

On the smallest convex cover of a simple arc of space-curve.

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A bounded convex point set may be defined as a point set which is identical to the set of its chords. More explicitly, if $\omega(S)$ denotes the set of all points contained at least in one chord (= segment of straight line joining two points of S) of the point set S , then S is convex if and only if

$$(1) \quad \omega(S) = S.$$

Adopting this definition of the process ω , the smallest convex cover $\Omega(S)$ of a bounded point set S in the *three*-dimensional space R_3 is evidently given by

$$(2) \quad \Omega(S) = \omega(\omega(S)),$$

while in general $\Omega(S) \neq \omega(S)$.

Considering that in general the iteration of the process ω is necessary, it is natural to ask, whether there are point sets whose smallest convex cover coincides with the set of their chords, i. e. for which

$$(3) \quad \Omega(S) = \omega(S)$$

holds.

The question as to the validity of (3) is easily decided in the case when the set $\omega(S)$ of the chords of S is convex. Indeed in this case we infer from (1) and (2) that

$$\Omega(S) = \omega(\omega(S)) = \omega(S).$$

A well-known example of a set S , for which $\Omega(S) = \omega(S)$ holds is a pair of convex point sets in R_3 . In this example however every interior point of $\Omega(S)$ is contained in an infinity of chords of S .

In the present note I shall discuss a classe $\{\gamma\}$ of space-curves in R_3 which possess the following two properties

I $\Omega(\gamma) = \omega(\gamma),$

II *every interior point of $\Omega(\gamma)$ is contained in one and only one chord of γ .*

It is obvious that a space-curve containing four coplanar points cannot possess the property II. All the more surprising seems to be the fact, (see p. 69) that if γ contains no four coplanar points then it possesses both properties I

and II. Thus one of the results of the present paper is that the properties I and II together is equivalent to the fact that γ does not contain four coplanar points. The arcs of space-curves not containing four coplanar points, or according to the terminology of HJELMSJEV, the *simple* arcs of space-curves may be regarded as the three-dimensional analoga of the convex plane-curves (not containing three collinear points). Their properties stated in I and II admit the following more intuitive formulation:

The smallest convex cover of a simple arc of space-curve is filled up simply and without gap by its chords.

This result enables us to establish a simple formula for the calculation of an integral extended over the smallest convex cover of a simple arc of space-curve. Let us consider a simple arc given by the equations

$$x = x(t), y = y(t), z = z(t); a \leq t \leq b,$$

where $x(t)$, $y(t)$, $z(t)$ have continuous first derivatives with respect to t . Let us coincide the axis z with the terminating chord AB of the arc (i. e. the chord joining the terminating points of the arc):

$$(4) \quad x(a) = x(b) = y(a) = y(b) = 0; z(a) < z(b).$$

Any point of the chord joining $[x(t_1), y(t_1), z(t_1)]$ and $[x(t_2), y(t_2), z(t_2)]$ may be specified by the equations

$$(5) \quad \begin{aligned} x &= \frac{1+t_3}{2} x(t_1) + \frac{1-t_3}{2} x(t_2) \\ y &= \frac{1+t_3}{2} y(t_1) + \frac{1-t_3}{2} y(t_2) \\ z &= \frac{1+t_3}{2} z(t_1) + \frac{1-t_3}{2} z(t_2), \end{aligned}$$

and in consequence of the property II these equations establish a 1—1 correspondence between the interior points of $\Omega(\gamma)$ and the interior points of the region characterised by the inequalities

$$(6) \quad a \leq t_1 < t_2 \leq b; -1 \leq t_3 \leq 1.$$

Hence, owing to (6)

$$\int \int \int_{\Omega(\gamma)} f(x, y, z) dx dy dz = \frac{1}{2} \int_a^b \int_a^b \int_{-1}^{+1} \bar{f}(t_1, t_2, t_3) \frac{\partial(x, y, z)}{\partial(t_1, t_2, t_3)} dt_1 dt_2 dt_3.$$

Especially for the volume V of the smallest convex cover of γ we get

$$\begin{aligned} V &= \int \int \int_{\Omega(\gamma)} dx dy dz = \frac{1}{2} \int_{-1}^{+1} \int_a^b \int_a^b \frac{\partial(x, y, z)}{\partial(t_1, t_2, t_3)} t_1 dt_1 dt_2 dt_3 = \\ &= \frac{1}{16} \int_a^b \int_a^b \left| \begin{array}{ccc} \dot{x}(t_1) & \dot{x}(t_2) & x(t_1) - x(t_2) \\ \dot{y}(t_1) & \dot{y}(t_2) & y(t_1) - y(t_2) \\ \dot{z}(t_1) & \dot{z}(t_2) & z(t_1) - z(t_2) \end{array} \right| dt_1 dt_2 \cdot \int_{-1}^{+1} (1+t_3)(1-t_3) dt_3 = \end{aligned}$$

$$= \frac{1}{12} \int_a^b \int_a^b \begin{vmatrix} \dot{x}(t_1) & \dot{x}(t_2) & x(t_1) \\ \dot{y}(t_1) & \dot{y}(t_2) & y(t_1) \\ \dot{z}(t_1) & \dot{z}(t_2) & z(t_1) \end{vmatrix} dt_1 dt_2 - \frac{1}{12} \int_a^b \int_a^b \begin{vmatrix} \dot{x}(t_1) & \dot{x}(t_2) & x(t_2) \\ \dot{y}(t_1) & \dot{y}(t_2) & y(t_2) \\ \dot{z}(t_1) & \dot{z}(t_2) & z(t_2) \end{vmatrix} dt_1 dt_2.$$

Furthermore changing t_1 and t_2 we obtain according to (4)

$$(7) \quad V = \frac{1}{6} \int_a^b \int_a^b \begin{vmatrix} \dot{x}(t_1) & \dot{x}(t_2) & x(t_1) \\ \dot{y}(t_1) & \dot{y}(t_2) & y(t_1) \\ \dot{z}(t_1) & \dot{z}(t_2) & z(t_1) \end{vmatrix} dt_1 dt_2 =$$

$$= -\frac{1}{6} \int_a^b \dot{x}(t_2) dt_2 \cdot \int_a^b \begin{vmatrix} \dot{y}(t_1) & y(t_1) \\ \dot{z}(t_1) & z(t_1) \end{vmatrix} dt_1 + \frac{1}{6} \int_a^b \dot{y}(t_2) dt_2 \cdot \int_a^b \begin{vmatrix} \dot{x}(t_1) & x(t_1) \\ \dot{z}(t_1) & z(t_1) \end{vmatrix} dt_1 -$$

$$- \frac{1}{6} \int_a^b \dot{z}(t_2) dt_2 \cdot \int_a^b \begin{vmatrix} \dot{x}(t_1) & x(t_1) \\ \dot{y}(t_1) & y(t_1) \end{vmatrix} dt_1 = \frac{z(b) - z(a)}{6} \int_a^b \begin{vmatrix} x(t) & y(t) \\ \dot{x}(t) & \dot{y}(t) \end{vmatrix} dt.$$

As an application I shall discuss a special case of the following maximum-problem referring to space-curves which is as far as I know not yet solved hitherto. What is the maximum of the volume of the smallest convex covers of all arcs of space-curves which have a prescribed length L ?

If we restrict ourselves to the subset of all *simple* arcs of space-curves having the length L , then the maximum of the volume will be easily found to be

$$(8) \quad V_{\max} = \frac{L^3}{18\sqrt{3}\pi},$$

and this maximum is reached only for the arc of helix given by the equations

$$x = \frac{L}{\sqrt{6}\pi} \cos t$$

$$y = \frac{L}{\sqrt{6}\pi} \sin t \quad 0 \leq t \leq 2\pi.$$

$$z = \frac{L}{2\sqrt{3}\pi} t$$

* * *

Let be γ a simple arc with the terminating chord AB , suppose that γ has at each of its points P a tangent which varies continuously with P , and consider the cone which projects γ from a point C collinear with A and B and separated from A by B .

This projecting cone is closed (i. e. its line of intersection with a sphere having its centre at the vertex is a closed curve) because the projecting lines of the terminating points A and B coincide.

Apart from the double generator-line passing through A and B this

cone cannot contain any more double (or multiple) generator-line. Indeed, a double generator-line would contain two points of γ , which were coplanar with A and B , contrary to our assumption on γ .

Further the closed projecting cone must be convex (i. e. its line of intersection with any plane must be a convex curve), because in the contrary case the cone would have four coplanar generators and the points of γ projected by these four lines would be coplanar, contrary to our assumption.

Obviously the same statement holds if the point C , being collinear with A and B , is separated from B by A .

In this way it is proved that from any point of the extension of the terminating chord the simple arc γ is projected by a closed convex cone having only one double generator-line.

From this result we infer that the projecting cylinder of γ parallel to AB is a closed convex cylinder.

The cones which project the arc γ from one of its terminating points A, B , are not determined and their rôle will be assumed by two cones which will turn out to be the supporting cones (Stützkegel) of γ at A and B . The cone at A consists of all lines projecting the interior points of γ from A , and of an angular region of the supporting plane passing through AB and the tangent of γ at A . This cone and the corresponding one at B are both closed, convex cones in consequence of the same considerations as discussed above, and thus they constitute indeed the supporting cones of γ at A and B .

Let now the convex point set consisting of all points interior to both supporting cones be denoted by $D(\gamma)$. I assert that $D(\gamma)$ is identical to the smallest convex cover of γ .

To prove this assertion I will first show that

$$\omega(\gamma) = D(\gamma),$$

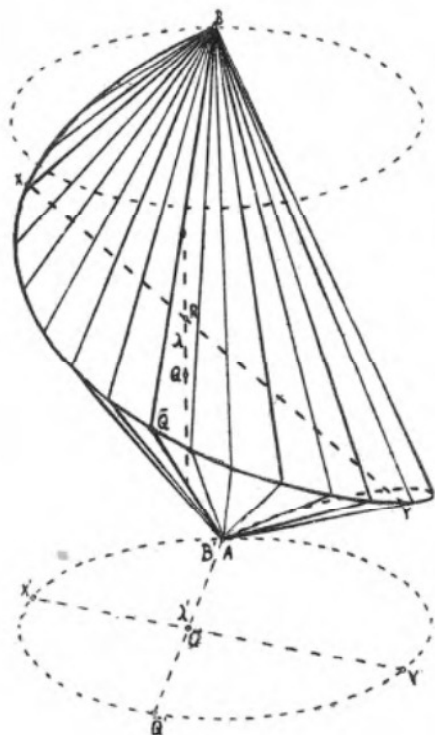
further, $D(\gamma)$ being convex

$$\omega(D(\gamma)) = D(\gamma)$$

and hence we draw the conclusion that

$$\Omega(\gamma) = \omega(\omega(\gamma)) = \omega(D(\gamma)) = D(\gamma).$$

In order to prove the relation $\omega(\gamma) = D(\gamma)$ take an arbitrary point Q in the interior of $D(\gamma)$. Draw the plane QAB , which cuts γ in a single point $\bar{Q}$, and draw a line λ through Q parallel to AB . Let now the plane $\bar{Q}AB$ rotate monotonously round λ until it has made a half-revolution. During this rotation the two points of intersection of γ with the rotating plane $XY\lambda$ are well de-



terminated and move continuously along γ . Hence the chord of γ joining the points of intersection X, Y of the rotating plane and γ as well as its intersection R with λ move continuously along λ . In the initial position of $XY\lambda$ this moving chord coincides with $\overline{QB}$, while in the final position it coincides with $\overline{AQ}$. Hence the point of intersection of the moving chord and the line λ must pass (at least once) through the arbitrary interior point Q .

But, γ being a simple arc, the moving chord can pass only once through Q , because in the contrary case γ would contain four coplanar points.

In this way we have proved that through every point of the convex region there is one and only one chord of γ , i. e.

$$\omega(\gamma) = D(\gamma),$$

therefore

$$\Omega(\gamma) = \omega(\omega(\gamma)) = \omega(D(\gamma)) = D(\gamma), \quad \text{q. e. d.}$$

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In order to determine the simple arc γ^* of a prescribed length $L = \sqrt{3} \cdot l > 0$ whose smallest convex cover has a maximal volume, let us remark that owing to (7)

$$V = \frac{z(b) - z(a)}{6} \int_a^b [x(t)y(t) - y(t)\dot{x}(t)] dt = \frac{q \cdot T}{3},$$

where q denotes the length of the terminal chord and T is the area bounded by the orthogonal projection of γ on a plane perpendicular to the terminal chord.

We establish the following inequalities.

1. If p is the length of the normal section of the projecting cylinder of γ parallel to AB , then by the extremal property of the circle

$$T \leq \frac{p^2}{4\pi}.$$

2. If the projecting cylinder of γ parallel to AB will be developed in a plane then from the extremal property of the straight line we infer that

$$p^2 + q^2 \leq 3l^2.$$

3. If $l > 0, q > 0$ then

$$(3l^2 - q^2)q \equiv 2l^3 - (l - q)^2(2l + q) \leq 2l^3.$$

Hence V satisfies the relations

$$V = \frac{qT}{3} \leq \frac{p^2q}{12\pi} \leq \frac{(3l^2 - q^2)q}{12\pi} \leq \frac{l^3}{6\pi} = \frac{L^3}{2 \cdot 3^{3/2}\pi}$$

and the equalities are valid if and only if

1. the projecting cylinder is circular, 2. the projecting cylinder being developed on a plane, γ becomes transformed into a straight line, 3. the length q of the terminating chord satisfies

$$q = l = \frac{L}{\sqrt{3}}.$$

The only space-curve satisfying these conditions is obviously an arc of a helix traced on a circular cylinder of radius $\frac{L}{\sqrt{6}\pi}$ and corresponding to a rotation 2π and a translation $\frac{L}{\sqrt{3}}$. Its smallest convex cover has the maximal value (8).

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