

Families of mappings and fixed points

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Abstract. In this paper, some necessary and sufficient conditions for the existence of fixed points of a family of self-mappings of a metric space are given and a fixed point theorem for a compact mapping is established.

1. Introduction

JUNGCK [1] first gave a necessary and sufficient condition for the existence of fixed points of a continuous self-mapping of a complete metric space. Afterwards, PARK [2] and KHAN and FISHER [3] established a few theorems similar to that of Jungck. JANOS [4] and PARK [5] proved fixed point theorems for compact self-mappings of a metric space.

The purpose of this paper is to offer some characterizations for the existence of fixed points of a family of self-mappings of metric spaces. We also obtain a fixed point theorem for a compact mapping, which extends properly the results of JANOS [4] and PARK [5].

ω and $\mathbb{N}$ denote the sets of nonnegative and positive integers, respectively. Let f be a self-mapping of a metric space (X, d) . Following FURI and VIGNOLI [6], f is said to be condensing if for every bounded subset A of X with $\alpha(A) > 0$, we have $\alpha(fA) < \alpha(A)$, where $\alpha(A)$ denotes the measure of noncompactness in the sense of Kuratowski. Define $C_f = \{g \mid g: X \rightarrow X \text{ and } fg = gf\}$, $H_f = \{g \mid g: X \rightarrow X \text{ and } g \bigcap_{n \in \omega} f^n X \subset \bigcap_{n \in \omega} f^n X\}$, $O(x, f) = \{f^n x \mid n \in \omega\}$ and $O(x, y, f) = O(x, f) \cup O(y, f)$ for $x, y \in X$. Let $\mathcal{F}$ be a family of self-mappings of X . A point $x \in X$ is said to be a fixed point of $\mathcal{F}$ if $fx = x$ for all $f \in \mathcal{F}$. Set $\mathcal{F} = \{F \mid F \text{ is a real-valued lower semi-continuous function of } X \times X \text{ into } [0, \infty) \text{ such that } F(x, y) = 0$

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if and only if $x = y$. For $A \subset X$, $\delta(A)$ and $\overline{A}$ denote the diameter and closure of A , respectively. Let $M(X)$ denote the set of all metrics on X that are topologically equivalent to d for a given metric space (X, d) .

Remark. Clearly, $H_f \supset C_f \supset \{f^n \mid n \in \omega\}$.

2. Fixed point theorems

Theorem 1. *Let $\mathcal{F}$ be a family of self-mappings of a bounded metric space (X, d) . Then $\mathcal{F}$ has a fixed point if and only if there exists a continuous compact self-mapping f of X such that $f \in \bigcap_{g \in \mathcal{F}} C_g$ and*

$$(*) \quad d(fx, fy) < \delta \left(\bigcup_{h \in H_f} O(x, y, h) \right) \text{ for all } x, y \in X \text{ with } x \neq y.$$

PROOF. To see that the stated condition is necessary, suppose that $\mathcal{F}$ has a fixed point $w \in X$. Define a mapping $f : X \rightarrow X$ by $fx = w$ for all $x \in X$. Then $fgx = w = gw = gfx$ for all $g \in \mathcal{F}$ and all $x \in X$; i.e., $f \in \bigcap_{g \in \mathcal{F}} C_g$. Clearly, $(*)$ holds.

On the other hand, suppose there exists a continuous compact self-mapping f of X such that $f \in \bigcap_{g \in \mathcal{F}} C_g$ and $(*)$ hold. Since f is compact, there exists a compact set Y with $fX \subset Y \subset X$. Consequently, $X \supset Y \supset \dots \supset f^n X \supset f^n Y \supset f^{n+1} X \supset f^{n+1} Y \supset \dots$ for $n \in \omega$. Set $A = \bigcap_{n \in \omega} f^n X$ and $B = \bigcap_{n \in \omega} f^n Y$. Note that $A = \bigcap_{n \in \mathbb{N}} f^n X \subset B \subset A$. It follows that $A = B$. Since f is continuous and Y is compact and $f^{n+1} Y \subset f^n Y$ for $n \in \omega$, it follows that B is a nonempty compact set and $fB \subset \bigcap_{n \in \omega} f^{n+1} Y \subset B$. We now show that $fB \supset B$. Given $b \in B = \bigcap_{n \in \omega} f^n Y$, there exists $x_n \in f^n Y$ with $b = fx_n$ for $n \in \mathbb{N}$. Note that $\{x_n\}_{n \in \mathbb{N}} \subset Y$. Hence we can extract a subsequence $\{x_{n_i}\}_{i \in \mathbb{N}}$ converging to $p \in Y$. For every $m \in \mathbb{N}$, there exists $i > m$ such that $\{x_{n_i}, x_{n_i+1}, x_{n_i+2}, \dots\} \subset f^m Y$. Compactness of $f^m Y$ implies $x_{n_i} \rightarrow p \in f^m Y$ as $i \rightarrow \infty$. Therefore $p \in \bigcap_{m \in \mathbb{N}} f^m Y = B$. By the continuity of f , we have $b = fx_{n_i} \rightarrow fp$ as $i \rightarrow \infty$; i.e., $b = fp \in fB$. This proves $fB \supset B$. Hence $fB = B$. Thus A is a nonempty compact set and $fA = A$. Consequently, we can find $x, y, u, v \in A$ with $\delta(A) = d(u, v)$, $u = fx$ and $v = fy$. We next show that A is a singleton. If not, then $\delta(A) > 0$, which

implies $x \neq y$. Using (*) we obtain

$$\delta(A) = d(fx, fy) < \delta \left(\bigcup_{h \in H_f} O(x, y, h) \right) \leq \delta(A),$$

a contradiction, and hence $A = \{w\}$ for some $w \in X$. Obviously, w is a fixed point of f . If z is another fixed point of f , then $z \in \bigcap_{n \in \omega} f^n X = A = \{w\}$; i.e., $z = w$. Hence w is the only fixed point of f .

Note that $f \in \bigcap_{g \in \mathcal{F}} C_g$. Thus we have $fgw = gfw = gw$ for $g \in \mathcal{F}$. Since f has a unique fixed point w , $gw = w$ for $g \in \mathcal{F}$; i.e., w is a fixed point of $\mathcal{F}$. This completes the proof.

In order to extend Janos' and Park's results to a mapping satisfying (*), we need the following

Theorem (MEYERS [7]). *Let f be a continuous self-mapping of a metric (X, d) with the following properties:*

- (i) f has a unique fixed point $w \in X$;
- (ii) For any $x \in X$, $f^n x \rightarrow w$ as $n \rightarrow \infty$,
- (iii) There exists an open neighborhood U of w with the property that given any open set V containing w there exists $k \in \mathbb{N}$ with $f^n U \subset V$ for $n > k$.

Then for any $\alpha \in (0, 1)$ there exists a metric $d' \in M(X)$ relative to which f satisfies $d'(fx, fy) \leq \alpha d'(x, y)$ for $x, y \in X$.

Theorem 2. *Let f be a continuous compact self-mapping of a bounded metric space (X, d) satisfying (*). Then f has a unique fixed point, and furthermore, for any $\alpha \in (0, 1)$ there exists a metric $d' \in M(X)$ relative to which f satisfies $d'(fx, fy) \leq \alpha d'(x, y)$ for all $x, y \in X$.*

PROOF. Let $A = \bigcap_{n \in \omega} f^n X$. As in the proof of Theorem 1, we have $A = \{w\}$, which implies that (i) and (ii) of Meyers' theorem hold. To prove that (iii) holds we take $U = X$ and observe that $f^{n+1}X \subset f^n X$, the diameter of which diminishes to zero as $n \rightarrow \infty$. Thus $f^n X$ squeezes into any neighborhood of w and the proof is complete.

The following simple example reveals that our Theorem 2 extends properly Theorem 1.1 of JANOS [4] and Theorem 1 of PARK [5].

Example. Let $X = \{1, 2, 4, 5, 8\}$ with the usual metric. Define a mapping $f : X \rightarrow X$ by $f1 = f4 = f5 = 5$, $f2 = 1$ and $f8 = 2$. Then f is a

continuous compact self-mapping of X . It is easy to check that

$$d(fx, fy) \leq 4 < 7 = \delta \left(\bigcup_{h \in H_f} O(x, y, h) \right)$$

for all $x, y \in X$ with $x \neq y$. Hence the conditions of Theorem 2 are satisfied. But Theorem 1.1 of JANOS [4] and Theorem 1 of PARK [5] are not applicable because

$$d(f2, f4) = 4 \not\leq 1 = \frac{1}{2}[d(2, f2) + d(4, f4)]$$

and

$$d(f2, f4) = 4 \not\leq 4 = \delta(O(2, 4, f)).$$

Theorem 3. Let $\mathcal{F}$ be a family of self-mappings of a complete metric space (X, d) . Then the following statements are equivalent:

- (1) $\mathcal{F}$ has a fixed point;
- (2) There exists $x_0 \in X$ and a continuous condensing mapping $f \in \bigcap_{g \in \mathcal{F}} C_g$ such that $O(x_0, f)$ is bounded and

$$d(fx, fy) < \delta(O(x, y, f)) \text{ for all } x, y \in X \text{ with } x \neq y;$$

- (3) There exists $x_0 \in X$ and $F \in \mathcal{F}$ and a continuous condensing mapping $f \in \bigcap_{g \in \mathcal{F}} C_g$ such that $O(x_0, f)$ is bounded and

$$F(fx, fy) < \max \left\{ F(x, y), F(x, fx), F(y, fy), \frac{F(x, fx)F(y, fy)}{F(x, y)} \right\}$$

for all $x, y \in X$ with $x \neq y$.

PROOF. Let (1) hold and let w be a fixed point of $\mathcal{F}$. Define $f: X \rightarrow X$ by $fx = w$ for $x \in X$. Let $x_0 = w$ and $F \in \mathcal{F}$. It is easy to show that (2) and (3) hold.

Assume that (2) holds. Set $B = O(x_0, f)$ and $A = \bigcap_{n \in \omega} f^n \overline{B}$. Since f is condensing and $\alpha(B) = \max\{\alpha(\{x_0\}), \alpha(fB)\} = \alpha(fB)$, we have $\alpha(B) = 0$, which implies that B is precompact. Since X is complete, $\overline{B}$ is compact. By the continuity of f we get $f\overline{B} \subset \overline{fB} \subset \overline{B}$. As in the proof of Theorem 1, we conclude that A is a nonempty compact subset and $fA = A$. We assert that A is a singleton. Otherwise $\delta(A) > 0$. Since

A is compact and f maps A into itself, there exist $a, b, x, y \in A$ with $d(a, b) = \delta(A)$, $a = fx$, $b = fy$ and $x \neq y$. From (2) we have

$$\delta(A) = d(fx, fy) < \delta(O(x, y, f)) \leq \delta(A),$$

a contradiction, and hence $A = \{w\}$ for some $w \in X$. Clearly, w is a fixed point of f . Suppose that f has another fixed point $v (\neq w)$, then by (2) we have

$$d(w, v) = d(fw, fv) < \delta(O(w, v, f)) = d(w, v)$$

which is impossible. Consequently w is a unique fixed point of f . It is easy to check that w is a fixed point of $\mathcal{F}$. Hence (1) holds.

Assume that (3) holds. Set $B = O(x_0, f)$. As above we may show that $\overline{B}$ is compact and f -invariant. Since the function F is lower semi-continuous, the function h defined by $hx = F(x, fx)$ for $x \in \overline{B}$ is lower semi-continuous and so assumes its minimum value at some $w \in \overline{B}$. Thus if $fw \neq w$, then $fw \in \overline{B}$ and

$$\begin{aligned} hfw &= F(fw, f^2w) \\ &< \max \left\{ F(w, fw), F(w, fw), F(fw, f^2w), \frac{F(w, fw)F(fw, f^2w)}{F(w, fw)} \right\} \\ &= \max\{hw, hfw\} \end{aligned}$$

which implies that $hfw < hw$, contradicting the definition of w . It follows that w is a fixed point of f . If f has a second distinct fixed point u , then by (3) we have

$$\begin{aligned} F(w, u) &= F(fw, fu) \\ &< \max \left\{ F(w, u), F(w, w), F(u, u), \frac{F(w, w)F(u, u)}{F(w, u)} \right\} = F(w, u), \end{aligned}$$

a contradiction. Therefore w is the only fixed point of f . It is a simple matter to show that w is a fixed point of $\mathcal{F}$. Hence (1) holds. This completes the proof.

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