n-1}, \text{ where } X'_0 = X^* \text{ and } X'_1 = x_1 X^* - dy_1 Y^*,$$

$$(2.4) \quad Y'_{n+1} = 2x_1 Y'_n - Y'_{n-1}, \text{ where } Y'_0 = -Y^* \text{ and } Y'_1 = y_1 X^* - x_1 Y^*.$$

PROOF. It is easily seen, because of Proposition 1.2, that the non-negative solutions of A and $\bar{A}$ satisfy the recursive formulae (2.1), (2.2) and (2.3), (2.4) respectively. We now use Proposition 1.2 to prove the reverse side of the theorem. It will be proved that

$$(2.5) \quad \begin{aligned} X_n + Y_n \sqrt{d} &= (X^* + Y^* \sqrt{d})(x_n + y_n \sqrt{d}) \\ &= (x_n X^* + dy_n Y^*) + (y_n X^* + x_n Y^*) \sqrt{d} \end{aligned}$$

for all $n = 0, 1, \dots$.

Clearly (2.5) is true for $n = 0, 1$. Suppose that (2.5) holds true for every index less than $n + 1$. (Induction hypothesis). It will be proved that

(2.5) holds true for $n + 1$. In fact;

$$\begin{aligned} X_{n+1} + Y_{n+1}\sqrt{d} &= 2x_1X_n - X_{n-1} + (2x_1Y_n - Y_{n-1})\sqrt{d} \\ &= (X^* + Y^*\sqrt{d})(x_n + y_n\sqrt{d})(2x_1 - (x_1 - y_1\sqrt{d})) \\ &= (X^* + Y^*\sqrt{d})(x_{n+1} + y_{n+1}\sqrt{d}). \end{aligned}$$

Evidently $X_n > 0$ and $Y_n \geq 0$. Hence, every pair (X_n, Y_n) is a non-negative integral solution of (F) belonging to A .

In a similar way to the proof of (2.5) it can be proved that

$$X'_n + Y'_n\sqrt{d} = (X^* - Y^*\sqrt{d})(x_n + y_n\sqrt{d}) \quad \text{for all } n = 1, 2, \dots$$

Furthermore, by (1.9) and (1.10), we deduce that $X'_n > 0$ and $Y'_n > 0$ for all $n = 1, 2, \dots$. Hence, every pair (X'_n, Y'_n) is a non-negative (positive) integral solution of (F) belonging to $\bar{A}$.

The set of *all* non-negative integral solutions of (F) , for $C > 0$, is determined in Theorem 2.3 whose proof is based (inter alia) on Proposition 2.2. A similar determination for $C < 0$ is described in Theorem 2.4.

Proposition 2.2. *Consider the Diophantine equation (F) , $C > 0$. Let A be a class of solutions with $X^* > 0$. Then the following hold true:*

- (i) $Y_{n+1} > Y_n \geq 0$ for every $n = 0, 1, \dots$.
- (ii) Let $Y^* > 0$. Then $Y'_{n+1} \geq Y_n > Y'_n > 0$ for every $n = 1, 2, \dots$.
- (iii) Let $Y^* = 0$. Then $Y_n = Y'_n$ for every $n = 0, 1, \dots$.
- (iv) Let A be genuine. Then

$$Y'_{n+1} > Y_n > Y'_n > 0 \quad \text{for all } n = 1, 2, \dots$$

- (v) Let A be ambiguous. Then for every m there exists n such that:

$$X'_m = X_n \quad \text{and} \quad Y'_m = Y_n.$$

PROOF. i)

$$\begin{aligned} Y_{n+1} &= y_{n+1}X^* + x_{n+1}Y^* = (x_1y_n + x_ny_1)X^* + (x_1x_n + dy_1y_n)Y^* \\ &= y_n(x_1X^* + dy_1Y^*) + x_n(y_1X^* + x_1Y^*) > y_nX^* + x_nY^* = Y_n \geq 0, \\ &\text{that is } Y_{n+1} > Y_n \geq 0 \quad \text{for every } n = 0, 1, \dots \end{aligned}$$

ii)

$$(2.6) \quad Y'_{n+1} = y_n(x_1X^* - dy_1Y^*) + x_n(y_1X^* - x_1Y^*).$$

Also, $-X^* + Y^*\sqrt{d}$ is the fundamental solution of $\bar{A}$, while,

$$x_1X^* - dy_1Y^* + (y_1X^* - x_1Y^*)\sqrt{d} = (-X^* + Y^*\sqrt{d})(-x_1 - y_1\sqrt{d})$$

is (by (1.9) and (1.10)) a positive integral solution of (F) belonging to $\bar{A}$. Hence, by Definition of fundamental solution, we obtain:

$$(2.7) \quad y_1 X^* - x_1 Y^* \geq Y^* \quad (\text{and equivalently} \quad x_1 X^* - dy_1 Y^* \geq X^*).$$

From (2.6) and (2.7) we deduce:

$$Y'_{n+1} \geq Y_n > Y'_n > 0 \quad \text{for every} \quad n = 1, 2, \dots$$

iii) By the definition of Y_n and Y'_n .

iv) It will be proved that

$$(2.8) \quad x_1 X^* - dy_1 Y^* > X^* \quad \text{and} \quad y_1 X^* - x_1 Y^* > Y^*.$$

In fact; by (2.7) we have:

$$x_1 X^* - dy_1 Y^* \geq X^* > 0 \quad \text{and} \quad y_1 X^* - x_1 Y^* \geq Y^* \geq 0.$$

Assume that (2.8) is not true. Then

$$x_1 X^* - dy_1 Y^* = X^* \quad \text{and} \quad y_1 X^* - x_1 Y^* = Y^*,$$

$$\text{i.e.} \quad (X^* - Y^* \sqrt{d})(x_1 + y_1 \sqrt{d}) = X^* + Y^* \sqrt{d},$$

which contradicts the assumption, because A is genuine. Hence (2.8) holds true. Thus by (2.6) we obtain:

$$Y'_{n+1} > Y_n > Y'_n > 0.$$

v) Evident by the assumption $A = \bar{A}$.

Theorem 2.3. Consider the Diophantine equation (F) , $C > 0$. Let $X_r^* + Y_r^* \sqrt{d}$, where $r = 1, 2, \dots, m$, be the only integral solutions of (F) such that:

$$0 < X_r^* \leq \sqrt{(x_1 + 1)C/2} \quad \text{and} \quad 0 \leq Y_r^* \leq y_1 \sqrt{C}/\sqrt{2(x_1 + 1)}.$$

Let

$$X_n + Y_n \sqrt{d} \equiv (X_r^* + Y_r^* \sqrt{d})(x_n + y_n \sqrt{d}) \quad \text{for all} \quad n = 0, 1, \dots,$$

$$X'_n + Y'_n \sqrt{d} \equiv (X_r^* - Y_r^* \sqrt{d})(x_n + y_n \sqrt{d}) \quad \text{for all} \quad n = 1, 2, \dots$$

(For a typical r).

Then the set of all non-negative integral solutions of (F) consists of all pairs (X_n, Y_n) together with all pairs (X'_n, Y'_n) for all respective genuine classes A_r in addition to all pairs (X_n, Y_n) for all respective ambiguous

classes B_r . Moreover, X_n, Y_n, X'_n and Y'_n are determined by the following recursive formulae:

$$\begin{aligned}
X_{n+1} &= 2x_1X_n - X_{n-1} \quad \text{for } n = 1, 2, \dots \quad \text{with} \\
X_0 &= X_r^*, \quad X_1 = x_1X_r^* + dy_1Y_r^* \quad \text{and } r = 1, 2, \dots, m. \\
Y_{n+1} &= 2x_1Y_n - Y_{n-1} \quad \text{for } n = 1, 2, \dots \quad \text{with} \\
Y_0 &= Y_r^*, \quad Y_1 = y_1X_r^* + x_1Y_r^* \quad \text{and } r = 1, 2, \dots, m. \\
X'_{n+1} &= 2x_1X'_n - X'_{n-1} \quad \text{for } n = 1, 2, \dots \quad \text{with} \\
X'_0 &= X_r^*, \quad X'_1 = x_1X_r^* - dy_1Y_r^* \quad \text{and } r = 1, 2, \dots, m. \\
Y'_{n+1} &= 2x_1Y'_n - Y'_{n-1} \quad \text{for } n = 1, 2, \dots \quad \text{with} \\
Y'_0 &= -Y_r^*, \quad Y'_1 = y_1X_r^* - x_1Y_r^* \quad \text{and } r = 1, 2, \dots, m.
\end{aligned}$$

PROOF. By using Proposition 2.2 and Theorems 1.1 and 2.1.

Theorem 2.4. Consider the Diophantine equation (F) , $C < 0$. Let $X_r^* + Y_r^*\sqrt{d}$, where $r = 1, 2, \dots, m$, be the only integral solutions of (F) such that:

$$0 \leq X_r^* \leq \sqrt{(x_1 - 1)(-C)/2} \quad \text{and} \quad 0 < Y_r^* \leq y_1\sqrt{(-C)}/\sqrt{2(x_1 - 1)}.$$

Let

$$\begin{aligned}
X_n + Y_n\sqrt{d} &\equiv (X_r^* + Y_r^*\sqrt{d})(x_n + y_n\sqrt{d}) \quad \text{for all } n = 0, 1, \dots, \\
X''_n + Y''_n\sqrt{d} &\equiv (-X_r^* + Y_r^*\sqrt{d})(x_n + y_n\sqrt{d}) \\
&\quad \text{for all } n = 1, 2, \dots \quad (\text{For a typical } r)
\end{aligned}$$

Then the set of all non-negative integral solutions of (F) consists of all pairs (X_n, Y_n) together with all pairs (X''_n, Y''_n) for all respective genuine classes A_r in addition to all pairs (X_n, Y_n) for all respective ambiguous classes B_r . Moreover, X_n, Y_n, X''_n and Y''_n are determined by the following recursive formulae:

$$\begin{aligned}
X_{n+1} &= 2x_1X_n - X_{n-1} \quad \text{for } n = 1, 2, \dots \quad \text{with} \\
X_0 &= X_r^*, \quad X_1 = x_1X_r^* + dy_1Y_r^* \quad \text{and } r = 1, 2, \dots, m. \\
Y_{n+1} &= 2x_1Y_n - Y_{n-1} \quad \text{for } n = 1, 2, \dots \quad \text{with} \\
Y_0 &= Y_r^*, \quad Y_1 = y_1X_r^* + x_1Y_r^* \quad \text{and } r = 1, 2, \dots, m. \\
X''_{n+1} &= 2x_1X''_n - X''_{n-1} \quad \text{for } n = 1, 2, \dots \quad \text{with} \\
X''_0 &= -X_r^*, \quad X''_1 = -x_1X_r^* + dy_1Y_r^* \quad \text{and } r = 1, 2, \dots, m.
\end{aligned}$$

$$Y''_{n+1} = 2x_1 Y''_n - Y''_{n-1} \quad \text{for } n = 1, 2, \dots \quad \text{with}$$

$$Y''_0 = Y_r^*, \quad Y''_1 = -y_1 X_r^* + x_1 Y_r^* \quad \text{and} \quad r = 1, 2, \dots, m.$$

PROOF. Similar to the proof of Theorem 2.3.

Our next Theorem 2.5 provides a recursive determination of all Y^2 for the elements $X + Y\sqrt{d}$ comprising the set of all *absolutely* distinct solutions of a class of (F). [Any two solutions $X + Y\sqrt{d}$ and $X' + Y'\sqrt{d}$ of (F) are considered as *absolutely* the same whenever $|X| = |X'|$ and $|Y| = |Y'|$]. A similar recursive determination of all $Y^2 + k^2$, for a fixed integer k (and Y etc. as above) is provided by Theorem 2.6 whose proof is a direct consequence of that of Theorem 2.5.

Theorem 2.5. *Consider the Diophantine equation (F). Let*

$$X_n + Y_n\sqrt{d} \equiv (X^* + Y^*\sqrt{d})(x_1 + y_1\sqrt{d})^n,$$

$$X'_n + Y'_n\sqrt{d} \equiv (X^* - Y^*\sqrt{d})(x_1 + y_1\sqrt{d})^n \quad \text{for all } n = 0, 1, \dots$$

Let $P_n \equiv Y_n^2$ and $P'_n \equiv Y'^2_n$ for all $n = 0, 1, \dots$. Then the numbers P_n, P'_n are determined by the following recursive formulae:

$$(2.9) \quad P_{n+1} = 2x_2 P_n - P_{n-1} + 2y_1^2 C, \quad \text{where } P_0 = Y^{*2} \text{ and}$$

$$P_1 = (x_1 Y^* + y_1 X^*)^2,$$

$$(2.10) \quad P'_{n+1} = 2x_2 P'_n - P'_{n-1} + 2y_1^2 C, \quad \text{where}$$

$$P'_0 = Y^{*2} \text{ and } P'_1 = (y_1 X^* - x_1 Y^*)^2.$$

PROOF. First we prove that the numbers P_n, P'_n satisfy the above mentioned recursive formulae. Let $Z^* = X^* + Y^*\sqrt{d}$, $z_n = (x_1 + y_1\sqrt{d})^n = z_1^n = x_n + y_n\sqrt{d}$, $Z_n = Z^* z_n$ and $Z'_n = \bar{Z}^* z_n$.

The following hold true:

$$Z_n^2 = Z^{*2} z_1^{2n} = Z^{*2} z_{2n} \quad \text{and} \quad Z^{*2} = X^{*2} + dY^{*2} + 2X^*Y^*\sqrt{d}.$$

Let $X_2^* \equiv X^{*2} + dY^{*2}$ and $Y_2^* \equiv 2X^*Y^*$. Then

$$(X_n + Y_n\sqrt{d})^2 = (X_2^* + Y_2^*\sqrt{d})(x_{2n} + y_{2n}\sqrt{d}).$$

Hence

$$(2.11) \quad X_n^2 + dY_n^2 = X_2^* x_{2n} + dY_2^* y_{2n}.$$

Also,

$$X_n^2 - dY_n^2 = C.$$

Therefore

$$2dY_n^2 = X_2^*x_{2n} + dY_2^*y_{2n} - C.$$

Let

$$(2.12) \quad Q_{2n} \equiv X_2^*x_{2n} + dY_2^*y_{2n}.$$

But $P_n = Y_n^2$, then

$$(2.13) \quad Q_{2n} = 2dP_n + C.$$

Also,

$$z_{m+2} = z_m z_2 \quad \text{and} \quad z_{m-2} = z_m \bar{z}_2.$$

Hence we deduce:

$$(2.14) \quad x_{m+2} = 2x_2 x_m - x_{m-2} \quad \text{and} \quad y_{m+2} = 2x_2 y_m - y_{m-2}.$$

From (2.12) and (2.14) we obtain:

$$(2.15) \quad Q_{2n+2} = 2x_2 Q_{2n} - Q_{2n-2}.$$

By (2.13) we have:

$$2x_2 Q_{2n} - Q_{2(n-1)} = 2dP_{n+1} + C,$$

that is

$$2x_2(2dP_n + C) - 2dP_{n-1} - C = 2dP_{n+1} + C,$$

and so

$$P_{n+1} = 2x_2 P_n - P_{n-1} + C(x_2 - 1)/d.$$

Also $x_2 = x_1^2 + dy_1^2$, that is $(x_2 - 1)/d = 2y_1^2$. Hence

$$P_{n+1} = 2x_2 P_n - P_{n-1} + 2Cy_1^2.$$

In a similar way as above we deduce:

$$P'_{n+1} = 2x_2 P'_n - P'_{n-1} + 2Cy_1^2.$$

Also, the initial conditions $P_0 = Y^{*2}$ etc are proved directly by the definitions of P_n and P'_n for $n = 0, 1$.

Consider now the sequences P_n, P'_n defined by (2.9) and (2.10). We shall prove that $P_n = Y_n^2$ and $P'_n = Y_n'^2$. Clearly

$$(2.16) \quad P_n = Y_n^2$$

is true for $n = 0, 1$. Suppose that (2.16) holds true for every index less than $n + 1$. (Induction hypothesis). It will be proved that (2.16) holds true for $n + 1$. In fact;

$$2y_1^2 = (x_2 - 1)/d,$$

hence

$$2dP_{n+1} = 2x_2 2dP_n - 2dP_{n-1} + 2(x_2 - 1)C.$$

Hence, by the induction hypothesis, we have

$$(2.17) \quad 2dP_{n+1} + C = 2x_2(X_n^2 + dY_n^2) - (X_{n-1}^2 + dY_{n-1}^2).$$

The following holds true:

$$(2.18) \quad x_{2n+2} = 2x_2x_{2n} - x_{2n-2} \quad \text{and} \quad y_{2n+2} = 2x_2y_{2n} - y_{2n-2}$$

From (2.11), (2.17) and (2.18) we obtain:

$$2dP_{n+1} + C = X_{n+1}^2 + dY_{n+1}^2.$$

Thus we deduce:

$$P_{n+1} = Y_{n+1}^2.$$

In a similar way as above we deduce that:

$$P'_n = Y'^2_n \quad \text{for every } n = 0, 1, \dots$$

Theorem 2.6. *Consider the Diophantine equation (F). Let $R_n \equiv Y_n^2 + k^2$ and $R'_n \equiv Y'^2_n + k^2$, where k is a fixed integer. Then the numbers R_n, R'_n are determined by the following recursive formulae:*

$$R_{n+1} = 2x_2R_n - R_{n-1} - 2k^2(x_2 - 1) + 2y_1^2C,$$

where $R_0 = Y^{*2} + k^2$ and $R_1 = (y_1X^* + x_1Y^*)^2 + k^2$.

$$R'_{n+1} = 2x_2R'_n - R'_{n-1} - 2k^2(x_2 - 1) + 2y_1^2C,$$

where $R'_0 = Y'^2 + k^2$ and $R'_1 = (y_1X^* - x_1Y^*)^2 + k^2$.

PROOF. It is actually a direct consequence of the proof of Theorem 2.5.

3. An application of Theorem 2.6

A special case of Theorem 2.6 ($d = 2$ and $C = 2k^2 - 1$, $k = 0, 1, 2, \dots$) is the following

Theorem 3.1. *The Diophantine equation*

$$(F_k) \quad X^2 - 2Y^2 = 2k^2 - 1, \quad \text{where } k = 0, 1, \dots$$

has at least one class of solutions A . Moreover, if $R_n \equiv Y_n^2 + k^2$ and $R'_n \equiv Y_n'^2 + k^2$, then the numbers R_n, R'_n are determined by the following recursive formulae:

$$(3.1) \quad R_{n+1} = 34R_n - R_{n-1} - 8(2k^2 + 1) \quad \text{for all } n = 1, 2, \dots$$

with $R_0 = Y^{*2} + k^2$ and $R_1 = (2X^* + 3Y^*)^2 + k^2$.

$$(3.2) \quad R'_{n+1} = 34R'_n - R'_{n-1} - 8(2k^2 + 1) \quad \text{for all } n = 1, 2, \dots$$

with $R'_0 = Y^{*2} + k^2$ and $R'_1 = (2X^* - 3Y^*)^2 + k^2$.

PROOF. It suffices to prove the existence of the class A . The other assertions are evident by Theorem 2.6, since $3 + 2\sqrt{2}$ is the fundamental solution of $x^2 - 2y^2 = 1$.

The fundamental solution of (F_0) is $1 + \sqrt{2}$. Also $2k - 1 + (k - 1)\sqrt{2}$ is a solution of (F_k) for $k = 1, 2, \dots$. In fact it is the fundamental solution of its class, since satisfies the inequalities (1.3) and (1.4). This proves the Theorem.

Let $X + Y\sqrt{2}$ be a non-negative integral solution of (F_k) (see Theorem 2.3 for $k \geq 1$ or Theorem 2.4 for $k = 0$). Hence, we have $X^2 = 2(Y^2 + k^2) - 1 \geq 1$ and so X is an old natural number. In case $X + Y\sqrt{2}$ is the fundamental solution of (F_0) or (F_1) we have $X = 1$. We set $N(w) \equiv w^2 + (w + 1)^2$; If $w = (X - 1)/2$ [w is an integer > 0 if $X + Y\sqrt{2}$ is not the fundamental solution of (F_0) or (F_1)] it follows that $N(w) = Y^2 + k^2$. Hence, by Theorem 3.1 the numbers R_n, R'_n , $n = 1, 2, \dots$ [see (3.1) and (3.2)] are of the form $w^2 + (w + 1)^2$.

Example. We consider the Diophantine equation

$$(F_0) \quad X^2 - 2Y^2 = -1.$$

From Theorems 2.4 and 3.1 we obtain: $X^* + Y^*\sqrt{2} = 1 + \sqrt{2}$ and

$$R_{n+1} = 34R_n - R_{n-1} - 8 = Y_{n+1}^2, \quad n = 1, 2, \dots \quad \text{with} \\ R_0 = 1 \quad \text{and} \quad R_1 = 25.$$

[It follows that $R_1 = 25$, $R_2 = 841$, $R_3 = 28561$, $R_4 = 970225$, $R_5 = 32959081$, $R_6 = 1119638521$, $R_7 = 38034750625$, $R_8 = 1292061882721$, $R_9 = 43892069261881$, ...]

The numbers $R_n = 1, 2, \dots$, are square (composite) numbers of the form $w^2 + (w+1)^2$.

Remark. Let $X^* + Y^*\sqrt{2}$ be the fundamental solution of a class A of integral solutions of (F_k) , with $X^* > 0$. If A is genuine, then (by Proposition 2.2, (iv) and Theorem 3.1) $R'_n < R_n < R'_{n+1}$ for all $n = 1, 2, \dots$. But if A is ambiguous, then for every m there exists n such that $R'_m = R_n$.

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