

On Schreier extension of rings without zero-divisors.

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1. In the theory of rings an important rôle is played by the extensions of rings. C. J. EVERETT [1]¹⁾ has considered and completely solved the ring-theoretic analogue of O. SCHREIER's extension theory for groups, that is, the following problem: Let R and S be given rings; the problem is to construct all rings T containing an ideal S' isomorphic to S such that

$$(1) \quad T/S' \approx R$$

holds. We call every solution T of this problem a *Schreier extension* (or briefly an extension) of S by R .

The present paper gives a necessary and sufficient conditions under which a Schreier extension contains no zero-divisors²⁾, resp. a Schreier extension without zero-divisors has a unity. By making use of these theorems we shall get a new proof of a former result of mine stating that an arbitrary ring without zero-divisors has always a minimal extension with unity and without zero-divisors [5].

2. Recently L. RÉDEI has introduced a fundamental method for constructing new structures from given ones. The new structure has been called by him the *skew product* of the given structures [2], [3]. He has also described Schreier's extension theory for rings by making use of the notion of skew product by which one may easily get a survey over all extensions [4]. In what follows we shall use RÉDEI's treatment, by the aid of which EVERETT's extension theorem for rings may be formulated as follows.

Let $R = 0, a, b, \dots$ and $S = 0, \alpha, \beta, \dots$ be two arbitrary rings. We consider a Rédeian skew product $T = R \circ S$ of the rings R and S . The elements of T are all pairs (a, α) with $a \in R, \alpha \in S$, further addition and multiplication in T are defined as follows:

$$(2) \quad (a, \alpha) + (b, \beta) = (a + b, [a, b] + \alpha + \beta),$$

$$(3) \quad (a, \alpha)(b, \beta) = (ab, \{a, b\} + \alpha b + a\beta + \alpha\beta)$$

where

$$(4) \quad [a, b], \{a, b\}, \alpha b, a\beta (\in S)$$

¹⁾ The numbers in brackets refer to the Bibliography at the end of this paper.

²⁾ This problem is due to O. STEINFELD.

are functions of two variables with values in S and satisfying the following conditions:

$$(5) \quad [0, a] = [a, 0] = \{a, 0\} = \{0, a\} = a\underline{0} = 0a = \underline{0}a = a0 = \underline{0}.$$

Such four functions determine uniquely the skew product $T = R \circ S$. Conversely, by (2), (3) and (5) we obtain

$$\begin{aligned} (a, \underline{0}) + (b, \underline{0}) &= (a+b, [a, b]), & (a, \underline{0})(b, \underline{0}) &= (ab, \{a, b\}), \\ (0, \alpha)(b, \underline{0}) &= (0, \alpha b), & (a, \underline{0})(0, \beta) &= (0, a\beta), \end{aligned}$$

therefore the four functions (4) are uniquely determined by the skew product $T = R \circ S$.

The skew product $T = R \circ S$ is a Schreier extension of S by R obeying (2), (3) and (5) if and only if the following conditions are satisfied for all elements $a, b, c \in R$ and $\alpha, \beta, \gamma \in S$:

$$\begin{aligned} (6) \quad & a(\beta + \gamma) = a\beta + a\gamma, & (\alpha + \beta)c &= \alpha c + \beta c, \\ (7) \quad & (a+b)\gamma + [a, b]\gamma = a\gamma + b\gamma, & \alpha(b+c) + \alpha[b, c] &= \alpha b + \alpha c, \\ (8) \quad & a\beta\gamma = (a\beta)\gamma, & \alpha\beta c &= \alpha(\beta c), \\ (9) \quad & ab\gamma + \{a, b\}\gamma = a(b\gamma), & \alpha bc + \alpha\{b, c\} &= (\alpha b)c, \\ (10) \quad & (a\beta)c &= a(\beta c), \\ (11) \quad & (\alpha b)\gamma &= \alpha(b\gamma), \\ (12) \quad & \{ab, c\} + \{a, b\}c &= \{a, bc\} + a\{b, c\}, \\ (13) \quad & [a, b] &= [b, a], \\ (14) \quad & [a, b] + [a+b, c] &= [a, b+c] + [b, c], \\ & [a, b]c + \{a+b, c\} &= [ac, bc] + \{a, c\} + \{b, c\}, \\ (15) \quad & a[b, c] + \{a, b+c\} &= [ab, ac] + \{a, b\} + \{a, c\}. \end{aligned}$$

These rings T exhaust all Schreier extensions of S by R . The elements $(0, \alpha)$ form an ideal S' of T which is isomorphic to S under the isomorphism $(0, \alpha) \rightarrow \alpha$, further

$$T/S' \approx R \quad ((a, \underline{0}) + S' \rightarrow a)$$

holds. If the functions (4) satisfy conditions (6)–(15), then the first two of them are called *additive* and *multiplicative factor system*, respectively, while the two last ones are said to be *left* and *right operator set*, respectively. (It is important to emphasize that these operations differ obviously from the usual ones.)

If $[a, b] = \underline{0}$ and $\{a, b\} = \underline{0}$ for all $a, b \in R$ then the extensions equivalent to T are said to be *splitting extensions of S by R* . (See [4]¹⁴.)

3. In the discussions of Schreier extensions without zero-divisors we shall need the following

Lemma. *Let S be a ring without zero-divisors and T a Schreier extension of S by the ring R . If for the elements $a \neq 0, b \neq 0$ ($\in R$) we have*

$ab = 0$, then

$$(16) \quad \{a, b\} + a\beta + \alpha b + \alpha\beta = \underline{0}$$

in S if and only if one of the equations

$$(16') \quad a\xi = \alpha\xi \quad \text{or} \quad b\xi = \beta\xi \quad (a \neq 0, b \neq 0)$$

is satisfied by some element $\xi \neq \underline{0}$.

Proof. Let us multiply (16) by $\xi (\neq \underline{0})$ on both sides. Since $ab = 0$, by (9), (5) and (11) we have

$$\xi\{a, b\}\xi = \xi(ab\xi + \{a, b\}\xi) = \xi(a(b\xi)) = \xi a \cdot b\xi.$$

Therefore, from (16), in view of (11), we get

$$(\xi a + \xi \alpha)(b\xi + \beta\xi) = \underline{0}.$$

The ring S has no zero-divisors, consequently, one of the factors must be zero. This means that one of the equations (16') has a non-trivial solution, in fact.³⁾ One may write α and β instead of $-\alpha$ and $-\beta$ respectively.

The converse of the assertion is clear.

Now we are going to prove the following

Theorem 1. *A Schreier extension T of the ring S by the ring R contains no zero-divisors if and only if S has this property and the equation*

$$(17) \quad a\xi = \alpha\xi \quad (a \neq 0)$$

admits the only solution $\xi = \underline{0}$.

Proof. In order to prove the necessity of the conditions, let us suppose that the Schreier extension T has no zero-divisors. It is obvious that neither S contains zero-divisors, since S is an ideal in T . On the other hand, if the equation (17) is satisfied by an element $\xi \neq \underline{0}$, then according to

$$(a, -\alpha)(0, \xi) = (0, a\xi - \alpha\xi) = (0, \underline{0})$$

T has zero-divisors. This contradiction implies that the only solution of (17) is $\xi = \underline{0}$, indeed.

Conversely, the stated conditions are sufficient too. Supposing

$$(18) \quad (a, \alpha)(b, \beta) = (0, \underline{0})$$

we shall prove that one of the factors must be zero. (18) is equivalent to

$$(19) \quad ab = 0, \{a, b\} + a\beta + \alpha b + \alpha\beta = \underline{0}.$$

a) If $a = b = 0$, then $\alpha\beta = \underline{0}$ and hence either $\alpha = \underline{0}$ or $\beta = \underline{0}$, i. e. one of the factors in (18) is zero.

b) If in (19₁) just one of the factors vanishes, say, $b = 0$, then we get from (19₂)

$$a\beta = (-\alpha)\beta \quad (a \neq 0).$$

By (17) this means $\beta = \underline{0}$, consequently, one of the factors in the product (18) is zero.

³⁾ Since S has no zero-divisors, the equation $r\xi = \alpha\xi$ implies $\xi r = \xi\alpha$ ($r \in R$; $\alpha, \xi \in S$).

c) If $ab=0$ holds for $a \neq 0$, $b \neq 0$, then, equation (17) having no solution except $\xi=0$, the lemma implies

$$(a, \alpha)(b, \beta) = (0, \varrho) \neq (0, 0) \quad (a \neq 0, b \neq 0, ab=0; \varrho \neq 0).$$

Consequently, T has no zero-divisors and the proof of Theorem 1 is completed.

As an immediate consequence of Theorem 1 we get the following

Corollary. A Schreier extension of a ring without zero-divisors by an arbitrary ring contains no zero-divisors if and only if equation (17) has no solution other than the trivial one.

4. In this section we turn our attention to the proof of the following

Theorem 2. A Schreier extension $T=R \circ S$ without zero-divisors has a unity if and only if R contains a unity e and there exists an element $r_1 \in S$ such that

$$(20) \quad e\xi = \xi - r_1\xi$$

holds for every element ξ of S . Then (e, r_1) is the unity of T .

Proof. The conditions are necessary. For, let us denote by (e, r_1) the unity of T and by (a, α) an arbitrary element in T . Then

$$(a, \alpha)(e, r_1) = (ae, \{a, e\} + ae + ar_1 + \alpha r_1) = (a, \alpha).$$

Hence it is clear that e must be the unity of R and

$$(21) \quad \{a, e\} + ae + ar_1 + \alpha r_1 = \alpha.$$

By (5) $a=0$ implies

$$ae + \alpha r_1 = \alpha,$$

whence we have (20)⁴⁾. If, however, $a \neq 0$, then from (21) multiplied by an element $\xi (\neq 0)$ on both sides, with regard to (9) and (11), we conclude

$$(\xi a + \xi \alpha)(e\xi + r_1\xi - \xi) = 0 \quad (a \neq 0, \xi \neq 0).$$

Since T has no zero-divisors, by Theorem 1 we have

$$e\xi = \xi - r_1\xi.$$

In order to prove the sufficiency of the conditions, let us consider the product

$$(a, \alpha)(e, r_1) = (ae, \{a, e\} + ae + ar_1 + \alpha r_1).$$

Introducing the shorter notation

$$\varrho = \{a, e\} + ae + ar_1 + \alpha r_1,$$

we have only to verify that $\varrho = \alpha$. Multiplying the last equation by $\xi (\neq 0)$ on both sides, it follows (using (9))

$$\xi \varrho \xi = \xi a (e\xi - \xi + r_1\xi) + \xi \alpha (e\xi + r_1\xi).$$

By (20) we have

$$\xi \varrho \xi = \xi a \xi$$

and this implies, as asserted, that $\varrho = \alpha$ which completes the proof.

⁴⁾ Similarly as in ³⁾.

5. We are going to give an example for a splitting Schreier extension without zero-divisors.

Let $S = \underline{0}, \alpha, \beta, \dots$ denote an arbitrary ring without zero-divisors. The ring of rational integers will be denoted by I , and the residue class ring of $I \bmod m$ by $I(m)$ where m is a rational integer. Let us denote by $\bar{0}, 1, \dots, m-1$ the elements of $I(m)$, if $m=0$, we have obviously $\bar{a}=a$. The integers r for which with a suitable element $\varrho (\in S)$

$$(22) \quad r\xi = \varrho \xi = \xi \varrho \quad (\xi \in S)$$

has a solution $\xi \neq \underline{0}$, form an ideal of I . Let the basis element of this ideal be $m (\geq 0)$. We construct the splitting Schreier extension T of S by $I(m)$. The elements of T are of the form $(\bar{a}, \alpha) (\bar{a} \in I(m), \alpha \in S)$. In the ring T we add and multiply according to the rules

$$\begin{aligned} (\bar{a}, \alpha) + (\bar{b}, \beta) &= (\overline{a+b}, \alpha + \beta), \\ (\bar{a}, \alpha) (\bar{b}, \beta) &= (\overline{ab}, \bar{b}\alpha + \bar{a}\beta + \alpha\beta) \end{aligned}$$

where $\bar{x}\xi = x\xi$, x denoting the least non-negative representative of the residue class $\bar{x}$.

Now it is easy to prove the following result [5]:

The splitting Schreier extension T of S by $I(m)$ is the minimal⁵⁾ extension with unity and without zero-divisors of the ring S .

It is readily seen that the only solution of the equation

$$\bar{a}\xi = \alpha\xi \quad (\bar{a} \neq 0)$$

is $\xi = \underline{0}$, therefore, by Corollary, T contains no zero-divisors. By Theorem 2 T has a unity, namely $(\bar{1}, \underline{0})$. Finally, the minimality of T follows from the fact that m is the minimal integer (≥ 0) satisfying (22). The proof is thus completed.

Bibliography.

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⁵⁾ This means that no proper subring of T is a ring with unity containing S .