Notes on the foundations of lattice theory.

By DAVID ELLIS in Florida (U. S. A.)*)

1. Introduction.

A *lattice* may be defined as a set closed under two binary single-valued operations. $a \vee b$ and $a \wedge b$, which satisfies the following postulates[1]):

1. *Double Commutativity.*

$$a \vee b = b \vee a \wedge a \wedge b = b \wedge a; \qquad (\forall a, b)^{2)}.$$

2. *Double Associativity.*

$$a \vee (b \vee c) = (a \vee b) \vee c \wedge a \wedge (b \wedge c) = (a \wedge b) \wedge c; \quad (\forall a, b, c).$$

3. *Alternation.*

$$a \vee b = a \cdot\!\sim\!\cdot a \wedge b = b; \qquad (\forall a, b).$$

4. *Double Idempotency.*

$$a \vee a = a \wedge a = a; \qquad (\forall a).$$

A lattice is called *distributive* if it also satisfies the postulate of

5. *Double Distributivity.*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \wedge$$
$$a \vee (b \wedge c = (a \vee b) \wedge (a \vee c); \qquad (\forall a, b, c).$$

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* Presented to the Annual Mee ing of the American Mathematical Society, Christmas, 1950.

[1]) This list of postulates is logically equivalent to any of the usual definitions of lattice either as an abstract algebra or as a partially ordered set with extrema for pairs of elements.

[2]) We employ the following logical symbols, most of which are due to E. H. MOORE in this paper:

$\exists$ is read "there exist(s)"
$\forall$ is read "for all"
$\ni$ is read "such that"
$\cdot\!\supset\!\cdot$ is read "implies"
$\cdot\!\sim\!\cdot$ is read "if and on y if"
$\wedge$ is read "and"
$\vee$ is read "or (conjunctive)"
$\in$ is read "is an element of".

However, KARL MENGER [3][3]) has shown that on the basis of 1.—4. the postulate 5. is implied by the weaker

6. *Single Distributivity.*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c); \qquad (\forall a, b, c)$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c); \qquad (\forall a, b, c).$$

Hence, distributive lattices may be characterized by postulates 1.—4. and 6.

Other lists of postulates characterizing distributive lattices have been given by G. D. BIRKOFF and GARRETT BIRKHOFF [1] in terms of the lattice operations for those distributive lattices possessing "last" elements, and by S. A. KISS and GARRETT BIRKHOFF [2] in terms of the ternary lattice operation for those distributive lattices possessing "first" and "last" elements.

In a recent discussion between Professor ROY UTZ and the author the question was raised: *What alteration, if any, would be effected by weakening idempotency in the lattice postulates to a condition of finite potency* (defined shortly)? In this paper we give an answer to this question in the distributive case and also obtain a new list of postulates, in terms of the lattice operations, for distributive lattices which does not contain idempotency explicitly.

We shall refer to a system satisfying the postulates 1.—3. (that is, satisfying the lattice postulates except for idempotency) as a *pseudo-lattice* and say that it is a *distributive pseudo-lattice* if postulate 5. is also satisfied[4]). Greek letters refer to natural integers throughout the paper and the other elements under discussion are to be construed as elements of a distributive pseudo-lattice. An element a is called *α-potent* for $\vee$ (for $\wedge$) provided $\vee_{\alpha+1} a = a$ $(\wedge_{\alpha+1} a = a)$[5]). We shall say that an element a is *finitely potent* if there is an α so that a is α-potent for $\vee$ or for $\wedge$.

2. Idempotency.

Lemma 1. *If a is α-potent for $\vee$ then* $\vee_3 a = \vee_2 a$.

Proof. From $a \vee (\vee_\alpha a) = a$ we obtain, by alternation, $a \wedge (\vee_\alpha a) = \vee_\alpha a$. Then $a \vee (a \wedge (\vee_\alpha a)) = (\vee_\alpha a) \vee a = a$. But $a \vee (a \wedge (\vee_\alpha a)) = (\vee_2 a) \wedge ((\vee_\alpha a) \vee a) = (\vee_2 a) \wedge a$, by distributivity. Hence, $(\vee_2 a) \wedge a = a$ and $\vee_3 a = (\vee_2 a) \vee a = \vee_2 a$, by alternation.

The postulates 1.—3. and 5. obviously form a selfdual set (with respect to the interchange of $\vee$ and $\wedge$) so that Lemma 1 is valid when $\vee$ is replaced by $\wedge$.

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[3]) Numbers in parentheses refer to the list of references concluding the paper.

[4]) Without the assumption of idempotency (that is, on the basis of 1.-3. alone) 6. does not, in general, imply 5. See the example given in the Remark of Part 2 of the paper.

[5]) $\wedge_\alpha x$ is an abbreviation for $x \wedge x \wedge \ldots \wedge x$ with α "factors". $\vee_\alpha x$ has the obvious similar meaning.

Lemma 2. *If a is finitely potent then* $\vee_2 a = \wedge_2 a$.

Proof. For convenience suppose that a is α-potent for $\vee$. Then $\vee_3 a = \vee_2 a$, by Lemma 1. By alternation, $(\vee_2 a) \wedge a = a$. But $(\vee_2 a) \wedge a = \vee_2(\wedge_2 a) = a$. Then, by direct computation, $\wedge_3 a = \wedge_3(\vee_2(\wedge_2 a)) = \vee_8(\wedge_6 a) = (\wedge_5 a) \wedge (\vee_8 a) = (\wedge_5 a) \wedge (\vee_2 a)$ [6] $= (\wedge_4 a) \wedge (\vee_2(\wedge_2 a)) = (\wedge_4 a) \wedge a = \wedge_5 a$. Hence, $\wedge_3 a = \wedge_5 a = (\wedge_3 a) \wedge (\wedge_2 a)$ and, by alternation, $(\wedge_3 a) \vee (\wedge_2 a) = \wedge_2 a$. Then $a \wedge ((\wedge_2 a) \vee a) = (\vee_2(\wedge_2 a)) \wedge ((\wedge_2 a) \vee a) = (\vee_2(\wedge_4 a)) \vee (\vee_2(\wedge_2 a)) = \wedge_2 a$, by distributivity. For convenience in later reference we label the last equality

$$(*) \qquad (\vee_2(\wedge_4 a)) \vee (\vee_2(\wedge_2 a)) = \wedge_2 a.$$

Now, by direct computation, $\wedge_2 a = \vee_4(\wedge_4 a)$ since $a = \vee_2(\wedge_2 a)$. But $\vee_4(\wedge_4 a) = (\wedge_3 a) \wedge (\vee_4 a) = (\wedge_5 a) \wedge (\vee_4 a) = \vee_4(\wedge_6 a) = (\wedge_2 a) \wedge (\vee_4(\wedge_4 a)) = (\wedge_2 a) \wedge (\wedge_2 a) = \wedge_4 a$. Hence, $\wedge_2 a = \wedge_4 a$ and, from (*),

$$\wedge_2 a = (\vee_2(\wedge_2 a)) \vee (\vee_2(\wedge_2 a)) = \vee_2 a.$$

The proof proceeds dually if a is α-potent for $\wedge$.

Theorem 1. *If a is finitely potent with respect to either operation, $\vee$ or $\wedge$, then a is idempotent with respect to each operation.*

Proof. In view of Lemma 2, it suffices to show that $\vee_2 a = a \vee \wedge_2 a = a$. We suppose again, for convenience, that a is α-potent for $\vee$ since the proof will merely be dualized in the contrary case. From Lemma 2, $\vee_2 a = \wedge_2 a$ so that $\vee_2(\wedge_2 a) = \vee_4 a = \vee_2 a$ [6]. However, $\vee_2(\wedge_2 a) = a$ (as in the proof of Lemma 2) so that $\vee_2 a = a$

Remark. The following example shows that Theorem 1 is not, in general, valid if distributivity is ommitted. Let P be the pseudo-lattice with elements a, b, c and with operation tables

$\vee$	a	b	c
a	c	a	b
b	a	b	c
c	b	c	a

$\wedge$	a	b	c
a	b	b	b
b	b	b	b
c	b	b	b.

For $\vee$, b is idempotent while a and c are 3-potent. For $\wedge$, b is again idempotent while a and c have no finite potencies. It should also be noted that this example shows that postulate 6. does not imply 5. on the basis of 1.—3. alone, since in this example, $\wedge$ distributes over $\vee$, but $\vee$ does not distribute over $\wedge$.

[6]) From $\vee_3 a = \vee_2 a$ we obtain $\vee_\alpha a = \vee_2 a$ for $\alpha \geqq 2$.

3. A new list of postulates for distributive lattices.

The necessity part of the following theorem being obvious and the sufficiency part being an immediate consequence of Theorem 1, we have finally:

Theorem 2. *Let P be a set closed under two binary single-valued operations, $a \vee b$ and $a \wedge b$. A necessary and sufficient condition that P form a distributive lattice under $a \vee b$ and $a \wedge b$ is that the following conditions be satisfied:*

1. *Double Commutativity.*

$$a \vee b = b \vee a \wedge a \wedge b = b \wedge a; \qquad (\forall a, b)$$

2 *Double Associativity.*

$$a \vee (b \vee c) = (a \vee b) \vee c \wedge$$
$$a \wedge (b \wedge c) = (a \wedge b) \wedge c; \qquad (\forall a, b, c).$$

3. *Alternation.*

$$a \vee b = a \cdot \sim \cdot a \wedge b = b; \qquad (\forall a, b).$$

4. *Finite Potency*

$$a \in P \cdot \supset \cdot \exists \alpha \ (a \text{ natural integer}) \ni \vee_{\alpha+1} a = a \vee \wedge_{\alpha+1} a = a.$$

5. *Double Distributivity.*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \wedge$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c); \qquad (\forall a, b, c).$$

References.

[1] G. D. Birkhoff and Garrett Birkhoff, Distributive postulates for systems like Boolean algebras. *Trans. Am. Math. Soc.*, **60** (1946), 3–11.

[2] S. A. Kiss and Garrett Birkhoff, A ternary operation in distributive lattices. *Bull. Am. Math. Soc.*, **53** (1947), 749–752.

[3] Karl Menger, New foundations of projective and affine geometry. *Annals of Math* (II. s.), **37** (1936), 356–482.

(Received July 19, 1950.)