

A note on the idealizer of a subring.

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A familiar notion in group theory is the normalizer of a subgroup H of G defined as the greatest subgroup of G in which H is a normal subgroup. The corresponding concept in ring theory: the "idealizer" of a subring has been recently introduced in a special case by Prof. L. KALMÁR¹⁾. The purpose of the present note is to give an application of this new concept and to show that it is a useful notion in algebra.

By the *idealizer* of a subring T of a commutative ring R we shall mean the greatest subring S of R in which T is an ideal. This definition of idealizer may be reformulated in a more direct way if we introduce the ring-residual $T:\sigma$ of a subring T by an arbitrary set σ of elements of R : let $T:\sigma$ be the set of all $x \in R$ satisfying $x\sigma \subseteq T$. (The ring-residual is readily seen to be a subgroup of R^+ .) Now the idealizer may alternatively be defined as the ring-residual $T:T$. It is immediate that the ring-residual and hence the idealizer always exists.

Let R be a commutative ring and Q the quotient-ring²⁾ of R . Any ideal A of R may be considered as a subring of Q and so we may form the idealizer of A in Q . This is a subring $I(A)$ of Q containing R .

Theorem 1. *A ring R is integrally closed³⁾ in its quotient-ring if and only if R is the idealizer of each finite regular ideal⁴⁾ A of R .*

¹⁾ L. KALMÁR, Über die Cantorsche Theorie der reellen Zahlen. *Publicationes Mathematicae*, 1 (1950), pp. 150–159.

²⁾ The quotient-ring Q of R contains, besides the elements of R , all fractions a/b with a, b in R such that b is regular (no divisor of zero). [The usual definition of quotient-rings (e. g. B. L. VAN DER WAERDEN, *Moderne Algebra*, vol. I (1937), p. 45, or W. KRULL, *Idealtheorie* (1935), p. 19) leaves out of consideration the degenerated case of rings without regular elements; in this case the quotient-ring is empty according to the usual definition, while by completion with the term "besides the elements of R " it coincides with the original ring R .]

³⁾ By definition, R is *integrally closed* in its quotient-ring Q , if an algebraic equation $x^n + c_1 x^{n-1} + \dots + c_n = 0$ with $x \in Q$, $c_1, \dots, c_n \in R^*$ implies $x \in R$. Here R^* denotes a least overring of R with a unit element (if $1 \in R$, $R^* = R$), that is, R^* consists of all pairs (r, n) ($r \in R$,

In the proof we may restrict ourselves to the case if R contains at least one regular element, since otherwise the statement is trivial.

Let R be integrally closed, hence having a unit element, and let $A = (a_1, \dots, a_n)$ be a finite regular ideal of R . If x belongs to $A:A = I(A)$, then each element of xA is of the form $r_1 a_1 + \dots + r_n a_n$ with $r_i \in R^* = R$. Therefore $xa_i = r_{i1} a_1 + \dots + r_{in} a_n$ with $r_{ik} \in R$, and hence the determinant $\Delta = |r_{ik} - \varepsilon_{ik} x|$ (where ε_{ik} is equal to 0 or 1 according as $i \neq k$ or $i = k$) is an annihilator of A . Consequently, by the regularity of A , we get $\Delta = 0$. Hence we have an algebraic equation for x with coefficients in R and leading term x^n ; thus x is integral over R . By integral closure, $x \in R$, that is, $I(A) = R$.

Conversely, if the idealizer of each finite regular ideal is R , then for each $x \in Q$ satisfying an algebraic equation $x^n + c_1 x^{n-1} + \dots + c_n = 0$ ($c_i \in R^*$), we form the (fractional⁵⁾) ideal $A = (1, x, \dots, x^{n-1})$ which is finite and regular. Now $xA = (x, x^2, \dots, x^n) = (x, x^2, \dots, -c_1 x^{n-1} - \dots - c_n) \subseteq A$, and since by hypothesis $A:A = R$, we find $x \in R$, i. e. R is integrally closed, in fact. The proof is completed.

We define a prime ideal P of R to be *complete* if it is divisorless and at the same time $I(P) = R$; we may then prove

Theorem 2. *Let A be an ideal of an integral domain R with maximal condition. If all prime overideals of A are complete, then A may be represented uniquely as the product of prime ideals.*

For the proof we may proceed on the lines of B. L. VAN DER WAERDEN'S proof in his cited book³⁾ § 102. It is only to be noted that for a complete prime ideal P always $P \cdot P^{-1} = R$ holds. Indeed, otherwise $P \cdot P^{-1} = P$, and this would imply $P^{-1} \subseteq P:P = I(P) = R$, i. e., $P^{-1} = R$ which is absurd, considering that P^{-1} for each prime ideal P necessarily contains elements not in R^* .

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n a rational integer) with the composition rules $(r_1, n_1) + (r_2, n_2) = (r_1 + r_2, n_1 + n_2)$ and $(r_1, n_1) \cdot (r_2, n_2) = (r_1 r_2 + n_2 r_1 + n_1 r_2, n_1 n_2)$. The subrings $(r, 0)$ and $(0, n)$ can be identified with R and the ring of all rational integers, respectively, so that $(r, n) = r + n$ is actually a sum in R^* . Our definition of integral closure coincides with VAN DER WAERDEN'S terminology (Moderne Algebra, vol. II (1940), pp. 76–77) for rings with maximal condition. It is immediate that, whenever R contains at least one regular element, integral closure implies the existence of a unit element as the root of the equation $x^2 - x = 0$.

⁴⁾ An ideal A is *finite*, if it has a finite base and is *regular* if it contains at least one regular element.

⁵⁾ A fractional ideal may be made into an integral ideal by multiplication by a regular element. Hence if $I(A) = R$ holds for all finite integral regular ideals A , then the same must hold for all fractional ideals of the same type.

⁶⁾ See loc. cit.³⁾, § 102, Hilfssatz 3.