

On arbitrary systems of linear equations.

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In a recent paper [2]¹⁾ S. GACSÁLYI has shown that any compatible system of linear equations (with an arbitrary cardinal number of the equations resp. of the unknowns) over a skew field F admits a solution in F .²⁾ ³⁾ A system of linear equations is said to be compatible if every linear dependence relation for a finite number of linear forms on the left hand sides of the equations is satisfied also by the given elements of F on the corresponding right hand sides. This theorem seems to be new and gives rise to the problem of extending the classical methods for solution of systems of linear equations (see [3], p. 106) from the finite case to the infinite case. The possibility of this extension will be shown in the present note.

Let us consider an arbitrary compatible system

$$(1) \quad f_\nu(\dots, x_\lambda, \dots) = b_\nu \in F$$

of linear equations over a skew field F , where λ and ν run resp. over an arbitrary set of indices (which is not assumed to be ordered). Here

$$(2) \quad f_\nu(\dots, x_\lambda, \dots) = \sum_\lambda a_{\nu, \lambda} x_\lambda$$

is a linear form of the x_λ 's and the coefficients $a_{\nu, \lambda} \in F$ are $= 0$ for all but a finite number of λ . The unknowns x_λ can also be regarded as indeterminates and then the set of all x_λ 's generates a module M of linear forms over F . Let N be the submodule of M generated by the f_ν 's. The set of all f_ν 's contains, by ZORN's lemma (or the equivalent lemma of TUKEY), a maximal linearly independent subsystem S , and, by the compatibility of the system (1), the set of all solutions of the system $f_\nu = b_\nu$ ($f_\nu \in S$) coincides with the set

¹⁾ The numbers in brackets refer to the Bibliography at the end of this note.

²⁾ Each single equation, however, contains a finite number of unknowns only.

³⁾ This seems to be in contradiction with a statement in [1] (see „Remarque“ p. 56). This seeming contradiction arises from the definition of a system of linear equations given in [1] which implies that a system $x_\lambda = \xi_\lambda \in F$ can be a solution only if $\xi_\lambda = 0$ holds for all but a finite number of the λ 's.

of all solutions of the whole system (1). Hence we may assume in the sequel that the given system of all linear forms f_v in (1) is independent.

Now, it is easy to show that there exists a (possibly void) subset $X' = \dots, x'_\mu, \dots$ of the x_v 's such that the f_v 's and the elements of X' form a linearly independent generating system (i. e. a basis) of M or, in other words, M is the direct sum of N and the submodule (X') of M generated by X' :

$$(3) \quad M = N + (X').$$

This means that for each indeterminate x_λ an identity of the form

$$(4) \quad x_\lambda = \sum_{i=1}^{n_\lambda} c_{\lambda v_i} f_{v_i} + \sum_{j=1}^{m_\lambda} d_{\lambda \mu_j} x'_{\mu_j}$$

($c_{\lambda v_i} \in F, d_{\lambda \mu_j} \in F, m_\lambda$ and n_λ are integers ≥ 0)

holds (with use of (2)). Substituting (4) into (2) (as well as (2) into (4)) we obtain identities in the indeterminates x_λ ,⁴⁾ and so it is evident that the solutions of the given system (1) coincide with those of the system consisting of all the equations

$$(5) \quad x_\lambda = \sum_{i=1}^{n_\lambda} c_{\lambda v_i} b_{v_i} + \sum_{j=1}^{m_\lambda} d_{\lambda \mu_j} x'_{\mu_j}.$$

It is clear, however, that *any solution of the system (5) can be obtained by substituting for the indeterminates x'_μ arbitrary elements of F , and by determining the values of the other unknowns x_λ from (5).*

Thus we have shown that the classical method for solving systems of linear equations can be extended from the finite to the infinite case. Note that the cardinal number $\bar{f}$ of the unknowns to be "freely chosen" (i. e. the cardinal number of the set X') is given by the rank of the factor module M/N (see (3)). Consequently, the cardinal number u of the unknowns and the cardinal number e of the equations of the given system (1) (which is assumed to be independent) verify the relation

$$u = e + \bar{f}$$

in analogy with the finite case.

The above statement implies, in particular, the result of GACSÁLYI and the following corollary to this: the compatible system (1) of linear equations admits exactly one solution if and only if (the set X' is void i. e.) the linear forms $f_v(x)$ generate the whole module M of the x_λ 's over F .

Finally we prove the validity of (3). Let the system $X' = \dots, x'_\lambda, \dots$ be a maximal mod N independent subset of the set of all indeterminates x_λ . [A subset of indeterminates x_λ is called mod N independent if it contains no

⁴⁾ This is assured by the independence of the system $\dots, f_v, \dots, x'_\mu, \dots$ in M .

finite subset $x_1, \dots, x_k$ ($k > 0$) such that a relation $e_1 x_1 + \dots + e_k x_k \in N$ with $0 \neq e_i \in F$ ($i = 1, \dots, k$) holds.] Such a maximal system X' always exists (by TUKEY's lemma) and by definition satisfies (3).

Bibliography.

- [1] N. BOURBAKI, *Éléments de mathématique*. I. Livre II. Algèbre. Chapitre II. Algèbre linéaire. (Paris, 1947.)
- [2] S. GACSÁLYI, On algebraically closed abelian groups. *Publ. Math. Debrecen*, **2** (1952) 292—296.
- [3] B. L. VAN DER WAERDEN, *Moderne Algebra*. II. (Berlin, 1940.)

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