

On completely integrable systems

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The objective of this paper is to give a geometric formulation, using contact theory, for differential systems of order k and dimension n over a differentiable manifold. In the case $k = 1$ we obtain the classical Frobenius theorem.

Let (M, N, π) be a fibred manifold and $J^k M$ the k -jets bundle of cross-sections of (M, N, π) . Two functions defined on a neighborhood of $X \in J^k M$ are identified, and the equivalence class is called a germ of functions at X . The set of all germs at X is denoted by $\Omega_X J^k M$. Given an open set $\mathbf{U} \subset J^k M$, let $\Omega \mathbf{U} = \bigcap \{\Omega_X, X \in \mathbf{U}\}$.

A System of Partial Differential Equations (P.D.E.) of order k in (M, N, π) is defined as an open set $\mathbf{U}$ in $J^k M$, together with a locally, finitely generated, subsheaf of ideals Φ of $\Omega \mathbf{U}$ [3]. The set $\mathbf{U}$ is called the domain of the S.P.D.

A cross-section f of (M, N, π) is said to be a solution of the equation $\Phi = 0$ (or a solution of Φ , for simplicity) if and only if, for any $x \in \text{Dom } f$, the jet $j_x^k f$ is in the domain of Φ and $F(j_x^k f) = 0$, for any F belonging to Φ . If $j_{x_0}^k f = X_0$ we say that f is a solution at X_0 .

An integral jet X of a S.P.D. Φ , with domain $\mathbf{U}$, and order k is a k -jet $X \in \mathbf{U}$ such that $F(X) = 0$, for all F belonging to Φ_x . The set of all the integral jets of Φ is denoted by $J\Phi$. A cross-section f of (M, N, π) is then a solution of Φ if and only if $j_x^k f \in J\Phi$, for all $x \in \text{Dom } f$.

A system Φ of order k defined on a fibred manifold (M, N, π) is said to be completely integrable at $X_0 \in J\Phi$ if there exists a solution f of Φ at X_0 .

Denote by $C^{k,n} M$, the contact bundle of order k of n -dimensional submanifolds in M , and by $C_x^k S$ the contact element of order k at $x \in S \subset M$ [2].

Given an imbedded submanifold $W \subset C^{k,n}M$, we will define an associated system of partial differential equations Φ in (V, U, π) , where $V \subset M$ is an open subset fibered over the open set $U \subset \mathfrak{R}^n$. We will give sufficient conditions on W for the complete integrability of Φ . Moreover, if $X_0 \in W$ and $S \subset M$ is an n -submanifold such that $C_x^k S \in W$, for all $x \in S$ and $C_{x_0}^k S = X_0$ then, we can obtain a solution of Φ using a local parametrization of S in a neighborhood of x_0 .

This allows us to give a geometrical interpretation of a completely integrable system of order k , using contact theory.

For $k = 1$, the submanifold W defines a integrable distribution on M , yielding the classical Frobenius Theorem.

1. Jet theory and completely integrable systems

Let (M, N, π) be a fibred manifold, and f, g two cross-sections whose domains contain $x_0 \in N$. Let k be an integer, $k \geq 0$. We say that f and g are k -equivalent at x_0 if the following condition is satisfied:

For any fibred chart (x, y) , where the domain of (x) contains x_0 , and for any partial derivative ∂^l in (x) of order $l \leq k$ we have: $\partial^l f^j(x_0) = \partial^l g^j(x_0)$, where $f^j = y^j \circ f$ (resp. $g^j = y^j \circ g$) is the expression of f (resp. of g) in terms of (x, y) .

If f is a cross-section whose domain contains x_0 , the equivalence class containing f is called k -jet of f at x_0 and will be denoted by $j_{x_0}^k f$. Denote by $J_x^k(M, N, \pi)$ the set of all k -jets at x of cross-sections of (M, N, π) , and by $J^k(M, N, \pi)$ the set of all k -jets of cross-sections of (M, N, π) . We shall write $J_x^k M$ (resp. $J^k M$) when there is no possibility of confusion.

If $X = j_x^k f$, we set $\alpha(X) = x$ and $\beta(x) = f(x)$, thus α (resp. β) is a map of $J^k M$ into N (resp. N), which is called the source map (resp. target map).

Denote by $I_l = (i_1, \dots, i_l)$ an ordered l -uple of integers $1, \dots, n$, with $n = \dim N$.

Let (x_i, y^j) be a fibred chart defined on $U \subset N$. If $X = j_{x_0}^l f$ is in $\beta^{-1}(U)$ and $f^j(x)$ is the expression of f , then $(x, y, p_{I_l}^j(X))$, $1 \leq l \leq k$, will be called the chart of $J^k M$ associated with (x, y) , where

$$p_{I_l}^j(X) = \frac{\partial^l}{\partial x_{i_1} \dots \partial x_{i_l}} f^j(x_0).$$

We consider the manifold structure on $J^k M$ given by these charts.

We have that $(J^k M, N, \alpha)$ and $(J^k M, N, \beta)$ are fibred manifolds.

For $l \leq k$, consider the map

$$\pi_l^k : j_x^k f \in J^k M \longmapsto j_x^l f \in J^l M,$$

then $(J^k M, J^l M, \pi_l^k)$ is also a fibred manifold [1].

Two functions defined on a neighborhood of $w \in M$ are identified if they coincide in some neighborhood of w . The equivalence class is called germ of functions at w . The germ of a function ϕ at w will be denoted by $[\phi]_w$. The set of germs of functions at w is a ring, and will be denoted by $\Omega_w M$.

We set $\Omega M = \bigcup \{\Omega_w M; w \in M\}$. If Φ is a subset of ΩM , we set $\Phi_w = \Phi \cap \Omega_w M$.

A function ϕ defined on an open set $U \subset M$ is said to be in Φ (denoted $\phi \in \Phi$) if $[\phi]_w \in \Phi$, for any $w \in U$.

The set Φ is called a subsheaf of ΩM if the following two conditions are satisfied for any $w \in M$:

- 1) Φ is not empty,
- 2) For any $a \in \Phi_w$ we can find a function ϕ defined on a neighborhood of w such that $a = [\phi]_w$ and $\phi \in \Phi$.

If U is an open subset of M and Φ is a subsheaf, $\bigcup \{\phi_x; w \in U\}$ is a subsheaf of ΩU and is called the restriction of Φ to U . A subsheaf Φ of ΩM is called a subsheaf of ideals when Φ_w is an ideal of $\Omega_w M$ for any $w \in M$.

Let $F_1, \dots, F_s$ be a finite set of functions defined on an open set $U \subset M$, and Φ_w , $w \in U$, the ideal of $\Omega_w M$ generated by $[F_1]_w, \dots, [F_s]_w$. Then $\Phi = \bigcup \{\phi_w; w \in U\}$ is a subsheaf of ideals of ΩU which is said to be generated by ΩU .

A subsheaf of ideals is said to be locally finitely generated when its restriction to an open neighborhood of each point is finitely generated.

A *system of partial differential equations* (P.D.E.) of order k in (M, N, π) is an open set $\mathbf{U} \in J^k M$, together with a locally finitely generated subsheaf of ideals of $\Omega \mathbf{U}$. The set $\mathbf{U}$ is called the domain of the P.D.E. .

An *integral jet* of a P.D.E. Φ , with domain $\mathbf{U}$, of order k , is a k -jet $X \in \mathbf{U}$ such that $F(X) = 0$, for all $F \in \Phi$. We denote by $J\Phi$ the set of integral jets of Φ .

A *solution* of Φ is a cross-section f of (M, N, π) , defined over an open set $U \subset \alpha(\mathbf{U})$ such that, for any $x \in U$, the jet $j_x^k f \in \mathbf{U}$. If $X_0 \in \mathbf{U}$ and $j_{x_0}^k f = X_0$, with $x_0 \in U$, we say that f is a solution of Φ at x_0 .

Let $F_1, \dots, F_s$ be a finite set of functions on an open set $\mathbf{U} \in J^k M$. Then the subsheaf Φ of ideals of $\Omega \mathbf{U}$ generated by $F_1, \dots, F_s$ is a S.P.D. of order k .

Assume that $\mathbf{U}$ is contained in the domain of the chart $(x, y, p_{I^l}^j)$ associated with a fibred chart (x, y) . Then a cross-section $f = (x, f^j)$ is a solution of the equation $\Phi = 0$ if $f^j(x)$ is a solution of the system of partial differential equations

$$F_r(j_x^k f) = 0, \quad 1 \leq r \leq s.$$

The system Φ is *complete* at $X \in \mathbf{U}$ if for any function F , defined on an open set $\mathbf{V} \subset U$, which vanishes on $J\Phi \cap \mathbf{V}$, the restriction of F to an open neighborhood of X belongs to Φ .

Let F be a function defined on an open set $\mathbf{U} \subset J^k M$ and θ a vector field on $\alpha(\mathbf{U}) \subset N$. The *formal derivative* of F with respect to θ , denoted by $\partial_\theta^\# F$, is a function on $(\pi_k^{k+1})^{-1}(\mathbf{U})$ defined as follows:

$$\partial_\theta^\# F : j_{x_0}^{k+1} f \longmapsto \theta_{x_0}(F \circ j_x^k f),$$

where $j_x^k f : \text{Dom } f \longrightarrow J^k M$ is defined by $x \longmapsto j_x^k f$. This definition is independent of the choice of representative f [3].

Let Φ be a P.D.E. with domain $\mathbf{U} \subset J^k M$. For any $X^{k+1} \in (\pi_k^{k+1})^{-1}(\mathbf{U}) = \mathbf{U}'$, let $(p\Phi)_{X^{k+1}}$ be the ideal of germs at X^{k+1} generated by

$$\{F \circ \pi_k^{k+1}, \partial_\theta^\# F; \quad F \in \Phi_{X^k}, \theta \in \chi(\alpha(\mathbf{U}))\}.$$

The subsheaf of ideals of $\Omega\mathbf{U}'$ generated by $(p\Phi)_{X^{k+1}}$, with $X^{k+1} \in \mathbf{U}'$, is called *the prolongation* of Φ and denoted by $P\Phi$.

Given $X^k \in J^k M$ denote by $Q_{X^k} J^k M$ (or Q_{X^k} when there is no possibility of confusion) the set,

$$Q_{X^k} = \text{kernel } \{d\pi_{k-1}^k : T_{X^k} J^k M \longrightarrow T_{X^{k-1}} J^{k-1} M\}$$

and $C_{X^k} \Phi$ the vector subspace defined by

$$C_{X^k} \Phi = \{v \in T_{X^k} Q_{X^k} : v(F) = 0; \quad F \in \Phi_{X^k}\}.$$

This subspace is called *the Symbol* of Φ at X^k .

A system of partial differential equations Φ is said to be *completely integrable* at $X \in J\Phi$ if the following conditions are satisfied:

- 1) $C_X \Phi = 0$,
- 2) The image of $J(P\Phi)$ by π_k^{k+1} is a neighborhood of X in $J\Phi$,
- 3) Φ is complete at X .

Theorem 1. Assume that a P.D.E. Φ of order k is completely integrable at $X \in J\Phi$. Then there is a solution f of Φ at X ; moreover, the germ of f at $\alpha(X)$ is uniquely determined.

PROOF. (see [3]).

A system of partial differential equations Φ , with domain $\mathbf{U}$, is regular at $X \in \Phi$ if:

- i) $J\Phi$ is a submanifold on a neighborhood of X ,
- ii) there exist functions $\{F_1, \dots, F_s, \quad F_i \in \Phi\}$, where $s + \dim J\Phi = \dim \mathbf{U}$, such that, $\{dF_1, \dots, dF_s\}$ are linearly independent at X (as elements in $T_X^* \mathbf{U}$).

Proposition 1. Let Φ be a regular S.P.D. defined on an open set $\mathbf{U} \subset J^k M$. Suppose that $J\Phi$ is a regular submanifold in $J^k M$, fibered on $U \subset N$ by α , then

$$J(P\Phi) = J^{k+1}M \cap J^1(J\Phi, U, \alpha).$$

PROOF. Let $X \in J(P\Phi)$ be an integral jet of the prolongation of Φ , defined by $X = j_u^{k+1}f$ and

$$H_X = T_u(j^k f)(T_u U).$$

It is clear that if $Y \in J(P\Phi)$ then,

$$X = Y \iff H_X = H_Y,$$

then X can be identified with the subspace H_X . Moreover, $X \in J(P\Phi)$ if and only if $H_X \subset T_{\pi_k^{k+1}(X)} J\Phi$, indeed:

$$\begin{aligned} X \in J(P\Phi) &\Leftrightarrow F(X) = 0, \text{ and } \partial_{x_i}^\#(F \circ j^k f)|_X = 0, \quad F \in \Phi \\ &\Leftrightarrow \frac{\partial}{\partial x_i} \Big|_u (F \circ j^k f) = 0 \quad \& \quad F(X) = 0, \quad F \in \Phi \\ &\Leftrightarrow T_u(j^k f) \left(\frac{\partial}{\partial x_i}(F) \right) = 0 \quad \& \quad F(X) = 0 \\ &\Leftrightarrow T_u(j^k f) \left(\frac{\partial}{\partial x_i} \right) \in T_X(J\Phi). \end{aligned}$$

On the other hand, a $(k+1)$ -jet $X \in P\Phi$ if and only if there exists a section $\sigma : U \longrightarrow J\Phi$ such that $X = j_{\alpha(X)}^1 \sigma$. This is clear because, as we

have shown, $H_X \in T_{\pi_k^{k+1}(X)} J\Phi$. In consequence, we have:

$$\begin{aligned} X \in J(P\Phi) &\Leftrightarrow X = j_u^{k+1} f, \quad f : U \longrightarrow V \subset M \quad \text{section} \\ &\& \quad X = j_u^1 \sigma, \quad \sigma : U \longrightarrow J\Phi \quad \text{section} \\ &\Leftrightarrow X \in J^{k+1}M \cap J^1(J\Phi, U, \alpha). \quad \square \end{aligned}$$

2. Contact manifolds and jet theory

Let M be a smooth $(n+m)$ -dimensional manifold, k an integer, $k \geq 0$, and $\tilde{J}_0^k(\mathfrak{R}^n, M)$ be the k -jets at 0 of maximal rank maps from $\mathfrak{R}^n$ into M .

Given $X_1, X_2 \in \tilde{J}_0^k(\mathfrak{R}^n, M)$ with $\beta(X_1) = \beta(X_2) = x \in M$. We say that X_1 and X_2 are equivalent at $x \in M$ if there exist $Y \in \tilde{J}_0^k(\mathfrak{R}^n, \mathfrak{R}^n)$ such that $X_2 = X_1 \circ Y$.

A class of this equivalence relation is called *the contact element* of order k and dimension n at x , denoted by $[j_x^k f]$. Let $C_x^{k,n}M$ denote the set of all contact elements of order k at x , and $C^{k,n}M$ the set of all k -contact elements of dimension n on M .

Let $S \subset M$ be an imbedded n -dimensional submanifold and f, g , two local parametrizations of S at $x \in M$ over a neighborhood $V \subset M$, with $f(0) = g(0) = x$, then

$$g^{-1}|_{V \cap S} : V \cap S \longrightarrow \mathfrak{R}^n$$

is a local diffeomorphism and

$$h = g^{-1} \circ f : A \subset \mathfrak{R}^n \longrightarrow B \subset \mathfrak{R}^n$$

is a local diffeomorphism such that $j_0^k f = j_0^k g \circ j_0^k h$, thus $[j_0^k f] = [j_0^k g]$.

The equivalence class $[j_0^k f]$ is called the contact element of order k of S at $x \in S$ and denoted $C_x^k S$.

Two imbedded submanifolds S_1, S_2 have contact of order k at $x \in S_1 \cap S_2$ if there exist local parametrizations given by imbeddings

$$f_1, f_2 : U \subset \mathfrak{R}^n \longrightarrow M$$

and a local coordinate system (V, y^j) , $1 \leq j \leq m$, about $x \in M$ such that $f_1(0) = f_2(0) = x$ and the partial derivatives at 0 of $y^j \circ f_1$ and $y^j \circ f_2$ are equal up to order k .

Remarks.

1. Clearly $C_x^1 S_1 = C_x^1 S_2$ if and only if $T_x S_1 = T_x S_2$.

2. Given a map $f : U \subset \mathfrak{R}^n \longrightarrow M$ of maximal rank defined on a neighborhood $U \subset \mathfrak{R}^n$, with $f(u) = x \in M$, and τ_u the traslation in $\mathfrak{R}^n$ such that $\tau_u(0) = u$. Then,

$$j_0^k(f \circ \tau_u) \in \tilde{J}_0^k(\mathfrak{R}^n, M) \quad \& \quad [j_0^k(f \circ \tau_u)] \in C_x^{k,n} M.$$

Let $X \in C^{k,n} M$ and $f : U_f = \text{Dom } f \longrightarrow V \subset M$ such that $[j_0^k f] = X$.

Consider a local coordinate system $(V, \varphi = (x^i, y^j))$, $1 \leq i \leq n$, $1 \leq j \leq m$, such that:

$x^i \circ f = \xi$, the canonical coordinates in $\mathfrak{R}^n$.

Denote by $\pi : \mathfrak{R}^{n+m} \longrightarrow \mathfrak{R}^n$ the canonical projection.

Then (V, U_f, ρ) , with $\rho = \pi \circ \varphi$, is a fibred manifold, said to be associated to X , and f is a cross-section.

Define a neighborhood $\mathbf{V}$ of X as:

$$\mathbf{V} = \{Y = C_u^k h(U) \in C^{k,n} V : h : U \longrightarrow V \text{ section, } u \in U\}.$$

Let

$$\Psi : J^k(V, U, \rho) \longrightarrow \mathbf{V}$$

given by

$$\Psi(j_u^k f) = [j_0^k(f \circ \tau_u)],$$

this map is a bijection and will be denoted by Ψ , when there is no possibility of confusion.

A coordinate neighborhood at $X = [j_0^k f]$ is given by:

$$(\Psi^{-1}(\mathbf{U}), \eta \circ \Psi^{-1}),$$

where $(\mathbf{U}, \eta)$ is a coordinate neighborhood at $j_0^k f \in J^k V$.

Consider the manifold structure in $C^{k,n} M$ given by all coordinate neighborhoods defined above.

With this differential structure the natural projections,

$$\rho_0^k : C_x^k S \in C^{k,n} M \longmapsto x \in M, \quad \& \quad \rho_l^k : C_x^k S \longmapsto C_x^l S \in C^{l,n} M$$

are submersions, and the natural injections,

$$i^k : x \in S \longmapsto C_x^k S \in C^{k,n} M$$

and

$$i^{k+1} : C_x^{k+1} S \in C^{k+1,n} M \longmapsto C_{C_x^k S}^1(C^k S) \in C^{1,n}(C^{k,n} M)$$

are immersions.

Moreover we have the differential map

$$\Psi^{1,k} : J^1(J^k V, \alpha, U) \longrightarrow C^{1,n}(C^{k,n} V)$$

defined by

$$j_u^1(\sigma) \longmapsto C_{\sigma(u)}^1(\Psi(\sigma(U))).$$

3. Differential systems of order k and dimension n

Definitions. By a differential system of order k and dimension n in M we mean an imbedded submanifold $W \subset C^{k,n} M$.

A solution of a differential system W at $X \in W$, is a n -dimensional imbedded submanifold $S \subset M$, with $x = \rho_0^k(X) \in S$, such that $C^k S \subset W$ and $C_x^k S = X$.

Example. Let D be a differentiable distribution of n -planes defined on M . By Remark 1, we can identify a plane $D_x \in D$ with a contact element $D_x \in C^{1,n} M$. Consider the map,

$$\vartheta : x \in M \longmapsto D_x \in C^{1,n} M.$$

If D is a differentiable distribution, then this map is an imbedding, and $\vartheta(M)$ is a differential system of order 1 and dimension n in M .

Moreover, if D is an involutive distribution, then the differential system W has solution.

Definition [5]. The first prolongation of a submanifold $W \subset C^{k,n} M$ is defined as:

$$PW = C^{1,n} W \cap C^{k+1,n} M,$$

where $C^{k+1,n} M$ is identified with its image by $i^{1,k}$ in $C^{1,n}(C^{k,n} M)$.

Theorem. Let $W \subset C^{k,n} M$ be an imbedded submanifold such that the following conditions are satisfied:

- 1) $\rho_{k-1}^k : W \longrightarrow C^{k-1,n} M$, is a local immersion in a neighborhood of $X \in W$.
- 2) $\rho_k^{k+1} : PW \longrightarrow W$ is a local submersion in a neighborhood of X ,

then there exists a solution $S \subset M$ of the differential system W passing through X . Moreover, if $\tilde{S}$ is another submanifold of W passing through X , then there exists an open set $A \subset M$, $x = \rho_0^k(X) \in A$, such that $S \cap A = \tilde{S} \cap A$.

PROOF. Let $f : U \subset \mathfrak{R}^n \longrightarrow V \subset M$, $x \in V$ be an immersion such that $X = [j_0^k f]$, and (V, U, ρ) the fibered manifold associated to X given above.

Let

$$\Psi : \mathbf{U} \subset J^k V \longrightarrow \mathbf{V} \subset C^{k,n} V$$

be the local diffeomorphism defined above.

Let $F_1, \dots, F_s$ be differentiable functions defined in a neighborhood of X (also written $\mathbf{V}$), such that:

$$\mathbf{V} \cap W = \{X \in \mathbf{V} : F_1 = \dots = F_s = 0\}.$$

Consider the system of partial differential equations Φ of order k in (V, U, ρ) with domain $\mathbf{U} = \Psi^{-1}(\mathbf{V})$, generated by $\{G_j = F_j \circ \Psi\}$. Let $Y = \Psi^{-1}(X)$.

We will verify that Φ satisfies the hypothesis of Theorem 1.

First, we observe that the integral jets $J\Phi$ are given by $\Psi^{-1}(W)$. Indeed:

$$Y \in J\Phi \Leftrightarrow G_j(Y) = 0 \Leftrightarrow (F_j \circ \Phi)(Y) = 0 \Leftrightarrow \Phi(Y) \in W.$$

Now, by hypothesis $\rho_{k-1}^k : (W \cap \mathbf{V}) \longrightarrow C^{k-1} V$ is an immersion, then

$$\pi_{k-1}^k : (J\Phi \cap \mathbf{U}) \longrightarrow J^{k-1} V,$$

is also an immersion and the kernel of $T_Y \pi_{k-1}^k$ vanishes, i.e.

$$T_Y \pi_{k-1}^k (T_Y Q_Y) = 0,$$

where Q_Y is defined above.

Then

$$T_Y \pi_{k-1}^k (T_Y J\Phi \cap T_Y Q_Y) = T_Y \pi_{k-1}^k (C_Y(\Phi)) = 0,$$

in consequence, the symbol $C_Y(\Phi)$ of Φ at $Y \in J\Phi$ vanishes, and condition 1) of Theorem 1 is verified.

To verify that $\pi_k^{k+1} : J(P\Phi) \longrightarrow J\Phi$ is a local submersion we consider the commutative diagram:

$$\begin{array}{ccc} C^{1,k} V & \xrightarrow{i^{1,k}} & C^1(C^k V) \\ \Psi^{k+1} \uparrow & & \uparrow \Psi^{1,k} \\ J^{k+1} V & \xrightarrow{\tilde{i}^{1,k}} & J^1(J^k V) \end{array}$$

where,

$$\tilde{i}^{1,k} : j_u^{k+1} g \in J^{k+1} V \longmapsto j_{j_u^k g}^1(j^k g) \in J^1(J^k, U, \alpha)$$

is the natural immersion [4].

Now using Proposition 1 we have,

$$Z \in J(P\Phi) \Leftrightarrow \Psi^{1,k}(Z) \in C^{1,n}W \cap C^{k+1,n}V = PW.$$

Hence $\Psi^{1,k}(J\Phi) = PW$.

Now, by hypothesis:

$$\rho_k^{k+1} : PW \longrightarrow W$$

is a local submersion and

$$\Psi^k \circ \pi_k^{k+1} = \rho_k^{k+1} \circ \Psi^{k+1}.$$

Consequently:

$$\pi_k^{k+1} : P(J\Phi) \longrightarrow J\Phi,$$

is a local submersion, and condition 2) of Theorem 1 is satisfied.

Finally, Φ is complete in Y because it is a sheaf of ideals generated by functions which vanish on the regular manifold $(\Psi)^{-1}(W \cap \mathbf{V})$.

It follows that Φ is a completely integrable system of partial differential equations. Hence, there exists a solution γ of Φ such that:

$$\gamma(0) = x, \quad j_0^k \gamma = X, \quad \& \quad j_u^k \gamma \in J\Phi, \quad u \in U_\gamma.$$

Let $S = \gamma(U_\gamma)$, then clearly S verifies:

$$x \in S, \quad C_x^k S = X, \quad C^k S \subset W,$$

in consequence S is a solution of W at $X \in W$.

If $\tilde{S}$ is another solution of W at X , then $C_x^k S = C_x^k \tilde{S}$.

In particular $T_x S = T_x \tilde{S}$ and therefore there are a fibred manifold (V, U, ρ) , associated to X , and parametrizations $\gamma, \tilde{\gamma}$ of S and $\tilde{S}$, respectively, which are sections of the fibred manifold (V, U, ρ) .

Since S (resp. $\tilde{S}$) is solution of W at X we have: $j^k \gamma \subset J\Phi$ (resp. $j^k \tilde{\gamma} \subset J\Phi$), and $j_0^k \gamma = j_0^k \tilde{\gamma}$. Then by uniqueness of germ solutions of completely integrable systems, there exists an open set $B \subset \mathfrak{R}^n$ such that $\gamma|_B = \tilde{\gamma}|_B$.

Let $A = \rho^{-1}(B)$, then $S \cap A = \tilde{S} \cap A$. $\square$

Remark. For $k = 1$ the submanifold W defines a Frobenius system. Indeed:

By condition 1) $\rho_0^1 : \longrightarrow M$ is a local immersion in a neighborhood $\mathbf{V}$ of X .

Then it is possible to define a section $\sigma = (\rho_0^1|_{W \cap \mathbf{V}})^{-1}$. Also we can find coordinate neighborhoods (V, x_i, j^j) and $(\mathbf{V}, x_i, y^j, p_i^j)$ of x and X respectively, such that the section σ can be expressed as,

$$\sigma(x_i, y^j) = (x_i, y^j, F_i^j(x_i, y^j)), \quad \text{with} \quad F_j^i : \mathbf{V} \longrightarrow \mathfrak{R}.$$

Then the manifold W defines a differentiable distribution D generated by:

$$L_i = \frac{\partial}{\partial x_i} + \sum F_i^j \frac{\partial}{\partial y^j}.$$

This distribution is involutive if and only if:

$$\frac{\partial F_i^j}{\partial x^k} + \sum_{l=1}^m \frac{\partial F_i^j}{\partial y^l} F_k^l = \frac{\partial F_k^j}{\partial x^i} + \sum_{l=1}^m \frac{\partial F_k^j}{\partial y^l} F_i^l.$$

By condition 2) we have:

$$\rho_1^2 : C^{1,n}W \cap C^{2,n}M \longrightarrow W.$$

is a local submersion.

Then given $Z \in W$, there exists $Z^2 \in C^{1,n}W \cap C^{2,n}M$ such that $\rho_1^2(Z^2) = Z$.

Since $Z \in C^2M$ we have:

$$p_{ik}^j(Z) = p_{ki}^j(Z).$$

Moreover, $Z \in C^{1,n}W$ then we have:

$$Z^2 = (x_i, y^j, F_i^j, \partial_{x_k}^\# F_i^j), \quad \partial_{x_k}^\# F_i^j = \frac{\partial F_i^j}{\partial x_k} + \sum_{l=1}^m \frac{\partial F_i^j}{\partial y^l} F_k^l.$$

In consequence,

$$p_{ik}^j(Z^2) = \partial_{x_k}^\# F_i^j(x_i, y^j) = \partial_{x_i}^\# F_k^j(x_i, y^j) = p_{ki}^j(Z^2),$$

and the involutivity of the distribution D defined above is verified.

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