

Nonoscillation theorems for second order quasilinear differential equations

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Abstract. Some nonoscillation criteria are obtained for second order quasilinear differential equations of the form

$$(E) \quad [r(t)|u'(t)|^{p-2}u'(t)]' + c(t)|u(t)|^{p-2}u(t) = 0,$$

where $p > 1$, $r(t) \in C^1([t_0, \infty); (0, \infty))$ and $c(t) \in C([t_0, \infty); \mathbb{R})$. These results extend some nonoscillation criteria of Hille, Wintner, Potter, Moore, Willett for the equation

$$[r(t)u'(t)]' + c(t)u(t) = 0$$

to equation (E).

1. Introduction

In this paper, we consider the following second order quasilinear differential equation

$$(E) \quad [r(t)|u'(t)|^{p-2}u'(t)]' + c(t)|u(t)|^{p-2}u(t) = 0,$$

where $p > 1$ is a constant, $r(t) \in C^1([t_0, \infty); (0, \infty))$ and $c(t) \in C([t_0, \infty); \mathbb{R})$ for some $t_0 \geq 0$. If $p = 2$, then equation (E) reduces to the linear differential equation

$$(E_1) \quad [r(t)u'(t)]' + c(t)u(t) = 0.$$

A solution of (E) is a function $u \in C^1([t_0, \infty), \mathbb{R})$ with $r|u'|^{p-2}u' \in C^1[t_0, \infty)$ and satisfies equation (E) on $[t_0, \infty)$. In [1], ELBERT established

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the existence and uniqueness of solutions to the initial value problem for (E) on $[t_0, \infty)$. It follows from (E) that any constant multiple of a solution of (E) is also a solution. A solution $u(t)$ of (E) is said to be nonoscillatory if there is a number $T \geq t_0$ such that $u(t) \neq 0$ for $t \geq T$. Equation (E) is said to be nonoscillatory if all its solutions are nonoscillatory.

There is a striking similarity in the oscillatory behavior between the second order quasilinear differential equation (E) and the corresponding linear equation (E_1) , see, for example, ELBERT [1,2], MIRZOV [9,10] and LI and YEH [7]. For example, Sturmian comparison and separation theorems for (E_1) , see for example [13], have been extended in a natural way to (E). Thus all solutions of (E) are either oscillatory or nonoscillatory, that is, the consistency of oscillatory and nonoscillatory solutions is excluded for equation (E).

For more recent papers of such similarity between (E) and (E_1) , we refer to KUSANO et al [4, 5, 6], LI and YEH [8].

In [7], the present authors established the following sufficient and necessary condition on the nonoscillation of (E).

Theorem 1.1 ([7], Theorem 3.2). *Equation (E) is nonoscillatory if and only if there are a number $T \geq t_0$ and a function $f \in C^1[T, \infty)$ satisfying*

$$c(t) + (p-1)r(t)|f(t)|^q - [r(t)f(t)]' \leq 0 \quad \text{for } t \geq T,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

The purpose of this paper is to establish some nonoscillation criteria of (E) by using Theorem 1.1. These results improve some nonoscillation criteria of HILLE [3], WINTNER [15], POTTER [12], MOORE [11], and WILLET [14], of (E_1) to equation (E).

2. Nonoscillation criteria

We assume, throughout this paper, that $\frac{1}{p} + \frac{1}{q} = 1$ and $\varepsilon = (q-1)q^{-p}$.

Theorem 2.1. *Let $g(t)$ and $\psi(t)$ be two continuously differentiable functions on $[t_0, \infty)$ satisfying $g(t) > 0$, $g'(t) \geq r^{1-q}(t)$ and $\psi'(t) \leq -c(t)$. If*

$$(2.1) \quad \limsup_{t \rightarrow \infty} g^{p-1}(t)|\psi(t)| < \varepsilon,$$

then equation (E) is nonoscillatory.

PROOF. By (2.1), there are two numbers $T \geq t_0$ and $k \in (0, \varepsilon)$ such that $|\psi(t)| < kg^{1-p}(t)$ for $t \geq T$. Let

$$f = -\frac{1}{\lambda r} \left(\lambda \psi + \frac{1 - \lambda k}{g^{p-1}} \right),$$

where $\lambda = q^{p-1}$. Then

$$(1) \quad \lambda k < \lambda \varepsilon = q^{p-1}(q-1)q^{-p} = \frac{q-1}{q} < 1,$$

$$(rf)' = -\psi' + \frac{(p-1)(1-\lambda k)g'}{\lambda g^p} \geq c + \frac{(p-1)(1-\lambda k)}{\lambda r^{q-1}g^p}$$

and

$$\begin{aligned} c + (p-1)r|f|^q - (rf)' &\leq c + (p-1)r \left| \frac{1}{\lambda r} \left(\lambda \psi + \frac{1 - \lambda k}{g^{p-1}} \right) \right|^q \\ -c - \frac{(p-1)(1-\lambda k)}{\lambda r^{q-1}g^p} &= (p-1)r^{1-q}g^{-p} \left\{ \left| \psi g^{p-1} + \frac{1 - \lambda k}{\lambda} \right|^q - \frac{1 - \lambda k}{\lambda} \right\} \\ &\leq (p-1)r^{1-q}g^{-p} \left\{ \left| k + \frac{1 - \lambda k}{\lambda} \right|^q - \frac{1 - \lambda k}{\lambda} \right\} \\ &= (p-1)r^{1-q}g^{-p} \left(\frac{1}{\lambda^q} - \frac{1}{\lambda} + k \right) = (p-1)r^{1-q}g^{-p}(k - \varepsilon) \leq 0. \end{aligned}$$

It follows from Theorem 1.1 that equation (E) is nonoscillatory.

If $p = 2$, $r(t) = 1$, $c(t) > 0$ for $t \geq t_0$ and $g(t) = t$, $\psi(t) = \int_t^\infty c(s)ds < \infty$, then Theorem 2.1 reduces to a Hille's criterion [3, p. 246].

Theorem 2.2. *Let $g(t)$ and $\psi(t)$ be two continuously differentiable functions on $[t_0, \infty)$ satisfying $g(t) > 0$, $g'(t) \leq -r^{1-q}(t)$ and $\psi'(t) \geq c(t)$. If (2.1) holds, then equation (E) is nonoscillatory.*

PROOF is the same as in Theorem 2.1 with the exception that $f = \frac{1}{\lambda r} \left(\lambda \psi + \frac{1 - \lambda k}{g^{p-1}} \right)$ and $\lambda k \leq \lambda \varepsilon$ in (1).

Theorem 2.3. *Let $g(t)$ and $\psi(t)$ be two continuously differentiable functions on $[t_0, \infty)$ satisfying $g(t) > 0$, $g'(t) \geq r^{1-q}(t)$ and $\psi'(t) \leq -c(t)$. If there exists a number $k > 0$ such that*

$$(2.2) \quad -k^{\frac{1}{q}} - k \leq g^{p-1}(t)\psi(t) \leq k^{\frac{1}{q}} - k \leq \varepsilon.$$

then equation (E) is nonoscillatory.

PROOF. Let

$$f = -\frac{1}{r} \left(\psi + \frac{k}{g^{p-1}} \right).$$

Then

$$(rf)' = -\psi' + \frac{(p-1)kg'}{g^p} \geq c + \frac{(p-1)k}{r^{q-1}g^p}$$

and

$$\begin{aligned} c + (p-1)r|f|^q - (rf)' &\leq c + (p-1)r \left| \frac{1}{r} \left(\psi + \frac{k}{g^{p-1}} \right) \right|^q - c - \frac{(p-1)k}{r^{q-1}g^p} \\ &= (p-1)r^{1-q}g^{-p} \left\{ \left| \psi g^{p-1} + k \right|^q - k \right\} \leq (p-1)r^{1-q}g^{-p}(k - k) = 0. \end{aligned}$$

It follows from Theorem 1.1 that equation (E) is nonoscillatory.

If $p = 2$, $g(t) = 1 + \int_{t_0}^t \frac{1}{r(s)} ds$ as $t \rightarrow \infty$ and $\psi(t) = \int_t^\infty c(s) ds < \infty$, then Theorem 2.3 reduces to a Moore's criterion [11, Theorem 6].

Theorem 2.4. *Let $g(t)$ and $\psi(t)$ be two continuously differentiable functions on $[t_0, \infty)$ satisfying $g(t) > 0$, $g'(t) \leq -r^{1-q}(t)$ and $\psi'(t) \geq c(t)$. If there exists a number $k > 0$ such that (2.2) holds, then equation (E) is nonoscillatory.*

PROOF. Let

$$f = \frac{1}{r} \left(\psi + \frac{k}{g^{p-1}} \right).$$

Then

$$(rf)' = \psi' - \frac{(p-1)kg'}{g^p} \geq c + \frac{(p-1)k}{r^{q-1}g^p}$$

and

$$\begin{aligned} c + (p-1)r|f|^q - (rf)' &\leq c + (p-1)r \left| \frac{1}{r} \left(\psi + \frac{k}{g^{p-1}} \right) \right|^q - c - \frac{(p-1)k}{r^{q-1}g^p} \\ &= (p-1)r^{1-q}g^{-p} \left\{ \left| \psi g^{p-1} + k \right|^q - k \right\} \leq (p-1)r^{1-q}g^{-p}(k - k) = 0. \end{aligned}$$

It follows from Theorem 1.1 that equation (E) is nonoscillatory.

If $p = 2$, $g(t) = \int_t^\infty \frac{1}{r(s)} ds < \infty$ and $\psi(t) = \int_{t_0}^t c(s) ds$, then Theorem 2.4 reduces to a MOORE's criterion [11, Theorem 6].

Example 2.5. Consider the equation

$$(E_2) \quad (t^\alpha |u'|^{p-2} u')' + \lambda t^{\alpha-p} |u|^{p-2} u = 0, \quad \text{for } t > 0$$

where $\alpha > p - 1$ and $\lambda > 0$ are two constants. Let

$$g(t) = \int_t^\infty s^{-\frac{\alpha}{p-1}} ds$$

and

$$\psi(t) = \int_0^t \lambda s^{\alpha-p} ds.$$

Then

$$g(t) = \frac{p-1}{\alpha-p+1} t^{\frac{p-1-\alpha}{p-1}}, \quad \psi(t) = \frac{\lambda}{\alpha-p+1} t^{\alpha-p+1}$$

and

$$g^{p-1}(t)\psi(t) = \frac{\lambda}{\alpha-p+1} \left(\frac{p-1}{\alpha-p+1} \right)^{p-1}.$$

If

$$\lambda \leq \left(\frac{\alpha-p+1}{p} \right)^p,$$

then $g^{p-1}(t)\psi(t) \leq \varepsilon$. It follows from Theorem 2.4 that equation (E_2) is nonoscillatory.

Theorem 2.6. Let $g(t)$ be a continuously differentiable function on $[t_0, \infty)$ such that $g(t) > 0$ and $g'(t) \geq r^{1-q}(t)$. If there exists a continuously differentiable function $\psi(t)$ on $[t_0, \infty)$ such that $\lim_{t \rightarrow \infty} \psi(t)$ exists and $\psi'(t) \leq -g^{p-1}(t)c(t)$, then equation (E) is nonoscillatory.

PROOF. Since $\lim_{t \rightarrow \infty} \psi(t)$ exists, there exist two real numbers $T \geq t_0$ and M such that $0 < M + \psi(t) \leq 1$ for $t \geq T$. Let

$$f = -\frac{M + \psi}{r g^{p-1}}.$$

Then

$$(rf)' = \frac{(p-1)(M + \psi)g'}{g^p} - \frac{\psi'}{g^{p-1}} \geq c + \frac{(p-1)(M + \psi)}{r^{q-1}g^p}$$

which implies

$$\begin{aligned} c + (p-1)r|f|^q - (rf)' &\leq c + (p-1)r \left| \frac{M+\psi}{rg^{p-1}} \right|^q - c - \frac{(p-1)(M+\psi)}{r^{q-1}g^p} \\ &\leq (p-1)r^{1-q}g^{-p}[(M+\psi) - (M+\psi)] = 0. \end{aligned}$$

It follows from Theorem 1.1 that equation (E) is nonoscillatory.

Theorem 2.7. *Let $g(t)$ be a continuously differentiable function on $[t_0, \infty)$ such that $g(t) > 0$ and $g'(t) \leq -r^{1-q}(t)$. If there exists a continuously differentiable function $\psi(t)$ such that $\lim_{t \rightarrow \infty} \psi(t)$ exists and $\psi'(t) \leq -g^{p-1}(t)c(t)$, then equation (E) is nonoscillatory.*

PROOF. Since $\lim_{t \rightarrow \infty} \psi(t)$ exists, there exist two real numbers $T \geq t_0$ and M such that $0 < M - \psi(t) \leq 1$ for $t \geq T$. Let

$$f = \frac{M - \psi}{rg^{p-1}}.$$

Then

$$(rf)' = -\frac{(p-1)(M-\psi)g'}{g^p} - \frac{\psi'}{g^{p-1}} \geq c + \frac{(p-1)(M-\psi)}{r^{q-1}g^p},$$

which implies

$$\begin{aligned} c + (p-1)r|f|^q - (rf)' &\leq c + (p-1)r \left| \frac{M-\psi}{rg^{p-1}} \right|^q - c - \frac{(p-1)(M-\psi)}{r^{q-1}g^p} \\ &\leq (p-1)r^{1-q}g^{-p}[(M-\psi) - (M-\psi)] = 0. \end{aligned}$$

It follows from Theorem 1.1 that equation (E) is nonoscillatory.

Theorem 2.8. *Let*

$$c(t) \leq \frac{1}{h^p(t)},$$

where $h(t) \in C^1([t_0, \infty); (0, \infty))$. If either

$$h'(t) - \frac{1}{r^{q-1}(t)} \geq \frac{1}{p-1} \quad \text{for all } t \text{ large enough,}$$

or

$$\lim_{t \rightarrow \infty} \left(h'(t) - \frac{1}{r^{q-1}(t)} \right) = L \text{ exists and } L > \frac{1}{p-1},$$

then equation (E) is nonoscillatory.

PROOF. It follows from the assumption that there is a number $T \geq t_0$ such that

$$h'(t) - \frac{1}{r^{q-1}(t)} \geq \frac{1}{p-1} \quad \text{for all } t \geq T.$$

Let

$$f = -\frac{1}{rh^{p-1}}.$$

Then

$$\begin{aligned} c + (p-1)r|f|^q - (rf)' &\leq \frac{1}{h^p} + \frac{p-1}{r^{q-1}h^p} - \frac{(p-1)h'}{h^p} \\ &= (p-1)h^{-p} \left(\frac{1}{p-1} + \frac{1}{r^{q-1}} - h' \right) \leq 0 \quad \text{for } t \geq T. \end{aligned}$$

Hence, by Theorem 1.1, equation (E) is nonoscillatory.

If $p = 2$ and $r(t) = 1$, then Theorem 2.7 reduces to a POTTER's criterion [12, Theorem 1.5].

Theorem 2.9. *Let $\psi(t)$ be a nonnegative continuously differentiable function on $[t_0, \infty)$ such that $\psi'(t) \leq -c(t)$. If*

$$\int_t^\infty \frac{\psi^q(s)}{r^{q-1}(s)} ds \leq p^{-q}\psi(t),$$

then equation (E) is nonoscillatory.

PROOF. Let

$$f = -\frac{1}{r(t)} \left(\psi(t) + (p-1)p^q \int_t^\infty \frac{\psi^q(s)}{r^{q-1}(s)} ds \right).$$

Then

$$[r(t)f(t)]' \geq c(t) + \frac{(p-1)p^q\psi^q(t)}{r^{q-1}(t)}.$$

Hence

$$\begin{aligned}
& c(t) + (p-1)r(t)|f(t)|^q - [r(t)f(t)]' \\
& \leq c(t) + (p-1)r(t) \left| \frac{1}{r(t)} \left(\psi(t) + (p-1)p^q \int_t^\infty \frac{\psi^q(s)}{r^{q-1}(s)} ds \right) \right|^q \\
& \quad - c(t) - \frac{(p-1)p^q \psi^q(t)}{r^{q-1}(t)} \\
& = (p-1)r^{1-q}(t) \left\{ \left| \psi(t) + (p-1)p^q \int_t^\infty \frac{\psi^q(s)}{r^{q-1}(s)} ds \right|^q - p^q \psi^q(t) \right\} \\
& \leq (p-1)r^{1-q}(t) \left[|\psi(t) + (p-1)p^q \cdot p^{-q} \psi(t)|^q - p^q \psi^q(t) \right] = 0.
\end{aligned}$$

It follows from Theorem 1.1 that equation (E) is nonoscillatory.

Theorem 2.10. *Let $\psi(t)$ be a nonnegative continuously differentiable function on $[t_0, \infty)$ such that $\psi'(t) \leq -c(t)$, and let*

$$\psi_1(t) = \int_t^\infty \frac{\psi^q(s)}{r^{q-1}(s)} \exp \left((p-1)p^{q-1} \int_t^s \frac{\psi^{q-1}(\xi)}{r^{q-1}(\xi)} d\xi \right) ds.$$

If $\psi_1(t) \leq p^{1-q}\psi(t)$, then equation (E) is nonoscillatory.

PROOF. Let

$$f = -\frac{1}{r} (\psi + (p-1)p^{q-1}\psi_1).$$

Then

$$\begin{aligned}
(rf)' &= -\psi' - (p-1)p^{q-1}\psi_1' \\
&\geq c + (p-1)p^{q-1}r^{1-q} ((p-1)p^{q-1}\psi^{q-1}\psi_1 + \psi^q).
\end{aligned}$$

Hence

$$\begin{aligned}
c + (p-1)r|f|^q - (rf)' &\leq c + (p-1)r^{1-q}[\psi + (p-1)p^{q-1}\psi_1]^q \\
&\quad - c - (p-1)p^{q-1}r^{1-q}[(p-1)p^{q-1}\psi^{q-1}\psi_1 + \psi^q] \\
&= (p-1)r^{1-q}[\psi + (p-1)p^{q-1}\psi_1] \{ [\psi + (p-1)p^{q-1}\psi_1]^{q-1} - p^{q-1}\psi^{q-1} \} \\
&\leq (p-1)r^{1-q}[\psi + (p-1)p^{q-1}\psi_1] \{ [\psi + (p-1)p^{q-1} \cdot p^{1-q}\psi]^{q-1} \\
&\quad - p^{q-1}\psi^{q-1} \} \\
&= (p-1)r^{1-q}[\psi + (p-1)p^{q-1}\psi_1][p^{q-1}\psi^{q-1} - p^{q-1}\psi^{q-1}] = 0.
\end{aligned}$$

It follows from Theorem 1.1 that equation (E) is nonoscillatory.

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