

A note on means of entire functions.

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1. Let $f(z)$ be an entire function of order ρ and lower order λ and let

$$\rho_1 = \overline{\lim}_{r \rightarrow \infty} \frac{\log n(r)}{\log r}$$

$$\lambda_1 = \lim_{r \rightarrow \infty} \frac{\log n(r)}{\log r}$$

where $n(r)$ denotes the number of zeros of $f(z)$ in $|z| \leq r$. Let $G(r)$ and $g(r)$ denote the geometric means of $|f(z)|$ on the circumference $|z| = r$ and the circle $|z| \leq r$ respectively. Then for $r > 0$,

$$G(r) = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \right\},$$

$$g(r) = \exp \left\{ \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} \log |f(\rho e^{i\theta})| \rho d\theta d\rho \right\}.$$

We have $G(r) \leq \exp(T(r))$, $g(r) \leq \exp \left\{ \frac{2}{r^2} \int_0^r T(\rho) \rho d\rho \right\}$. If $n(r) \equiv 0$ then $G(r) = |f(0)| = g(r)$.

Suppose now that $n(r) > 0$ for $r > r_0$ and write

$$K = \overline{\lim}_{r \rightarrow \infty} \left(\frac{g(r)}{G(r)} \right)^{1/\mu(r)}$$

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It is known that¹⁾ (I) if $0 < \rho_1 < \infty$, then

$$(1) \quad k \leq \exp \left(\frac{-1}{2 + \rho_1} \right) \leq K$$

¹⁾ See [1], II. p. 10, problems 64, 65, 66,

and that (II) if $n(r) \sim r^{\rho_1} l(r)$ or $r^{\rho_1}/l(r)$ where $l(r)$ is a monotone increasing function such that $l(2r) \sim l(r)$ then $k = K$. I prove here

Theorem 1. *If $\lim_{r \rightarrow \infty} n(r) > 0$, then*

$$(2) \quad \exp\left(-\frac{1}{2}\right) \leq k \leq \exp\left(\frac{-1}{2+\lambda_1}\right) \leq \exp\left(\frac{-1}{2+\rho_1}\right) \leq K \leq 1.$$

Theorem 2. *Let $\Phi(x)$ be positive and continuous for $x > 1$ and satisfy the condition $\Phi(ax) \sim \Phi(x)$, as $x \rightarrow \infty$, for every fixed positive a . If $n(r) \sim r^{\rho_1} \Phi(r)$ then $k = K$.*

2. Let $\tilde{M}(r)$ and $m(r)$ denote the arithmetic means of $|f(z)|^2$ on $|z| = r$ and $|z| \leq r$ respectively. Let

$$L = \lim_{r \rightarrow \infty} \left(\frac{m(r)}{\tilde{M}(r)} \right)^{1/\log r}$$

$$l = \lim_{r \rightarrow \infty} \left(\frac{m(r)}{\tilde{M}(r)} \right)^{1/\log r}.$$

It is known that¹⁾ when $\rho < \infty$, $l = e^{-\rho}$. I prove here

Theorem 3. *For $0 \leq \lambda \leq \infty$, $L = e^{-\lambda}$. When $\rho = \infty$, $l = 0$.*

3. *Proof of Theorem 1.* Let $(z_n)_1^\infty$ denote the zeros of $f(z)$, $|z_n| = r_n$ and suppose that $0 = r_1 = \dots = r_q < r_{q+1} \leq r_{q+2} \dots$ ($q \geq 0$).

(I) Let $n > q + 1$, $r_n \leq r < r_{n+1}$, $f(z) = z^q F(z)$. Then $G(r, f) = r^q G(r, F)$; $g(r, f) = r^q e^{-q/2} g(r, F)$,

$$(3) \quad \left(\frac{g(r)}{G(r)} \right)^{1/n(r)} = \exp \left(-\frac{1}{2} + \frac{1}{2n(r)} \sum_{r_v \leq r} \left(\frac{r_v}{r} \right)^2 \right).$$

Further

$$\sum_{r_v \leq r} (r_v)^2 = \int_0^r x^2 dn(x) = r^2 n(r) - 2 \int_0^r x n(x) dx.$$

Hence

$$(4) \quad \left(\frac{g(r)}{G(r)} \right)^{1/n(r)} = \exp \left(-\frac{1}{r^2 n(r)} \int_0^r x n(x) dx \right)$$

and so $k \geq \exp\left(-\frac{1}{2}\right)$, $K \leq 1$.

(II) We now prove $k \leq \exp\left(\frac{-1}{2+\lambda_1}\right)$. We suppose $\lambda_1 < \infty$, for if $\lambda_1 = \infty$ then this follows from the relation $g(r) \leq G(r)$. We also suppose that $n(r) \rightarrow \infty$,

for if $n(r) = O(1)$, it would follow from (3) that $k = K = \exp\left(-\frac{1}{2}\right)$. We have

$$\lambda_1 = \lim_{n \rightarrow \infty} \frac{\log n}{\log r_n}.$$

Hence if $\beta > \lambda_1$, $\lim_{n \rightarrow \infty} n/r_n^\beta = 0$. Hence we can find a sequence of integers n_p ($p = 1, 2, \dots$) such that²⁾

$$\frac{N}{r_N^\beta} < \frac{\mu}{r_\mu^\beta}, \quad \mu = q+1, q+2, \dots, N-1; \quad N = n_p.$$

Hence $r_N > r_{N-1}$. Choose r such that $r_{N-1} < r < r_N$ and

$$\frac{N}{r^\beta} \leq \frac{\mu}{r_\mu^\beta} \quad \text{for } \mu = q+1, q+2, \dots, N-1.$$

Then

$$\left(\frac{r_\mu}{r}\right)^2 \leq \left(\frac{\mu}{N}\right)^{2/\beta} \quad \text{for } \mu = 1, 2, \dots, N-1.$$

$$\sum_{\mu=1}^{N-1} \left(\frac{r_\mu}{r}\right)^2 \leq \frac{(N-1)^\beta}{2+\beta} + o(N).$$

Hence for these values of r we have from (3)

$$\left(\frac{g(r)}{G(r)}\right)^{1, n(r)} \leq \exp\left[-\frac{1}{2} + \frac{1}{2(N-1)}\left(\frac{(N-1)^\beta}{2+\beta} + o(N)\right)\right]$$

and our statement follows.

(III) We now prove that when $\rho_1 = \infty$, $K = 1$. Since $\rho_1 = \infty$, we have for any (arbitrary large) positive constant H , $\lim_{x \rightarrow \infty} n(x)/x^H = \infty$. Hence we can choose a sequence of points R_n ($n = 1, 2, \dots$) such that

$$\frac{n(x)}{x^H} \leq \frac{n(R_n)}{R_n^H} \quad \text{for } r_{q+1} \leq x \leq R_n.$$

Set $R_n = R$. Then

$$\frac{1}{R^2 n(R)} \int_0^R x n(x) dx < \left(O(1) + \int_{r_{q+1}}^R \frac{x^{H+1} n(R) dx}{R^H}\right) \frac{1}{R^2 n(R)} < o(1) + \frac{1}{H+2}.$$

Hence

$$\lim_{r \rightarrow \infty} \frac{1}{r^2 n(r)} \int_0^r x n(x) dx = 0$$

and from (4) we get $K = 1$. This completes the proof of Theorem 1.

²⁾ See [1], I, p. 18, problem 107.

4. We now show that (2) is "best possible".

Example 1. Given ϱ , $0 < \varrho < \infty$, let

$$a_n = 2^{2^{n-1}} (n \geq 1), a_0 = 0, p = 1 + [\varrho],$$

$$\xi_n = [a_n^{\varrho/p}] - [a_{n-1}^{\varrho/p}], f(z) = \prod_{n=1}^{\infty} \left\{ 1 + \frac{z^p}{a_n} \right\}^{\xi_n}.$$

Then $f(z)$ is a canonical product of order ϱ and $\lambda_1 = \varrho/2$. For this function $k = \exp\left(-\frac{1}{2}\right)$, $K = 1$.

Example 2. Given λ_1 and ϱ_1 , $0 < \lambda_1 < \varrho_1 < \infty$, let $p = 1 + [\varrho_1]$, $x_1 = 3$, $x_{n+1} = (x_n!)^n$ ($n \geq 1$), $a_1 = a_2 = 1$,

$$a_m = m^{1/\varrho_1} \quad \text{for } x_n \leq m \leq x_{n+1}!$$

$$= m^{1/\lambda_1} \quad \text{for } x_n! < m < x_{n+1}! = z_n \text{ (say)}$$

$$= Am + B \quad \text{for } z_n \leq m \leq x_{n+1},$$

where A and B are chosen such that

$$a_{x_{n+1}} = (x_{n+1})^{1/\varrho_1}, a_{z_n} = (z_n)^{1/\lambda_1}; \quad (n = 1, 2, 3, \dots)$$

Let

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \left(\frac{z}{a_n} \right)^p \right).$$

Then $f(z)$ is a canonical product of order ϱ_1 for which

$$\lim_{r \rightarrow \infty} \frac{\log n(r)}{\log r} = \lambda_1, k = \exp\left(\frac{-1}{2 + \lambda_1}\right), K = \exp\left(\frac{-1}{2 + \varrho_1}\right).$$

5. *Proof of Theorem 2.* We have ³⁾

$$\int_0^r x n(x) dx \sim \int_1^r x^{1+\varrho_1} \Phi(x) dx \sim \frac{r^{2+\varrho_1}}{2+\varrho_1} \Phi(r) \sim \frac{r^2}{2+\varrho_1} n(r).$$

Hence from (4), the theorem follows.

6. *Proof of Theorem 3.* If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $M(r) = \text{Max}_{|z|=r} |f(z)|$ then we have

$$\tilde{M}(r) = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}, m(r) = \sum_{n=0}^{\infty} \frac{|a_n|^2 r^{2n}}{n+1}.$$

Let

$$H(z) = \sum_{n=0}^{\infty} \frac{|a_n|^2 z^{2n+2}}{n+1}; \mu(r) = \text{maximum term of } f(z).$$

Then

$$\frac{r^2 \{\mu(r)\}^2}{\nu(r) + 1} < H(r) < r^2 \{M(r)\}^2.$$

³⁾ See [2], p. 54, Lemma 5.

Hence $\varrho \equiv \varrho(f) = \varrho(H)$ and $\lambda \equiv \lambda(f) = \lambda(H)$, ($0 \leq \varrho \leq \infty$). Further

$$\frac{\tilde{M}(r)}{m(r)} = \frac{rH'(r)}{2H(r)}$$

and ⁴⁾

$$\overline{\lim}_{r \rightarrow \infty} \frac{1}{\log r} \log \frac{rH'(r)}{H(r)} = \varrho,$$

$$\lim_{r \rightarrow \infty} \frac{1}{\log r} \log \frac{rH'(r)}{H(r)} = \lambda.$$

Hence Theorem 3 follows.

References.

- [1] G. PÓLYA und G. SZEGŐ, Aufgaben und Lehrsätze aus der Analysis I, II. Berlin, 1925.
- [2] G. H. HARDY and W. W. ROGOSINKI, Notes on Fourier series. III. *Quarterly J. Math.* (Oxford series) 16 (1945), 49—58.
- [3] S. M. SHAH, A note on the derivatives of integral functions. *Bull. Amer. Math. Soc.*, 53 (1947), 1156—1163.

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⁴⁾ See [3], pp. 1156—59.